Non-standard limits for a family of autoregressive stochastic sequences

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Abstract

We consider a family of multivariate autoregressive stochastic sequences that restart when hit a neighbourhood of the origin, and study their distributional limits when the autoregressive coefficient tends to one, the noise scaling parameter tends to zero, and the neighbourhood size varies. We obtain a non-standard limit theorem where the limiting distribution is a mixture of an atomic distribution and an absolutely continuous distribution whose marginals, in turn, are mixtures of distributions of signed absolute values of normal random variables. In particular, we provide conditions for the limiting distribution to be normal, like in the case where there is no the restart mechanism.

The Model

Let $a \in (0, \infty)$. Let $\{\alpha_m\}_{m \geq 1}$ be *positive* random variables with $\mathbb{E}\alpha_m^2 < \infty$ and $\{\beta_m\}_{m \geq 1}$ constants: as $m \to \infty$,

$$1 - \mathbb{E}\alpha_m \sim \frac{a}{m} \quad \text{and} \quad 1 - \mathbb{E}\alpha_m^2 \sim \frac{2a}{m} \quad \text{as well as} \quad \beta_m \sim 1/\sqrt{m}.$$

Let $\{\xi_t\}_{t \geq 0}$ be i.i.d. copies of a random vector ξ in $\mathbb{R}^d$, where $\mathbb{E}\xi = 0$ and $\mathbb{E}||\xi||^2 < \infty$.
For $m \geq 1$, let $\{\alpha_{m,t}\}_{t \geq 0}$ be i.i.d. copies of α_m that do not depend on the ξ's.
Autoregressive model in $\mathbb{R}^d$: for any $m = 1, 2, \ldots$,

$$X_{t+1}^{(m)} = \alpha_{m,t+1} X_t^{(m)} + \beta_m \xi_{t+1}, t = 0, 1, \ldots \text{ with } X_0^{(m)} = 0.$$

Basic facts:

(1) For m large: There exists a unique stationary/limiting distribution $\pi^{(m)}$ for $X_t^{(m)}$ as $t \to \infty$ and, for $X^{(m)} \sim \pi^{(m)}$,

$$X^{(m)} \stackrel{d}{=} \alpha_m X^{(m)} + \beta_m \xi.$$

(2) Further, $X^{(m)} \xrightarrow{d} \frac{1}{\sqrt{2a}} N_\Sigma$ as $m \to \infty$, where N_Σ is a centred Gaussian random vector with covariance matrix $\Sigma := \mathbf{E}\,\xi\xi^T$.

Restart mechanism:

$\{\gamma_m\}_{m\geq 1}$, a sequence of positive real-valued numbers
$A \subseteq \mathbb{R}^d$, a measurable set such that

$$B^d(\underline{r}) \subseteq A \subseteq B^d(\overline{r})$$

with some $\underline{r}, \overline{r} \in (0, \infty)$. Here $B^d(r)$ is the d-dimensional radius-r open ball centred at 0.
Let

$$Y_{t+1}^{(m)} := \alpha_{m,t+1} Y_t^{(m)} \mathbf{1}\{Y_t^{(m)} \notin \gamma_m A\} + \beta_m \xi_{t+1}, t \geq 0 \text{ with } Y_0^{(m)} := 0.$$

The elements of the sequences $\{X_t^{(m)}\}_{t\geq 0}$ and $\{Y_t^{(m)}\}_{t\geq 0}$ coincide until time

$$\tau^{(m)} = \min\{t \geq 1 : \; X_t^{(m)} \in \gamma_m A\} \leq \infty,$$

and then $Y_t^{(m)}$ restarts from the origin.

$Y^{(m)}$: a random vector having the stationary distribution of $\{Y_t^{(m)}\}_{t\geq 0}$,

$$Y^{(m)} \stackrel{d}{=} \alpha_m Y^{(m)} \mathbf{1}\{Y^{(m)} \notin \gamma_m A\} + \beta_m \xi.$$

What is the limiting distribution for $Y^{(m)}$, as $m \to \infty$, and how does it differ from the normal distribution?

Main Result

THEOREM. Assume that the matrix $\boldsymbol{\Sigma} := \mathbf{E}\,\xi\xi^T$ has full rank. Assume further that

$$\mathbf{E}\,Y^{(m)} \to \mu \in \mathbb{R}^d \quad \text{as} \quad m \to \infty$$

and

$$\mathbf{E}\,\tau^{(m)} \to \widehat{\tau} \in [1, \infty] \quad \text{as} \quad m \to \infty.$$

If $\widehat{\tau} \in [1, \infty)$, assume additionally that there exists a random variable τ such that $\tau^{(m)} \xrightarrow{\mathbb{P}} \tau$ as $m \to \infty$.

Then

$$Y^{(m)} \xrightarrow{d} Y := B_1 \cdot Z \quad \text{as} \quad m \to \infty,$$

where B_1 and Z are mutually independent. The random variable B_1 takes values 0 and 1 with probabilities

$$p := \mathbb{P}(B_1 = 1) = 1 - \mathbb{P}(B_1 = 0) = \begin{cases} 1 - \mathbf{E}\,\tau/\widehat{\tau}, & \widehat{\tau} \in [1, \infty), \\ 1, & \widehat{\tau} = \infty, \end{cases}$$

and the d-dimensional random vector Z has an absolutely continuous distribution that is characterised by the following properties:

(i) The characteristic function of Z is

$$\varphi_Z(u) = \left(1 + \mathbf{i}\frac{\langle u, \mu\rangle}{p\sqrt{u^T\Sigma u}}\int_0^{\frac{\sqrt{u^T\Sigma u}}{\sqrt{2a}}} \exp\left(\frac{t^2}{2}\right) dt\right) \exp\left(-\frac{u^T\Sigma u}{4a}\right).$$

(ii) The density of Z is given by

$$f_Z(x) = \frac{\sqrt{a}^d}{\sqrt{\det(\Sigma)}\sqrt{\pi}^d} \exp(-ax^T\Sigma^{-1}x) + \widetilde{f}_Z(x), \quad x \in \mathbb{R}^d,$$

where, for odd dimensions $d = 1, 3, 5, \ldots$,

$$\widetilde{f}_Z(x) = \frac{(-2)^{\frac{d-1}{2}} a^{\frac{d+2}{2}}}{\sqrt{\det(\Sigma)}\kappa_{d-1}(d-1)!!p} \langle \Sigma^{-\frac{1}{2}}\mu, x\rangle h^{((d-1)/2)}(ax^T\Sigma^{-1}x),$$

and for even $d = 2, 4, \ldots$

$$\widetilde{f}_Z(x) = \frac{(-2)^{\frac{d}{2}} a^{\frac{d+3}{2}}}{\kappa_d d!!p} \langle \Sigma^{-\frac{1}{2}}\mu, x\rangle \int_{-\infty}^{\infty} h^{(d/2)}(a(x^T\Sigma^{-1}x + z^2))\, dz.$$

Here $h(s) = e^{-s}/\sqrt{s}$ for $s > 0$, $h^{(k)}$ is its kth derivative, and κ_d the volume of the d-dimensional unit ball.

(iii) For any $v \in \mathbb{R}^d$,

$$\langle v, Z\rangle \overset{d}{=} \frac{\sqrt{v^T \Sigma v}}{\sqrt{2a}} B_{2,v} |N|,$$

where N and $B_{2,v}$ are two independent random variables, with N having the standard normal distribution and $B_{2,v}$ having a two-point distribution,

$$\mathbb{P}(B_{2,v} = 1) = \frac{1}{2} + \frac{\sqrt{\pi a}\langle v, \mu\rangle}{2p\sqrt{v^T \Sigma v}} \quad \text{and} \quad \mathbb{P}(B_{2,v} = -1) = \frac{1}{2} - \frac{\sqrt{\pi a}\langle v, \mu\rangle}{2p\sqrt{v^T \Sigma v}}.$$

Remarks

Remark 1. Note that Z has a multivariate normal distribution if and only if $\mu = 0$. The latter condition holds if, say, the distribution of the ξ's is symmetric and the set A is symmetric too (e.g. a ball).

Remark 2. We get $\mu = 0$ again if assume that $\gamma_m \to \infty$ and the conditions of the theorem hold. If we assume in addition that ξ has more moments, then we get $\mu = 0$ underweaker assumptions. For example, if $||\xi||$ has a finite exponential moment, then $\mu = 0$ if $\gamma_m \cdot \frac{\sqrt{m}}{\log m} \to \infty$.

Remark 3. It follows that, for the limiting vector $Z = (Z_1, \ldots, Z_d)$ in the Theorem, the absolute value $|Z_i|$ of any of its coordinates has the same distribution as the absolute value of a center normal random variable.In addition, the marginal distribution of any projection of Z on an orthogonal to μ projection in normal, and the distribution of a non-orthogonal projection is a mixture of distributions of signed absolute values of normal random variables. Therefore, it would look plausible for the limiting vector Z to coincide in distribution with a random vector $(\psi_1 N_1, \ldots, \psi_d N_d)$ where $(N_1, \ldots, N_d)$ is a multivariate normal vector and $(\psi_1, \ldots, \psi_d)$ an independent random vector whose coordinates take values ± 1 only. However, as it follows from the formulae for the densities, this appears to be not the case if $d > 1$.

Remark 4. The assumption that the matrix $\Sigma = \mathbf{E}\,\xi\xi^T$ has full rank is not restrictive. If Σ is not regular, the components of ξ and, thus, the components of $\{X_t^{(m)}\}_{t\geq 0}$ are linearly dependent. In this case, it is sufficient to study a maximal subset of linearly independent components, for which the assumption on the covariance matrix takes place.

Remark 5. In the paper, we provide sufficient conditions for the main assumptions of the Theorem to hold and for the limiting distribution to be continuous.