Bill Bruce 60 and Terry Wall 75
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Symmetry properties of the roots of Bernstein-Sato polynomials and duality of D-modules

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The monic generator of the ideal of $\mathbb{C}[s]$ of such $b(s)$ is called the *Bernstein-Sato* polynomial, or the *b-function* of h, and it is denoted by $b_h(s)$.

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The b-function of h appears as the minimal polynomial of the action of s on $\mathscr{D}[s]h^s/\mathscr{D}[s]hh^s$.

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$(**)$ But the operator $P(s)$ in the Bernstein relation $b(k)h^k = P(k)(h^{k+1})$ becomes highly complicated!

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M. SAITO (1994): The roots of the b-function of any analytic germ $h : (\mathbb{C}^d, 0) \to (\mathbb{C}, 0)$ are contained in the interval $]-d, 0[$.

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DEFINITION: We say that a germ of divisor (= hypersurface) $(D, 0) \subset (\mathbb{C}^d, 0)$ given by a reduced equation $h = 0$ is *free* if the module of tangent vector fields (or logarithmic derivations) $\mathrm{Der}(-\log D)$ is a free $\mathscr{O}$-module (of rank d).

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- New examples by adding "adjoint divisors" (Mond, Schulze).

Some properties of free divisors

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(2) (Calderón-Moreno, N-M) Any locally quasi-homogeneous free divisor is Koszul and of linear Jacobian type (e.g. free hyperplanes arrangements; discriminants of stable maps in Mather's "nice dimensions").

Some properties of free divisors

Let $(D,0) \subset (\mathbb{C}^d,0)$ be a free germ given by a reduced equation $h=0$, a let $\delta_1,\dots,\delta_d$ be a basis of the logarithmic vector fields: $\delta_i(h)=\alpha_i h$. The *Jacobian ideal* of h (or of D) is $J_h := \langle h, h'_{x_1},\dots,h'_{x_d}\rangle$.

DEFINITION: (a) We say that D is *Koszul* if the symbols $\sigma(\delta_1),\dots,\sigma(\delta_d)$ form a regular sequence in $\operatorname{gr}\mathscr{D} = \mathscr{O}[\xi_1,\dots,\xi_d]$.

(b) We say that D is *strongly Koszul* if

$$\sigma(\delta_1)-\alpha_1 s,\dots,\sigma(\delta_d)-\alpha_d s, h$$

form a regular sequence in $\operatorname{gr}_T \mathscr{D}[s] = \mathscr{O}[\xi_1,\dots,\xi_d,s]$.

(c) We say that D is of *linear Jacobian type* if the Jacobian ideal is of linear type.

RESULTS:
(1) Any plane curve is a Koszul free divisor.

(2) (Calderón-Moreno, N-M) Any locally quasi-homogeneous free divisor is Koszul and of linear Jacobian type (e.g. free hyperplanes arrangements; discriminants of stable maps in Mather's "nice dimensions").

(3) (N-M) For a free divisor: linear Jacobian type $\Leftrightarrow$ strongly Koszul $\Rightarrow$ Koszul.

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DEFINITION (BUCHWEITZ-MOND): A reduced hypersurface $D \subset \mathbb{C}^d$ is a *linear free divisor (LFD)* if the (coherent) sheaf of logarithmic vector fields with respect to D, $Der(-\log D)$, has a global basis of homogeneous vector fields of degree 0, i.e. linear combination of the partial derivatives with coefficient linear forms.

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THEOREM (GRANGER-SCHULZE, 2010): The b-function of any reductive prehomogeneous determinant or of any regular special LFD satisfies the symmetry property

$$b(-s-2) = \pm b(s).$$

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Hypothesis ($\star$) is rather technical. But free divisors of linear Jacobian type (in particular, locally quasi-homogeneous free divisors) satisfy ($\star$).

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The existence of a Spencer logarithmic free resolution of $\mathscr{D}[s]h^s$ exactly means that:

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Now we can apply a duality formula of Calderón-Moreno, N-M to obtain that:

$$\mathrm{Dual}_{\mathscr{D}[s]} \left(\mathscr{D}[s] \overset{L}{\otimes}_{\mathscr{V}[s]} \mathscr{O}[s]h^s \right) \simeq \mathscr{D}[s] \overset{L}{\otimes}_{\mathscr{V}[s]} \mathscr{O}[s]h^{-s-1}$$

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By means of the Thom-Sebastiani join, it is possible to construct non-trivial examples where the conjecture is true with singular locus of arbitrary codimension.

Happy birthday for Terry and Bill, and thank you!