k-cut Model for the Brownian Continuum Random Tree*

Minmin Wang

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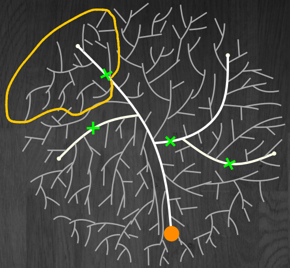
*Based on arXiv 2007.11080

Cutting down random trees

Meir & Moon '70s



- Imagine a network T_n (rooted tree with n nodes)
- At rate 1, a uniform node is attacked.
  It is then removed from T_n along with the subtree above.
- Iterate on the remaining tree until nothing left.
- $X(T_n)$ = total number of attacks $\leq n$

Let T_n be a uniform tree with n nodes. Then Panholzer 06, Janson 06 show that

$$\frac{X(T_n)}{\sqrt{n}} \xrightarrow{(d)} Z \sim \text{Rayleigh dist.}$$

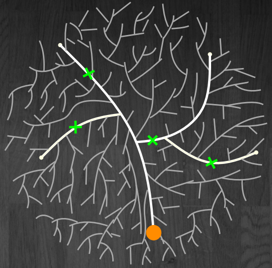
An invariance principle

- [Janson '06] Holds more generally as T_n can be replaced by a conditioned Galton–Watson tree with finite variance.
- [Aldous '93] The above conditioned Galton–Watson tree has a scaling limit: Brownian Continuum Random Tree (CRT).
- Question: Does the previous cutting process of T_n converge to a “cutting” of the CRT, so that $Z =$ functional of the CRT?
- Yes, according to Addario-Berry, Broutin & Holmgren '15, Bertoin & Miermont '13, Abraham & Delmas '13

Cutting down resilient random trees

Cai, Devroye, Holmgren & Skerman 2019

- Imagine a resilient network T_n (rooted tree with n nodes)
- At rate 1, a uniform node is attacked.
  It is then removed from T_n after k attacks.
- Iterate on the remaining tree until nothing left.
- $X_k(T_n)$ = total number of attacks



Let T_n be a conditioned Galton–Watson tree with variance σ^2. Then Berzunza, Cai & Holmgren '20 show that

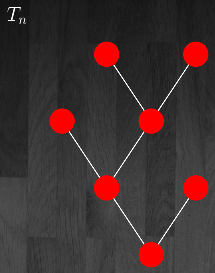
$$\frac{X_k(T_n)}{\sigma^{\frac{1}{k}} n^{1-\frac{1}{2k}}} \xrightarrow{(d)} Z_k$$

Question: Write Z_k as a functional of the CRT?

Overview

- Continuum Random Tree
- Cutting down Continuum Random Tree
- Scaling limit of $X_k(T_n)$

Continuum Random Tree



$T =$ G-W tree with offspring dist. $(p_k)_{k \geq 0}$

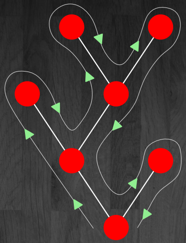
Suppose : $\sum_k k p_k = 1$

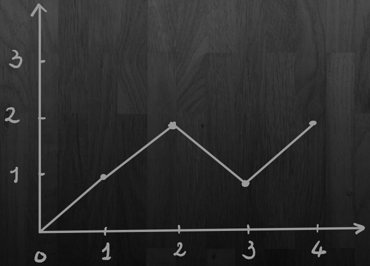
$\sigma^2 := \sum_k (k^2 - k)\, p_k < \infty$.

$T_n \overset{(d)}{=} T$ cond. on $\# T = n$.

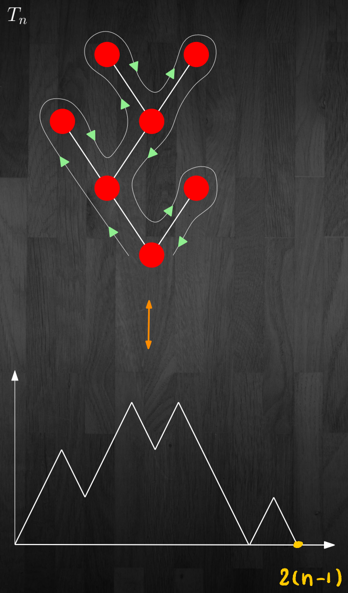
Continuum Random Tree

T_n

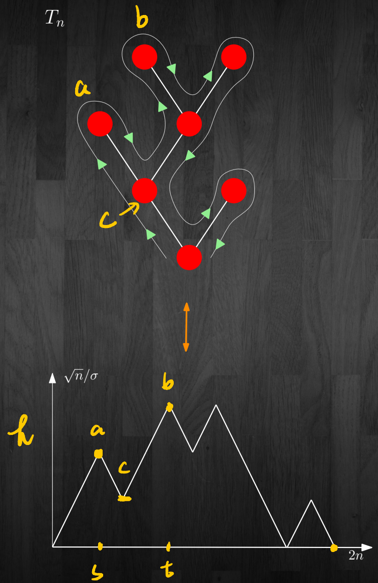




Continuum Random Tree

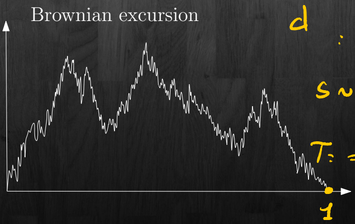


Continuum Random Tree



$n \to \infty$

$\Longrightarrow$



$$d(a,b) = d(a,c) + d(c,b)$$

$$= h(a) - h(c) + h(b) - h(c)$$

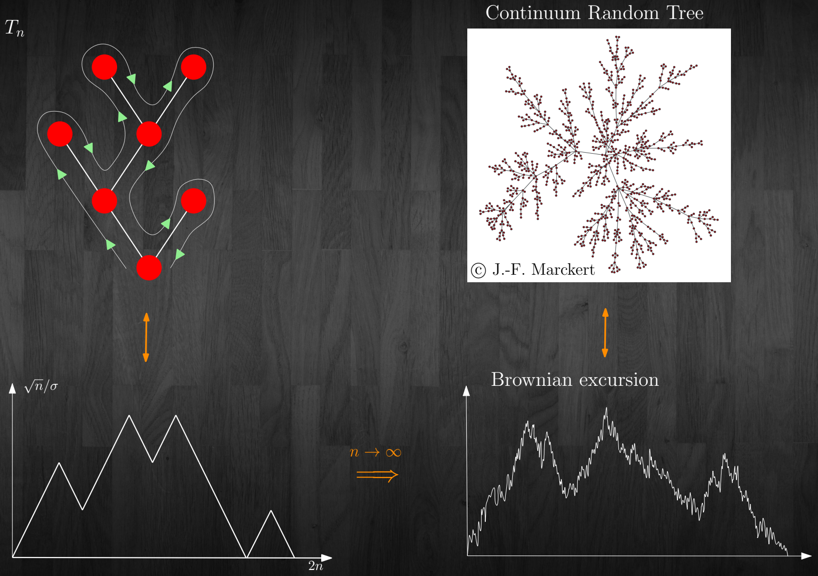
$$= h(s) + h(t) - 2\min_{u \in [s,t]} h(u)$$

d :

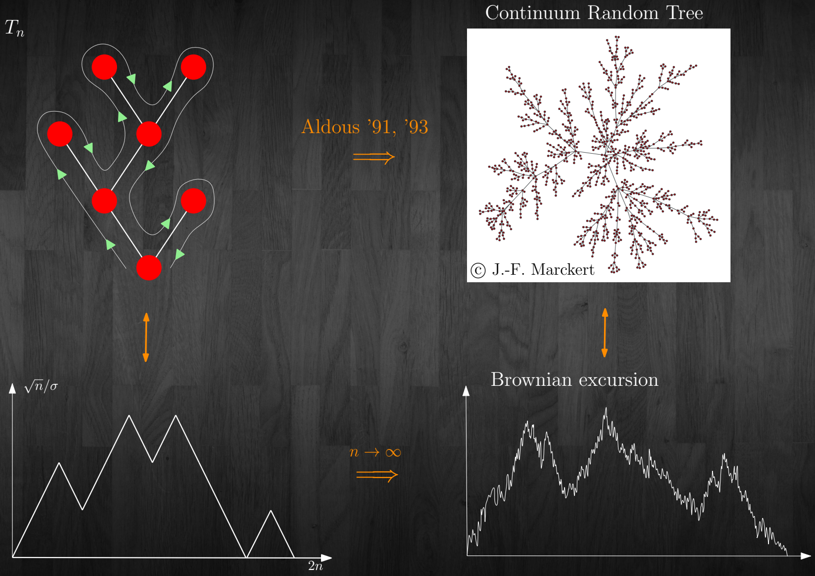
$s \sim t \Longleftrightarrow d(s,t) = 0$

$\mathcal{T} := \left([0,1]/\sim \,, d \right)$

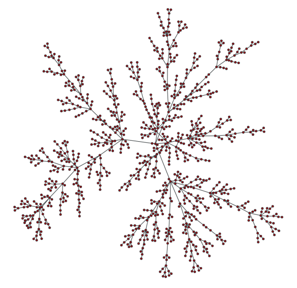
Continuum Random Tree



Continuum Random Tree



Some properties of CRT



by J-F. Marckert

- Tree-like: loop-free and unique geodesic.
- Of fractal dimension 2: inherited from BM.
- Countable number of branch points: In bijection with local minima of Br. exc.; each one of degree 3.
- Leaves are dense everywhere: Define μ as the pushforward of unif. measure on $[0, 1]$. Sample $U \sim \mu$; then U is a leaf a.s.

Cutting down CRT

Alternative formulation of cutting T_n

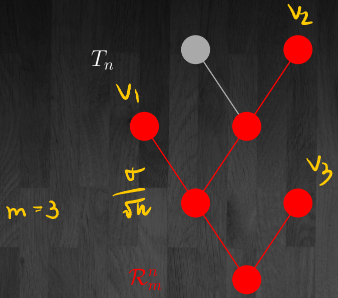
- Write μ_n = unif. measure on vertex set of T_n.
- Launch a Poisson point proc. $\{(t_i, x_i) : i \geq 1\}$ on T_n with intensity $n \cdot \mu_n$.
- At time t_i, attack x_i. This attack is counted in the tally $X_k(T_n)$ iff x_i is connected to the root at t_i.
- If a vertex has been attacked k times, remove it along with the subtree above.

Extend to the CRT? Look at spanning trees.

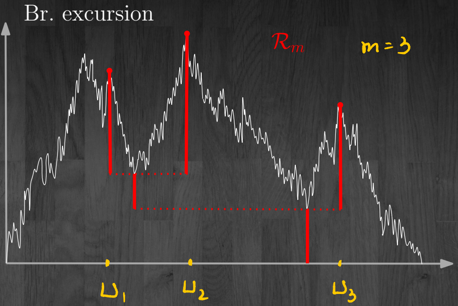
Skeleton $= T_n \setminus \{\text{leaves}\}$.

$= \bigcup \{\text{spanning trees}\}$

Scaling limit of spanning trees



$\Longrightarrow$



Let $\mathcal{R}_m^n$ = subtree of T_n spanned by m uniform vertices $V_1, \ldots, V_m$.
Rescale the edge-length of $\mathcal{R}_m^n$ by $\frac{\sigma}{\sqrt{n}}$. Then,

$$\frac{\sigma}{\sqrt{n}} \mathcal{R}_m^n \xrightarrow{(d)} \mathcal{R}_m$$

where $\mathcal{R}_m$ is the subtree of CRT spanned by m uniform points.

Cutting down CRT

- Rank the vertices of T_n in the order of their removal: $v_1, v_2, \ldots, v_n$ and let $\tau_1 < \tau_2 < \cdots$ be their corresponding removal times.
  Note that (v_i) is a uniform permutation and (τ_i) is the order statistics of n i.i.d. Gamma$(k, 1)$.

- Consider the sub-collection $\{(\tau_i, v_i) : v_i \in \mathcal{R}_m^n\}$. We have

$$\left(\sigma^{\frac{1}{k}} n^{-\frac{1}{2k}} \tau_i, v_i\right)_{i\geq 1} \xrightarrow{(d)} \left((k! t_i)^{\frac{1}{k}}, x_i\right)_{i\geq 1} \quad \text{in an appropriate sense,}$$

  where $((t_i, x_i))_{i\geq 1}$ is a Poisson point process of unit rate on $\mathcal{R}_m$.

- As m increases, $\mathcal{R}_m \nearrow$ skeleton of CRT; we can then extend the previous Poisson point proc. to the CRT and use the Poisson proc. to cut it down.

Understand the scaling...

$$\mathbb{P}(\Gamma(k,1) \le t) = \mathbb{P}(E_1 + \cdots + E_k \le t) = \mathbb{P}(\mathrm{Pois}(t) \ge k)$$

$$= \sum_{j=k}^{+\infty} e^{-t} \cdot \frac{t^j}{j!} \underset{t \to 0+}{\sim} \frac{t^k}{k!}$$

$$\mathbb{E}[\# \{ \text{removed vertices in } R_m^n \text{ in } [0,t] \}]$$

$$= \# \{ \text{vertices in } R_m^n \} \cdot \mathbb{P}(\Gamma(k,1) \le t)$$

$$\asymp m \cdot \frac{\sqrt{n}}{\sigma} \cdot \frac{t^k}{k!}$$

$$\sqrt{n} \cdot t^k = O(1) \quad \Rightarrow \quad t = O(n^{-\frac{1}{2k}})$$

Records & Number of cuts

- The r-th attack at a vertex v is called a r-record if v is still connected to the root when the attack occurs, $1 \leq r \leq k$, so that

$$X_k(T_n) = \#\{1\text{-records}\} + \cdots + \#\{k\text{-records}\}.$$

$\sim \#\{1\text{-records}\}$

- We have

$$\mathbf{E}[\#\{r\text{-records}\}] = \mathcal{O}(n^{1-\frac{r}{2k}}).$$

- So it suffices to look at the asymptotic of 1-records.

Asymptotic of 1-records

- Let $S_n(t)$ = remaining part of T_n at time t. Denote

$$a_n(t) = \#\{\text{vertices in } S_n(t) \text{ which have received no attack at time } t\}.$$

- Since 1-records arrive at $\text{Exp}(1)$, we have

$$\mathbf{E}\big[\#\,\{\text{1-records arriving in } [t, t+dt]\} \mid a_n(t)\big] = a_n(t)dt$$

  A second moment argument then implies

$$\#\,\{\text{1-records}\} \sim \int_0^\infty a_n(t)dt \quad \text{in prob.}$$

$$\sim \sigma^{\frac{1}{k}} n^{1-\frac{1}{2k}} \int_0^\infty \mu_n(S_n(n^{-\frac{1}{2k}} t))\, dt$$

- Given $\#S_n(t) = n \cdot \mu_n(S_n(t))$, we have $a_n(t) \sim \text{Binom}(\#S_n(t), e^{-t})$. Then,

$$\frac{1}{n} a_n(\sigma^{\frac{1}{k}} n^{-\frac{1}{2k}} t) \sim \mu_n(S_n(n^{-\frac{1}{2k}} t)) \quad \text{in prob.}$$

Scaling limit of $X_k(T_n)$

- Let $\mathcal{P} = \{(t_i, x_i) : i \geq 1\}$ be a Poisson point proc. of unit rate on the skeleton of CRT. Remove x_i and the subtree above at time $(k!t_i)^{1/k}$.
- Let $\mathcal{S}(t)$ be the remaining part of the CRT at time t and define

$$Z_k = \int_0^\infty \mu(\mathcal{S}(t))\, dt.$$

Theorem

As $n \to \infty$, we have

$$\left(\frac{\sigma}{\sqrt{n}} T_n,\ \frac{X_k(T_n)}{\sigma^{\frac{1}{k}} n^{1-\frac{1}{2k}}} \right) \xrightarrow{(d)} (\mathcal{T},\ Z_k),$$

Some final remarks

- For $k = 1$, we recover the construction in Addario-Berry, Broutin & Holmgren, Bertoin & Miermont, Abraham & Delmas.
- From the previous construction of Z_k, we deduce
  - comparison between $(Z_k)_{k\geq 1}$; in particular,

$$k \cdot (k!)^{-\frac{1}{k}} Z_k \leq k + Z_1.$$

  - direct computations of $\mathbf{E}[Z_k^j \mid \mathsf{CRT}]$.

THANK YOU!