INVARIANTS OF LINKS AND 3-MANIFOLDS FROM SKEIN THEORY AND FROM QUANTUM GROUPS.

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ABSTRACT. Starting with Kauffman's bracket polynomial the techniques of linear skein theory are used to present and package a family of polynomial invariants for a framed link. An equivalent family of invariants is derived from representations of the quantum group $SU(2)_q$. Specialisation of the variable q leads to invariants of a 3-manifold defined by surgery on a framed link, in terms of the invariants of the link. A similar programme is outlined relating the invariants constructed from the Homfly polynomial to those derived from the quantum groups $SU(k)_q$.

Introduction.

In this series of talks I shall start by discussing the knot invariants and algebra related to Kauffman's bracket polynomial, and the construction of 3-manifold invariants from them. The whole area can alternatively be viewed in terms of representations of the quantum group $SU(2)_q$; I shall exhibit descriptions which have a convenient interpretation in either light, and also give the means for translating between them. My presentation here is based on the bracket polynomial, and has much in common with the work of Lickorish, [16], and Blanchet, Habegger, Masbaum and Vogel, [2].

A direct approach on the quantum group route is given in my paper with Strickland, [23], which draws directly on the early work of Kirillov and Reshetikhin, [13]. A more general basis for the use of quantum group representations in constructing knot invariants is given in the work of Reshetikhin and Turaev, [31]. Detailed descriptions of representations for $SU(2)_q$ can be found in Kirby and Melvin, [12]; while these are based on specialisations of $SU(2)_q$ in which the parameter q is a root of unity they do present careful and explicit details which enable the less complicated case of generic q to be handled as well.

The reason for their treatment is to give an account of the invariants of a 3-manifold M which depend on the choice of a root of unity, and a quantum group (in this case $SU(2)_q$), in terms of the invariants of any framed link which determines the 3-manifold M by the process of surgery on the link. These 3-manifold invariants were first constructed in this way by Reshetikhin and Turaev, [32]; their existence and general properties were proposed originally by Witten, [43], based on interpretations of constructions from theoretical physics. Other accounts are given in [24], [16] and [2]. Those in [16] and [2] are based entirely on the bracket polynomial, while the account in [24] uses the quantum group representations at generic q as a means of establishing properties of the knot invariants, and then makes constructions based on

Appears in 'Topics in knot theory', the Proceedings of the NATO Summer Institute in Erzurum 1992, NATO ASI Series C 399, ed. M.Bozhüyük. Kluwer 1993, 107-156.

the evaluations of these at a given root of unity, without having to consider the more complicated representation theory which arises at the root of unity.

My presentation of the 3-manifold invariants uses the techniques appropriate to the bracket polynomial. I shall restate the point that, however the knot invariants are constructed, whether by quantum group representations or by bracket polynomials, there is a common halfway stage reached to which each of the constructions brings its own insights. The final attack on the question of manifold invariants can then be made from this point, no matter how it has been reached, although the representation theory provides invaluable guidance at this stage in, for example, setting up and choosing a suitable basis for a naturally occurring finite dimensional vector space.

I believe that a similar two-stage process is appropriate in constructing 3-manifold invariants from other quantum groups. Such a construction is done by slightly different means, for example, by Turaev and Wenzl, [37], and a general framework is given by Walker, [38], in the spirit of Segal's modular functors. It is possible to make a nice comparison of the knot invariants defined from the quantum groups $SU(k)_q$, for different k, with knot invariants based on the Homfly polynomial, [29], [41], [19]. This permits an analogous two-stage process, allowing the definition of 3-manifold invariants in terms of the knot invariants for generic q, with a root of unity substituted for q; the representation theory to be used in the first stage only requires the study of generic q, when the representations mirror directly those of the corresponding classical group. In the final section I shall give a description of the $SU(k)_q$ knot invariants from the point of view of Homfly polynomials, in a similar framework to the earlier talks, which can be thought of as dealing with the case $k=2$. More details will be found in [19]; this gives a preparation of the common ground which could be used for the production of manifold invariants by specialising q to be a root of unity.

Readers of earlier versions of this paper should note some minor amendments in section 6, where the substitutions $v=s^{-k}, x=s^{-1/k}$ replace those used previously.

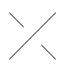
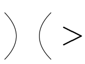
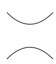
1. Knot invariants derived from Kauffman's bracket.

1.1 The bracket invariant.

In 1986, Kauffman showed how to construct an element $<D> \in \mathbf{Z}[A^{\pm 1}]$ for every plane diagram D of a knot or link in $\mathbf{R}^3$, which is determined (up to a constant) by two properties. These are

(1) $$<D_+> = A<D_0> + A^{-1}<D_\infty>,$$

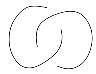
or more pictorially

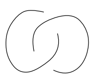
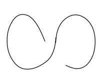
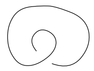
$<$  $> = A<$  $> + A^{-1}<$  $>,$

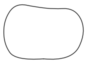
where D_+, D_0 and D_∞ are three link diagrams which only differ as shown.

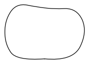
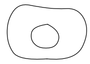
(2) $$<D \sqcup \mathrm{O}> = \delta<D>,$$

where $\delta = -A^2 - A^{-2}$, and $D \sqcup \mathrm{O}$ is a diagram containing one component O which has no self-crossings, or crossings with the rest of the diagram D.

Example. Properties (1) and (2) allow the simplification of $<$  $>$ as

$<$  $> = A <$  $> + A^{-1} <$  $>$

$= A^2 < \mathrm{O}\ \mathrm{O} > + <$  $>$

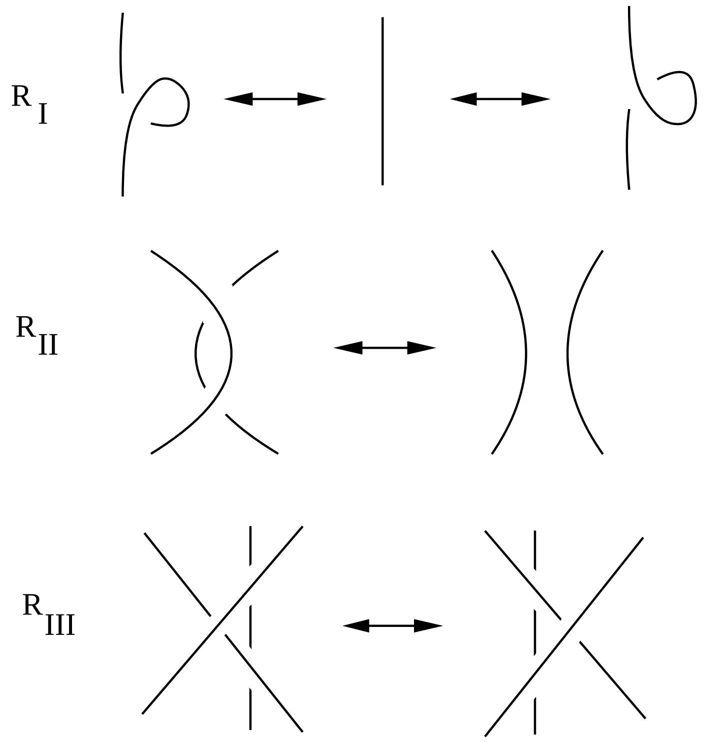
$+ <$  $> + A^{-2} <$  $>$

$$= (A^2 + A^{-2})\ \delta < \mathrm{O} > + 2 < \mathrm{O} >$$
$$= -(A^4 + A^{-4}) < \mathrm{O} > .$$

In a similar way, $< D >$ can be written in terms of $< \mathrm{O} >$ for any D; in Kauffman's original work the Laurent polynomial $< D >$ was normalised by taking $< \mathrm{O} > = 1$, but now it is more often chosen to include use of the empty diagram ϕ, with the condition that $< \phi > = 1$, and consequently $< \mathrm{O} > = \delta < \phi > = \delta$.

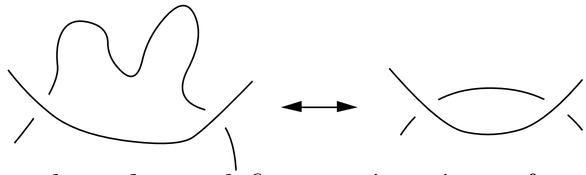
The reason for using properties (1) and (2) is given by Kauffman's theorem, which can be readily established.

Theorem 1.1 (Kauffman) . *When a diagram D is altered by one of the Reidemeister moves R_{II} or R_{III} the value of $< D >$ is unchanged.* □



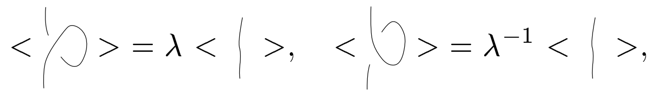
Reidemeister's moves

Reidemeister's moves R_I, R_{II} and R_{III} alter one diagram to another which represents a different view of the same knotted curve in space, up to a natural equivalence of closed curves in space corresponding to physical manipulations of pieces of rope. The classical theorem of Reidemeister states that any two diagrams D_1 and D_2 of two curves which are equivalent in space can be transformed from one to the other by a sequence of Reidemeister's moves, (allowing diagrams to be distorted between moves as shown).



Thus Kauffman's bracket *almost* defines an invariant of a curve C in space, by calculating $< D >$ for a diagram D of the curve. The element $< D > \in \Lambda$ would indeed depend only on the curve C if it were to be unaltered by all of the three Reidemeister moves. Now R_{II} and R_{III} have no effect on $< D >$. However the bracket, $< D >$, *is* altered when D is changed by a move of type R_I. All the same, this change is quite limited, and consists of multiplication by a fixed scalar $\lambda^{\pm 1}$, depending on whether a left-handed or right-handed curl is removed.

This can be summarised as

(3) 

where properties (1) and (2) show readily that $\lambda = -A^3$.

Kauffman's theorem is proved in [8]. It leads immediately, using property (3), to an invariant of oriented curves in space, which can be seen as follows.

In an oriented diagram each crossing c can be given a sign $\varepsilon(c) = \pm 1$, defined as shown,

 $\varepsilon = +1$,  $\varepsilon = -1$.

Now define the *writhe* $w(D)$ of the oriented diagram D to be $w(D) = \sum \varepsilon(c)$, the sum of the signs of the crossings in D. Since $w(D)$ is unaltered by Reidemeister moves R_{II} and R_{III}, and changes by ± 1 under move R_I, the function

$$\lambda^{-w(D)} < D >$$

is unaltered by all Reidemeister moves, and hence gives an invariant of an oriented curve C in space in terms of any choice of diagram D representing C. Kauffman showed that this invariant could be identified with Jones' invariant, introduced in 1984, which has been the foundation for much recent work in relating knot theory with other topics.

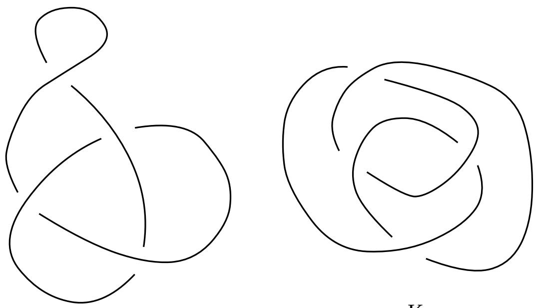
1.2 Linear skein theory for the Kauffman bracket.

In this section I shall develop the notation and ideas of linear skein theory in using diagrams of various sorts to define certain linear spaces, or more accurately Λ-modules, with the properties (1) and (2) of the bracket polynomial closely in mind. The general methods were first used by Conway in dealing with versions of the Alexander polynomial.

Notation. Let F be a planar surface, for example $\mathbf{R}^2$ itself, or an annulus $S^1 \times I \subset \mathbf{R}^2$, or a rectangular disc. When F has a boundary we also specify a finite, possibly empty, set of points on its boundary. A *diagram* in F consists of any number of closed curves, together with arcs joining the specified boundary points of F. As in the standard case of knot diagrams, the curves and arcs have a finite number of crossing points where two strands cross. At a crossing the strands are distinguished in the conventional way as an over-crossing and an under-crossing, so that the diagram can be interpreted as a view of some curves lying within $F \times I$.

Write Λ for the ring $\mathbf{Z}[A^{\pm 1}]$, and $\mathcal{D}(F)$ for the set of Λ-linear combinations of diagrams in F.

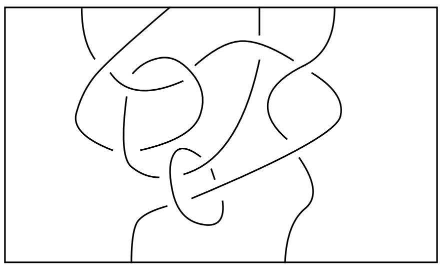
For example, when $F = \mathbf{R}^2$, $\mathcal{D}(F)$ consists of linear combinations of knot (and link) diagrams, such as $AK_1 - (A + 2A^{-1})K_2$ for the diagrams K_1 and K_2 shown.

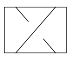
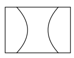
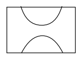


K_1 K_2

Notation. When F is a rectangular disc with m points specifed on the top edge, and n points on the bottom edge, denote F by R^m_n, and call a diagram in F an (m, n)-*tangle.*

An example of a $(4, 2)$-tangle is shown below.



The linear combination of $(2, 2)$-tangles $\sigma - AI - A^{-1}H$ is an element of $\mathcal{D}(R^2_2)$ for the tangles $\sigma =$  , $I =$  and $H =$  .

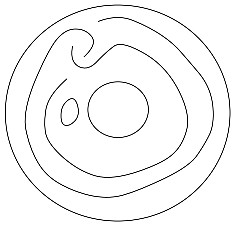
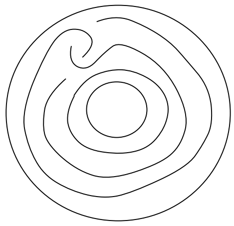
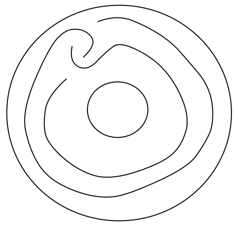
Definition. The *linear skein* $\mathcal{S}(F)$ of a planar surface F, with a distinguished finite, (possibly empty), subset of boundary points, is the quotient of $\mathcal{D}(F)$ by the linear relations

(1) $$D_+ = AD_0 + A^{-1}D_\infty$$

(2) $$D \amalg \mathrm{O} = \delta D \quad (= -(A^2 + A^{-2})D)$$

where D_+, D_0 and D_∞ are any three diagrams in F which differ only as in the bracket relation (1), and $D \amalg \mathrm{O}$ consists of a diagram D together with a disjoint simple closed curve O which is *null-homotopic in* F.

Thus condition (2) allows us to replace

 but not  by $\delta \times$ 

in the linear skein of the annulus, $\mathcal{S}(S^1 \times I)$.

THEOREM 1.2. *As a Λ-module, $\mathcal{S}(F)$ is spanned by diagrams with no crossings and no null-homotopic closed curves.*

Proof: By induction on the number of crossings and null-homotopic curves. Relation (1) in the definition of $\mathcal{S}(F)$ allows us to replace a diagram by a linear combination of two diagrams with fewer crossings, while relation (2) allows the removal of null-homotopic closed curves. $\square$

COROLLARY 1.3. *The linear skein $\mathcal{S}(\mathbf{R}^2)$ is spanned as a Λ-module by the empty diagram ϕ, (or, if the empty diagram is excluded, by the simple unknot diagram* O*).* $\square$

Remark. For any diagram D in $\mathbf{R}^2$ we can write $D = < D > \phi$ in $\mathcal{S}(\mathbf{R}^2)$; this provides an isomorphism $\mathcal{S}(\mathbf{R}^2) \cong \Lambda$, induced by mapping D to $< D >$.

THEOREM 1.4. *Two diagrams in F which differ by a Reidemeister move within F of type R_{II} or R_{III} represent the same element of $\mathcal{S}(F)$.*

Proof: Relations (1) and (2) in $\mathcal{S}(F)$ are exactly what is used in the proof of Kauffman's theorem. $\square$

1.3 SKEIN MAPS.

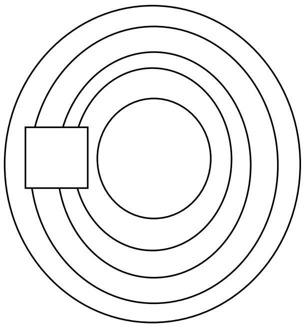
Conway's framework, as described by Lickorish [15], for relating skeins of different surfaces can be helpfully used here to provide a range of linear and multilinear maps between skeins.

The central idea is to place one planar surface F inside another F', and include some fixed 'wiring', W, in the region between F and F', consisting of one or more closed curves and arcs, arranged so that the boundary points of the arcs consist exactly of the distinguished boundary points of F and F'.

Definition. A *wiring* W of F into F' means a choice of inclusion of F in F', and a fixed diagram of curves and arcs in $F' - F$ whose endpoints consist of all the distinguished points on the boundaries of F and F'.

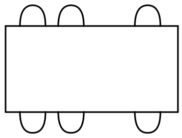
Any diagram D inserted in the surface F is then extended by W to give a diagram $W(D)$ in F'.

Examples. (1) The rectangle R_n^n can be wired into the annulus $S^1 \times I$ as shown. For a tangle T the extended diagram $W(T)$ in the annulus, or more usually in $\mathbf{R}^2$, is called the *closure* of T, and will be denoted by $\widehat{T}$.

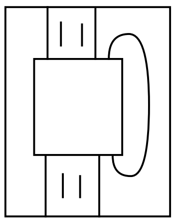


(2) The annulus itself can be wired into $\mathbf{R}^2$ by simple inclusion, without any extra curves.

(3) The *plat closure* of a $(2m, 2m)$-tangle is the diagram in $\mathbf{R}^2$ induced by the wiring shown.



(4) A partial closure of an (n, n)-tangle T is the $(n-1, n-1)$-tangle $W(T)$ induced by the wiring of R_n^n into R_{n-1}^{n-1} shown below.



Any wiring W of F into F' determines a linear map

$$\mathcal{D}(W) : \mathcal{D}(F) \to \mathcal{D}(F')$$

by $D \mapsto W(D)$. It is clear that this induces a linear map between the skeins $\mathcal{S}(F)$ and $\mathcal{S}(F')$.

THEOREM 1.5. *A wiring W of F into F' induces a linear map*

$$\mathcal{S}(W) : \mathcal{S}(F) \to \mathcal{S}(F'),$$

defined on a diagram D in F by $D \mapsto W(D)$.

Proof: It is enough to observe that if diagrams in F satisfy skein relations (1) or (2) then they continue to do so when extended by W to diagrams in F', so the relations in $\mathcal{S}(F)$ are respected by the map. □

It is clear from theorem 1.4 that the wiring W can be altered by Reidemeister moves R_{II} or R_{III} in $F' - F$ without changing the map $\mathcal{S}(W)$.

1.4 Multilinear extensions.

The wiring construction can be used to wire several surfaces at once, $F_1, \ldots, F_k$ say, into F'. Any such wiring will induce a map

$$\mathcal{S}(W) : \mathcal{S}(F_1) \times \ldots \times \mathcal{S}(F_k) \to \mathcal{S}(F')$$

which is multilinear.

For example, we can very simply wire the rectangles R_n^m and R_p^n into R_p^m, one above the other, inducing a bilinear product

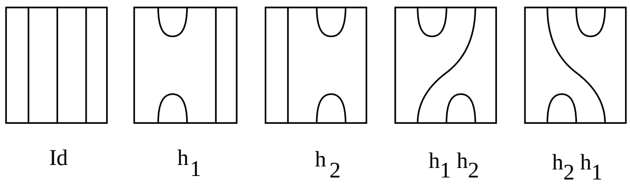
$$\mathcal{S}(R_n^m) \times \mathcal{S}(R_p^n) \to \mathcal{S}(R_p^m).$$

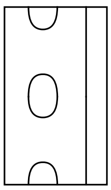
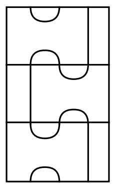
In the case $m = n = p$ this diagram-based product determines a multiplication which turns $\mathcal{S}(R_n^n)$ into an *algebra* over Λ.

Notation. Write $TL_n = \mathcal{S}(R_n^n)$ for this algebra, which is isomorphic to the n-th Temperley-Lieb algebra.

Theorem 1.2 shows that TL_n is spanned by diagrams in R_n^n with no closed curves, and no crossings.

When $n = 3$ there are just five such diagrams,



Note that $h_1^2 =$  $= \delta h_1$ and $h_1 h_2 h_1 =$  $= h_1$.

For general n, TL_n is spanned by $\binom{2n}{n}/(n+1)$ such diagrams; the number of diagrams is the n-th Catalan number.

Kauffman proved in [8] that TL_n can be presented as an algebra with generators $h_1, \ldots, h_{n-1}$, similar to h_1 and h_2 above, and only the obvious relations, namely

$$\begin{aligned} h_i h_j &= h_j h_i, \quad \mid i - j \mid \; > 1, \\ h_i^2 &= \delta h_i, \\ h_i h_{i\pm 1} h_i &= h_i. \end{aligned}$$

He was thus able to identify this algebra with the Temperley-Lieb algebra, which appears from a totally different viewpoint in Jones' original work.

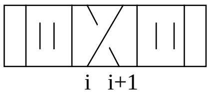
1.5 The braid groups.

An n-string braid is a diagram in R_n^n consisting only of n arcs, which all run monotonically from bottom to top. Two n-braids are composed by placing one below the other. Braids, up to Reidemeister moves R_{II} and R_{III}, form Artin's n-string braid group, B_n, described by him in [1].

Proposition 1.6. *There is a multiplicative homomorphism $B_n \to TL_n$ determined by representing $\beta \in B_n$ by a diagram in R_n^n and reading the diagram as an element of the skein TL_n.*

Proof: Diagrams which differ only by moves R_{II} and R_{III} represent the same element in the skein, so the map is well-defined. It is clearly a homomorphism, since composition is defined in the same way in each case. □

The image of B_n under this homomorphism spans TL_n, since each of the generators h_i of TL_n satisfies the relation $\sigma_i = A\,\mathrm{Id} + A^{-1}h_i$, where σ_i is the elementary braid

$\sigma_i \;=$ 

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and thus $h_i = A\,\sigma_i - A^2\,\mathrm{Id}$. The presentation of TL_n can be rewritten in terms of the generators σ_i. The relations then include the relations in B_n together with the additional relations

$$(\sigma_i - A)(\sigma_i + A^{-3}) = 0,$$

or in other words $(\sigma_i + A^{-3})h_i = 0$.

1.6 Calculational methods.

It is possible to make use of the algebra TL_n in calculating the bracket invariant of a link L which has been presented as a closed braid $\widehat{\beta}$ on n strings, simply by combining the map $B_n \to TL_n$ with the linear map $TL_n \to \Lambda = \mathcal{S}(\mathbf{R}^2)$ induced by the closure wiring on R^n_n. We must thus write the braid β as a linear combination $\beta = \sum \lambda_g T_g$ of the $\binom{2n}{n}/(n+1)$ spanning elements $\{T_g\}$ of TL_n, with $\lambda_g \in \Lambda$. It is then enough to know the bracket invariant $<\widehat{T}_g>$ of the closure of each T_g, to get

$$<L> = <\widehat{\beta}> = \sum \lambda_g <\widehat{T}_g>.$$

The expression of β in terms of $\{T_g\}$ can be built up from knowledge of β as a word in the elementary braids σ_i, by knowing simply how to write each product $T_g\sigma_i$, as defined in section 1.4, in terms of the basis $\{T_g\}$ of TL_n.

The amount of calculation required does not grow rapidly with the number of crossings, for braids on a fixed number of strings. Such calculations still give one of the quickest ways of handling invariants of quite complicated links; see Morton and Short, [21, 22], for further analysis and comments. In principle the bracket invariant of any knot can be found in this way, as every knot can be presented as a closed n-braid for some n, although calculations become rapidly more impracticable with increasing n.

For a simple related illustration, note first that TL_2 is spanned by just two elements, 1

The induced bilinear map

$$\mathcal{S}(W) : TL_2 \times TL_2 \to \Lambda$$

then evaluates the bracket invariant of the complete diagram, when applied to (σ^r, σ^k). We can write $\sigma^r = (A + A^{-1}h)^r = P_r + Q_r h \in TL_2$ in terms of the basis elements 1 and h, and similarly σ^k. Combine this information with the calculation of $\mathcal{S}(W)$ on pairs of basis elements, to complete the calculation. It is easy to see that $\mathcal{S}(W)(1,1) = \mathcal{S}(W)(h,h) = \delta$ while $\mathcal{S}(W)(1,h) = \mathcal{S}(W)(h,1) = \delta^2$, so that the required invariant can be written

$$\begin{pmatrix} P_r & Q_r \end{pmatrix} \begin{pmatrix} \delta & \delta^2 \\ \delta^2 & \delta \end{pmatrix} \begin{pmatrix} P_k \\ Q_k \end{pmatrix} .$$

In calculating $\sigma^r \in TL_2$ it can be more efficient to use a different basis of TL_2 which reflects better its algebraic structure. In each TL_n there is one element which will be of further algebraic use. This is related to one of the two non-zero homomorphisms from TL_n to Λ. It is clear from the presentation of TL_n that there is a Λ-linear homomorphism $\varphi : TL_n \to \Lambda$, defined by $\varphi(1) = 1$, $\varphi(h_i) = 0$. In terms of braids this corresponds to $\varphi(\sigma_i) = A$, $\varphi(1) = 1$. (The other homomorphism, ψ, is defined by $\psi(\sigma_i) = -A^{-3}$.)

In the next section I shall exhibit an element $f_n \in TL_n$ with the property that $Tf_n = f_n T = \varphi(T) f_n$ for every $T \in TL_n$. Before doing this, I shall look in further detail at the skein of the annulus.

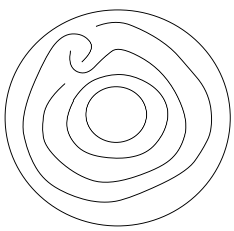
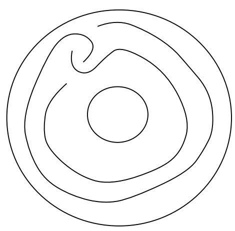
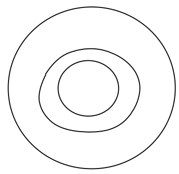
1.7 The skein of the annulus.

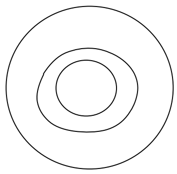
Notation. Write $\mathcal{B} = \mathcal{S}(S^1 \times I)$ for the skein of the annulus.

The linear map $\mathcal{B} \to \mathcal{S}(\mathbf{R}^2) \cong \Lambda$ induced by the inclusion as in example (2) above will sometimes be denoted simply by $v \mapsto \, < v >$ as it is induced on a diagram in the annulus by taking its bracket invariant when regarded as a diagram in the plane.

We can wire two copies of the annulus into the annulus itself by running one copy parallel to the other without adding extra wiring. This defines a bilinear product $\mathcal{B} \times \mathcal{B} \to \mathcal{B}$, under which $\mathcal{B}$ becomes an algebra over Λ.

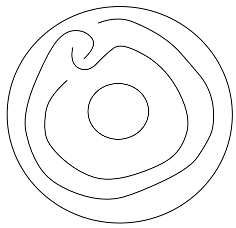
For example, the element of $\mathcal{B}$ represented by

 is the product of  and  .

Write α =  as an element of $\mathcal{B}$. Then k parallel curves represent α^k, while the empty diagram is the unit element, 1, of the algebra $\mathcal{B}$.

By theorem 1.2, $\mathcal{B}$ is spanned by diagrams with no crossings and no null-homotopic curves. Any such diagram is either empty, or consists of k parallel curves around the annulus, for some k, so that $\mathcal{B} \cong \Lambda[\alpha]$, the ring of polynomials in α.

For example, use of the skein relations shows that the diagram



is equal to $(-1 - A^{-4}) + (1 - A^4)\alpha^2$ in $\mathcal{B}$.

PROPOSITION 1.7. *The evaluation map* $< \cdot >: \mathcal{B} \to \Lambda \cong \mathcal{S}(\mathbf{R}^2)$ *is a ring homomorphism.*

Proof: This follows at once from the structure of $\mathcal{B}$ since $< \alpha^k > = \delta^k$ by skein relation (2). Even without this knowledge it is enough to observe that the two parallel copies of the annulus containing diagrams to be multiplied in $\mathcal{B}$ can be moved apart without change, using R_{II} and R_{III}, before evaluating each separately. □

2. Satellite knots.

Suppose that we want to use the bracket invariant to compare two knots C_1 and C_2. Let us draw diagrams of each knot and calculate its bracket invariant. If the knots are equivalent, and the diagrams used have the same writhe, then we will get the same answer in each case. Hence different answers, from diagrams with the same writhe, guarantee that the given knots are different.

We might, however, get the same answer from two knots which we suspect to be different. It is still possible that we may be able to show that the knots are different by a less direct use of the bracket invariant. First, 'decorate' the two knots in the same way, to give two more complicated knots K_1 and K_2. Make sure that if the decoration is done in the same way, and the two knots C_1 and C_2 are equivalent, then the decorated knots are equivalent. Then use the bracket invariant again to compare K_1 and K_2; if these give different answers then C_1 and C_2 must be different.

Such a project might be doomed to failure. If, for example, the bracket invariant of K_i could be calculated in terms of the bracket invariant of C_i and the decoration, as is the case for the classical Alexander polynomial, then two knots with the same bracket invariant would, after being decorated in the same way, still give two knots with the same invariant.

Happily, there is a chance of using the bracket invariant in this way. One of the early discoveries [21] about the recent knot invariants was the existence of pairs of knots with the same invariant which can be distinguished by calculating the invariant of the knots resulting from suitable decoration.

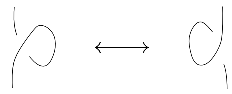
2.1 CONSTRUCTION OF SATELLITES.

I shall now describe how to decorate a knot. Starting with a given knot C we draw a diagram of it. This selects a 'parallel' curve to C, determined by keeping just to one side of C in the diagram. Altering the diagram by R_{II} or R_{III} does not change this 'diagrammatic' parallel curve when thought of as a curve in space relative to C, while R_I introduces a full twist of the parallel around C.

Definition. A *framed knot* is a curve C in $\mathbf{R}^3$, with a choice of a neighbouring parallel curve; a *framed link* has a choice of parallel for each component of the link.

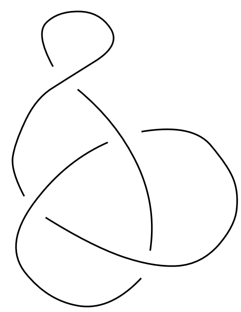
In much of what follows we shall be dealing with framed knots and links. I shall normally assume that any diagram of a framed link is drawn so that the chosen parallels agree with the diagrammatic parallels. Suitable insertion of curls in the diagram allows the diagrammatic parallels to be adjusted so that this is the case.

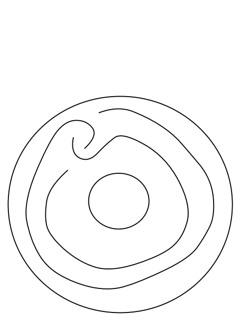
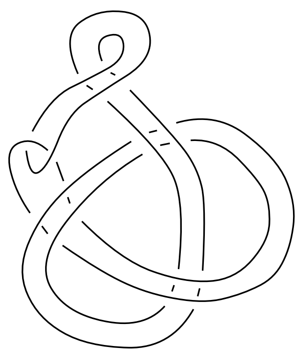
The study of framed knots and links is almost equivalent to the study of diagrams of the knots and links up to the moves R_{II} and R_{III}. As noted, the diagrammatic parallels are unaltered by R_{II} and R_{III} ; conversely we can pass between diagrams with the same parallel curves by using R_{II} and R_{III} if we are also allowed to move curls from one side of the string to the other, as shown.



See Kauffman [10] for further comments. In the applications given here this last move will be permissible, so I shall assume that any statements about framed links can be interpreted in terms of diagrams up to moves R_{II} and R_{III}, and vice versa.

To continue then with the construction, we shall assume that we have a diagram of C, or equivalently a framing of C. Now select a diagram P in the annulus. We decorate C with P as follows. Place the annulus with one edge following C and one following its parallel, and copy P into this annulus. The image of P forms a new diagram, which is the knot C decorated by P. Changing the exact positioning of the copy of P as it is placed to lie around C will alter this new diagram, but only by moves R_{II} and R_{III}. Write $C*P$ for this new diagram, defined up to R_{II} and R_{III}. For example, when C is the trefoil with framing as shown,

$C =$ 

and $P =$  then $C*P =$  .

Alteration of the diagram of C itself by R_{II} or R_{III} will alter $C*P$ only by a sequence of moves R_{II} or R_{III} respectively, and so $C*P$, as a framed knot, depends only on C as a framed knot and on P. Altering the framing of C, i.e. altering its diagram by a move R_I, will in general alter $C*P$ substantially; for this reason a framing of C must be specified in some way.

From a more 3-dimensional viewpoint, the decoration P can be viewed as lying in a solid torus, which is then embedded in $\mathbf{R}^3$ as a neighbourhood of the curve C. The resulting image of P is called a *satellite* of C, while C is known as its *companion*. Again, some specification, amounting to a decision on framing, is needed to describe exactly how the solid torus is to be embedded.

2.2 The total bracket invariant.

Our immediate study can be seen as the study of C by means of the bracket invariant of its various satellites, as we change the decoration pattern P. As with the wiring construction, we can show that the process of decorating a fixed diagram C by a pattern P in the annulus induces a linear map $\mathcal{B} \to \Lambda \cong \mathcal{S}(\mathbf{R}^2)$.

Theorem 2.1. *Let C be a knot diagram. Then there is a linear map $J_C : \mathcal{B} \to \Lambda \cong \mathcal{S}(\mathbf{R}^2)$ induced by mapping a diagram P in the annulus to the diagram $C * P$.*

Proof: When diagrams in the annulus satisfy skein relations (1) or (2) then the diagrams which result from decorating C will also satisfy the same skein relation. The map J_C is thus well-defined on the skein $\mathcal{B}$. □

As in the case of wiring diagrams, there is an extension of this result where C is replaced by a link diagram L with k components. Each component can be decorated independently, giving a multilinear map

$$J_L : \mathcal{B} \times \ldots \times \mathcal{B} \to \Lambda,$$

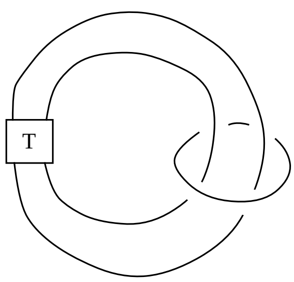
from k copies of $\mathcal{B}$. It is clear that if L is changed by R_{II} or R_{III} then the map J_L is unaltered; indeed J_L is an invariant of the framed link L, its *total bracket invariant*.

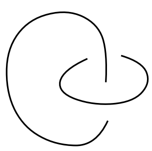
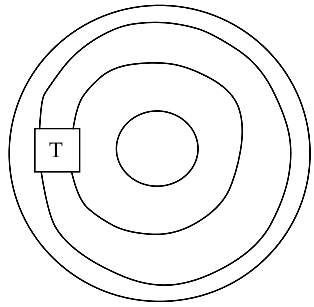
We can make a further generalisation on this construction to the case where D is a diagram with k closed components in a surface F. By decorating each component of D, following its diagrammatic parallel, with a linear combination of diagrams in the annulus, we induce a multilinear map

$$J_D : \mathcal{B} \times \ldots \times \mathcal{B} \to \mathcal{S}(F).$$

When a diagram, L say, arises by decoration of another diagram we can use such a map, taking F itself as an annulus, to write the total invariant of the diagram L as the composite of simpler maps.

For example, suppose that L is a link of $k+1$ components which can be drawn with one of the components, L_{k+1} say, as a simple closed curve. Then, after suitable adjustment by moves R_{II} and R_{III}, the remaining components can be arranged to form a diagram $D = \widehat{T}$ in an annulus, as shown, so that the link L itself is arranged as the Hopf diagram H, with one component decorated by D.

$L =$ 

$H =$ 
$D =$ 

THEOREM 2.2. *The invariant J_L is the composite*

$$\mathcal{B}\times\ldots\times\mathcal{B} \xrightarrow{J_D\times 1} \mathcal{B}\times\mathcal{B} \xrightarrow{J_H} \Lambda.$$

Proof: Decorate each component of L by diagrams $P_1,\ldots,P_{k+1}$. The decorations $P_1,\ldots,P_k$ determine a diagram in the annulus which is just the decoration of D. The final diagram is the Hopf diagram H with this complicated diagram, representing $J_D(P_1,\ldots,P_k)$ in $\mathcal{B}$, decorating one component, while the other component is decorated by P_{k+1}. Then $J_L(P_1,\ldots,P_{k+1}) = J_H(J_D(P_1,\ldots,P_k),P_{k+1})$. The result follows by linearity. □

2.3 THE SATELLITE FORMULA.

We may also use this framework to calculate the total invariant J_K of a knot $K = C*P$ which is a satellite of C constructed by decorating the framed knot C by a diagram P in the annulus. Assuming that P has one component we may decorate P by any diagram Q in the annulus, to get a diagram $P*Q$ also in the annulus. It is easy to see that, up to R_{II} and R_{III}, the diagrams $C*(P*Q)$ and $K*Q=(C*P)*Q$ are the same. It is then immediate that the invariant $J_K:\mathcal{B}\to\Lambda$ is the composite

$$\mathcal{B}\xrightarrow{J_P}\mathcal{B}\xrightarrow{J_C}\Lambda.$$

This equation, and its counterpart for links and patterns with more than one component, will be termed the *s*atellite formula. In this simple case we may also write it as

$$J_{C*P}=J_C\circ J_P.$$

The satellite formula shows that, unlike the bracket polynomial alone, we know the total invariant J of a satellite once we know J for the companion and for the annulus diagram P used in constructing the satellite. (Where C or P have more than one component, the corresponding multilinear maps should be used, and composed appropriately, depending on the component of the companion which is decorated.)

The total bracket invariant J_C contains all the information about bracket invariants of satellites of the knot C. It is known once its values $J_C(\alpha^k)$ on the basis $\{\alpha^k\}$ of $\mathcal{B}$ are known. To determine the bracket invariant of the satellite when C is decorated by a pattern P it is enough to write out $P=a_0+a_1\alpha+\ldots+a_r\alpha^r$ in $\mathcal{B}$ and calculate the bracket invariant of C decorated by α^k, for $0\le k\le r$. Then

$$J_C(P)=\sum_{k=0}^{r}a_kJ_C(\alpha^k).$$

Now $J_C(1)=1$ and $J_C(\alpha)=<C>$, since decoration of C by α just gives C again. However, as remarked earlier, there is in general no way to determine $J_C(\alpha^k)$ from $J_C(\alpha)$, when $k\ge 2$.

For the unknot and the Hopf link, and also for other torus knots and links, the map J_L is known, but not for any other knots. There are examples known, though, of inequivalent knots C_1 and C_2 for which $J_{C_1}=J_{C_2}$; these examples include all mutant pairs of knots, such as the famous pair of Conway and Kinoshita-Teresaka, [25].

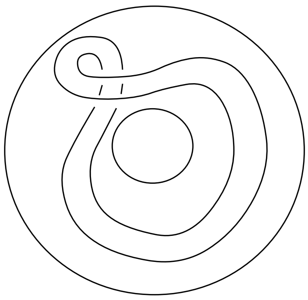
The relation given above for the invariant J of a satellite knot is equivalent to the 'satellite formula' of [23] which relates the total invariant of a given satellite to those of the companion, the Hopf link H and the 'pattern link', namely the satellite of H constructed from the same annulus diagram P as the given satellite. The pattern link consists of P and one extra component, which can be compared to the axis of a closed braid, and which gives the means for recovering P as a diagram in the annulus from the pattern link in S^3. To get the appropriate reinterpretation of [23] it is simply necessary to identify $\mathcal{B}$ with the representation ring $\mathcal{R}$ of the quantum group $SU(2)_q$.

In section 4 I shall give a brief account of the translation between the two viewpoints, but the important features of either approach are the existence of the multilinear invariant J_L for a framed link L, and its natural behaviour on satellites.

2.4 Framing change and the total invariant.

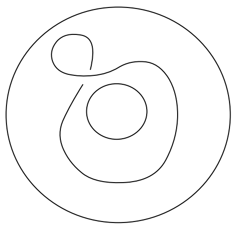
To complete this stage in the understanding of the invariant J_L for a framed link L we must discuss the behaviour of J when the framing of L is altered.

To see more clearly what happens I shall look at the case when L has one component. Suppose that L' has the same diagram as L, except for the addition of a single right-handed curl, so that the underlying knots are equivalent, but the framing has been altered by a single twist. If we use the simple decoration by α then $J_{L'}(\alpha) = < L' > = \lambda < L > = \lambda J_L(\alpha)$, where $\lambda = -A^3$. However, $J_{L'}(\beta)$ is not in general a simple multiple of $J_L(\beta)$. For example, we can calculate $J_{L'}(\alpha^2)$ in terms of J_L by using the diagram shown



to decorate L. This diagram represents $A^8\alpha^2 - (A^8 - 1)$ in $\mathcal{B}$.

In general the change of framing can be expressed in terms of the map $\mathcal{F} : \mathcal{B} \to \mathcal{B}$ induced by decorating the diagram T in the annulus.

$T =$ 

Theorem 2.3. *Let L' be a knot given from L by adding one right-hand twist to the framing. Then $J_{L'} = J_L \circ \mathcal{F}$, where $\mathcal{F} = J_T$, induced by the diagram shown above in the annulus.*

Proof: The diagram L' is just $L * T$, and so the result follows from the satellite formula. □

The map $\mathcal{F}$ has an inverse, induced similarly by the left-hand curl.

When the framing on a link of several components is altered, the total invariant J, as a multilinear map on $\mathcal{B}$, is changed by applying a suitable power of the automorphism $\mathcal{F}$ to each copy of $\mathcal{B}$, depending on the change of framing to be made on the corresponding link component.

To describe the effect of framing change it is enough to determine the map $\mathcal{F}$, or equivalently to find $\mathcal{F}(\alpha^k)$ for each k. As noted above, it is not true that $\mathcal{F}(\alpha^k)$ is a multiple of α^k when $k > 1$, although it is easy to see that, as a polynomial in α, it must have degree at most k, and indeed that its degree is exactly k. In the 3-dimensional view, $\mathcal{F}$ arises when the solid torus formed by thickening the annulus is mapped to itself by cutting along a meridian disc and regluing after a full twist.

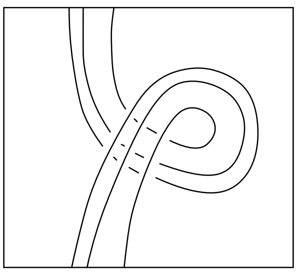
To handle the invariant J most readily, including its behaviour under framing change, it is natural to try to change the basis of $\mathcal{B}$ from $\{\alpha^k\}$ to one consisting of eigenvectors w_i of $\mathcal{F}$, if this is possible. Then $J_{L'}(w_i) = \lambda_i J_L(w_i)$, where λ_i is the eigenvalue of w_i, and the value of $J_{L'}(\beta)$ can be found readily in terms of J_L by writing β in terms of the basis w_i.

2.5 The Temperley-Lieb algebra.

I shall now use the Temperley-Lieb algebra to help construct enough eigenvectors of $\mathcal{F}$ to form a basis of $\mathcal{B}$. Some of the properties of these eigenvectors are most readily appreciated in the alternative view of $\mathcal{B}$ as the representation ring of $SU(2)_q$ in which the eigenvectors appear naturally as the irreducible representations. For this reason I shall index the eigenvectors as $w_1, \ldots, w_i, \ldots$, where w_i, which is a monic polynomial in α of degree $i-1$, will correspond to the irreducible representation of dimension i, in conflict with the notation used by Lickorish [16], who indexes by the degree of the polynomial. In the corresponding construction in [2], Blanchet et al focus heavily on the eigenvector property, without using the Temperley-Lieb algebra at all.

Using the closure wiring referred to earlier to map (n,n)-tangles into annulus diagrams I shall construct elements of $\mathcal{B}$ from the closure of elements in TL_n; in particular, the closure of the element f_n mentioned at the end of section 1.6 is a multiple of the desired eigenvector w_{n+1}.

It is easy to see the effect of $\mathcal{F}$ on any element of $\mathcal{B}$ which is in the closure of TL_n, in terms of the multiplication in TL_n. For suppose that X is an (n,n)-tangle. Then the closure of the tangle $Q_n X$, where Q_n is the right-handed curl on n parallel strings, as shown,



will represent $\mathcal{F}(\widehat{X})$ as an element of $\mathcal{B}$. Write $\varphi_A : TL_n \to \Lambda$ for the linear homomorphism defined by $\varphi_A(\sigma_i) = A$ for each i. In what follows I shall define the elements $f_n \in TL_n$ with the property that $Tf_n = \varphi_A(T)f_n$ for all $T \in TL_n$. It is then immediate that the closure of f_n is an eigenvector of $\mathcal{F}$ since we can write $Q_n f_n = \varphi_A(Q_n)f_n$. Take $X = f_n$; its closure $\widehat{f}_n$ is then an eigenvector with eigenvalue $\varphi_A(Q_n)$.

Now by removing n right-hand curls, one from each component, we can write Q_n as a multiple of the right-hand full-twisted braid Δ_n^2, as an element of TL_n, so we have $Q_n = (-1)^n A^{3n}\Delta_n^2$. Since Δ_n^2 is a braid it is easy to calculate $\varphi_A(\Delta_n^2)$ in terms of the crossings in the braid, as $\varphi_A(\sigma_i) = A$ for each i. Now after removal of the n curls from Q_n there remain $n^2 - n$ crossings in the braid Δ_n^2, all in the same sense, so we have $\varphi_A(\Delta_n^2) = A^{n^2-n}$ and thus the eigenvalue for $\widehat{f}_n$ is $(-1)^n A^{n^2+2n}$.

Define elements $w_i \in \mathcal{B} \cong \Lambda[\alpha]$ by the relations

$$w_1 = 1, \quad w_2 = \alpha,$$

$$w_{i+1} = \alpha w_i - w_{i-1},\ i > 1.$$

Each w_k is clearly a monic polynomial of degree $k-1$, and can be recognised as the Chebyshev polynomial of the second kind, resulting from writing $\sin k\theta/\sin\theta$ as a polynomial in $\alpha = 2\cos\theta$, (cf. Lickorish [16]).

The final result in this section is to establish that $\varphi_A(f_n)w_{n+1} = \widehat{f}_n$ so that each w_i is an eigenvector of $\mathcal{F}$ with eigenvalue $\lambda_i = (-1)^{i-1}A^{i^2-1}$, provided that $\varphi_A(f_n) \neq 0$.

While it appears more appealing to divide $\widehat{f}_n$ by $\varphi_A(f_n)$ in order to map exactly to w_{n+1} this can only be done by extending the ring Λ to allow suitable denominators. At the present stage this need cause no problems, but later developments which require substitution of the variable A in Λ then become more difficult as there is a chance that $\varphi_A(f_n)$ may become zero. Lickorish in fact uses carefully controlled denominators to define an element denoted by $f^{(n)}$ whose closure is exactly w_{n+1}. However, the definition of f_n without the factor, as given here, is also quite natural.

2.6 Positive permutation braids.

I shall construct the element $f_n \in TL_n$ by means of positive permutation braids. These have been used in [22] as a convenient basis for the Hecke algebra, and are discussed more fully in [4]. In the algebraic context the construction of f_n given here is a special case of a construction of Jones in the Hecke algebra [7]; this method has also been noted more recently by Kauffman [10].

Definition. For each permutation $\pi \in S_n$ there is an n-braid $w_\pi(\sigma_1, \ldots, \sigma_{n-1})$, called a *positive permutation braid*. It is uniquely determined by the following properties.

(1) String i joins the point numbered i at the bottom of the braid to the point numbered $\pi(i)$ at the top, $i = 1, \ldots, n$.

(2) At any crossing, string i always crosses *over* string j if $i < j$.

We may view the strings in the braid as lying in layers, with string 1 above string 2, and so on, so that each string can be moved independently of the others. This ensures the uniqueness of w_π, which can be drawn, if required, so that pairs of strings cross at most once. In this form, condition (2) is equivalent to asking that each crossing be positive, when all strings are oriented from bottom to top.

Let us now consider an algebra $\mathcal{A}$ in which the n-string braid group B_n is represented. In what follows, we shall be primarily interested in the Temperley-Lieb algebra, TL_n, but the arguments will work as well in a more general setting. I shall continue to write σ_i for the element of $\mathcal{A}$ which represents the elementary braid σ_i. We may then define an element $E_n(\sigma_1,\ldots,\sigma_{n-1})$ in the algebra $\mathcal{A}$ as the sum

$$E_n(\sigma_1,\ldots,\sigma_{n-1}) = \sum_{\pi\in S_n} w_\pi(\sigma_1,\ldots,\sigma_{n-1}).$$

Thus $E_3 = 1+\sigma_1+\sigma_2+\sigma_1\sigma_2+\sigma_2\sigma_1+\sigma_1\sigma_2\sigma_1$, noting that the last braid in the sum, corresponding to the permutation $(1\,3)$, could equally well be written as $\sigma_2\sigma_1\sigma_2$. It is a convenient property of the permutation braids that it is only necessary to remember them by their permutation of the strings, without having to specify each braid as a word in $\{\sigma_i\}$.

THEOREM 2.4. *For each i we can factorise E_n in the given algebra $\mathcal{A}$ as $E_n = E_n^{(i)}(\sigma_i+1)$.*

Proof: Given i, we can pair the permutations as follows. For each permutation π consider its composite $\pi' = \pi\circ(i\ i{+}1)$ with the transposition $(i\ i{+}1)$. Exactly one of the pair preserves the order of i and $i+1$. Suppose that it is π, so that $\pi(i)<\pi(i+1)$. Then the braid $w_\pi\sigma_i$ satisfies property (2) above, and so is itself a positive permutation braid. Since its permutation is π' we have $w_\pi\sigma_i = w_{\pi'}$. Then

$$\begin{aligned} E_n &= \sum_{\pi(i)<\pi(i+1)} w_\pi \quad + \sum_{\pi'(i)>\pi'(i+1)} w_{\pi'} \\ &= \sum_{\pi(i)<\pi(i+1)} w_\pi \quad + \sum_{\pi(i)<\pi(i+1)} w_\pi\sigma_i \\ &= E_n^{(i)}(\sigma_i+1), \end{aligned}$$

where $E_n^{(i)} = \sum_{\pi(i)<\pi(i+1)} w_\pi$. □

If λ is a scalar, then we may substitute $\lambda\sigma_i$ for σ_i and rewrite the element $w_\pi(\lambda\sigma_1,\ldots,\lambda\sigma_{n-1})$ as $\lambda^{l(\pi)}w_\pi(\sigma_1,\ldots,\sigma_{n-1})$ in $\mathcal{A}$, where $l(\pi)$ is the writhe of the braid w_π. This is the same as the length of w_π when written as a monomial in positive powers of the elementary braids σ_i. It is equal to the number of reversals of the permutation π, i.e. the number of pairs $i<j$ for which $\pi(i)>\pi(j)$.

Suppose now that all the elementary braids satisfy the quadratic equation

$$(\sigma_i-a)(\sigma_i-b)=0$$

in the algebra $\mathcal{A}$. Substitute $\lambda\sigma_i$ for σ_i in E_n, with $\lambda=-a^{-1}$ or $\lambda=-b^{-1}$, to define

$$a_n = E_n(-a^{-1}\sigma_1,\ldots,-a^{-1}\sigma_{n-1}),\quad b_n = E_n(-b^{-1}\sigma_1,\ldots,-b^{-1}\sigma_{n-1}).$$

THEOREM 2.5. *Suppose that the algebra $\mathcal{A}$ is spanned by braids, that $(\sigma_i-a)(\sigma_i-b)=0$ in $\mathcal{A}$ and that φ_a and φ_b are linear homomorphisms from $\mathcal{A}$ to the scalars defined by $\varphi_a(\sigma_i)=a$, $\varphi_b(\sigma_i)=b$. Then every $T\in\mathcal{A}$ satisfies*

$$a_nT = \varphi_b(T)a_n = Ta_n,\quad b_nT=\varphi_a(T)b_n = Tb_n.$$

Proof: To establish the left-hand equality in each case it is enough to show that $a_n\sigma_i = \varphi_b(\sigma_i)a_n = ba_n$ for each i, and similarly that $b_n\sigma_i = ab_n$. We can factorise a_n using the theorem above, as

$$a_n = E_n^{(i)}(-a^{-1}\sigma_1,\ldots,-a^{-1}\sigma_{n-1}) \times (-a^{-1}\sigma_i + 1) = Q_n(\sigma_i - a), \text{ say},$$

giving $a_n(\sigma_i - b) = Q_n(\sigma_i - a)(\sigma_i - b) = 0$, so that $a_n\sigma_i = ba_n$. Similarly $b_n(\sigma_i - a) = 0$.

The factorisation of E_n as $(\sigma_i + 1)E_n^{'(i)}$ is also possible, proving the right-hand equalities $Ta_n = \varphi_b(T)a_n$ and $Tb_n = \varphi_a(T)a_n$. □

Remark. When $\mathcal{A}$ is the group algebra of the symmetric group, $\mathbf{Z}[S_n]$, and each σ_i is represented as a transposition, the quadratic equation is $\sigma_i^2 - 1 = (\sigma_i - 1)(\sigma_i + 1) = 0$. The elements a_n and b_n are then the classical symmetriser and skew-symmetriser,

$$b_n = \sum_{\pi \in S_n} \pi, \ a_n = \sum_{\pi \in S_n} \varepsilon(\pi)\,\pi \ .$$

The Temperley-Lieb algebra TL_n is generated by the n-braids $\sigma_1,\ldots,\sigma_{n-1}$ which satisfy the relation $\sigma_i = A + A^{-1}h_i$ with $h_i^2 = \delta h_i$ $(= -(A^2 + a^{-2})h_i)$. Then $\sigma_i h_i = -A^{-3}h_i$, so that $(\sigma_i + A^{-3})(\sigma_i - A) = 0$.

Definition. In TL_n we define an element f_n by

$$f_n = E_n(A^3\sigma_1,\ldots,A^3\sigma_{n-1}) \quad (= \sum_{\pi \in S_n} A^{3l(\pi)} w_\pi(\sigma_1,\ldots,\sigma_{n-1}) \ ).$$

Corollary to theorem 2.5. *Every $T \in TL_n$ satisfies the multiplicative property*

$$f_nT = Tf_n = \varphi_A(T)f_n,$$

where $\varphi_A : TL_n \to \Lambda$ is the linear homomorphism defined by $\varphi_A(\sigma_i) = A$ for each i.

Proof: We can apply the theorem to TL_n with $a = -A^{-3}$ and $b = A$. Then $f_n = a_n$ and the result follows. □

Remarks. The element $b_n \in TL_n$ is identically zero for $n \geq 3$.

The general algebra to which the theorem applies is some quotient of the Hecke algebra. Jones [7] notes the elements a_n and b_n for the Hecke algebra when $a = q, b = -1$; any other case can be rewritten in this way if σ_i is replaced throughout by a suitable multiple.

2.7 The alternative basis for the skein of the annulus.

Having established the definition and multiplicative property of f_n in TL_n I now want to relate the closure of f_n in $\mathcal{B}$ to the element w_{n+1}, defined inductively above by $w_{n+1} = \alpha w_n - w_{n-1}$, with $w_1 = 1$ and $w_2 = \alpha$.

Theorem 2.6. *In $\mathcal{B}$, the skein of the annulus, we have $\widehat{f}_n = \varphi_A(f_n)w_{n+1}$ for all $n \geq 1$.*

Proof: We have $f_1 = 1$ in TL_1 as a braid on one string, so $\widehat{f}_1 = \alpha = w_2$. Indeed, we could consider $f_0 = \phi$ in TL_0, which gives $\widehat{f}_0 = \phi = w_1$, noting that the empty diagram in the annulus represents the identity element $w_1 = 1$ in the algebra $\mathcal{B}$. The rest of the proof is by induction on n, and depends on establishing the appropriate relation between $\widehat{f}_n$, $\widehat{f}_{n-1}$ and $\widehat{f}_{n-2}$. This in turn depends on rewriting some of the permutation braids w_π which appear in the sum E_n.

Corresponding to the inclusion $i : S_{n-1} \subset S_n$ in which $\pi' \in S_{n-1}$ is extended to $\pi' \in S_n$ by $\pi'(n) = n$ there is an inclusion of the braid group B_{n-1} in B_n by adjoining an n-th straight string. The same procedure defines an inclusion $i : TL_{n-1} \subset TL_n$; this can even be seen to come from a simple wiring of R^{n-1}_{n-1} into R^n_n which adjoins the extra string. The element $i(E_{n-1})$ is then $\displaystyle\sum_{\pi(n)=n} w_\pi$. Because of the extra string the closure of $i(T)$, for any $T \in TL_{n-1}$, can be written as $\alpha\widehat{T}$ in $\mathcal{B}$.

Define braids $\gamma_r \in B_n$, $r = 0, \ldots, n-1$, by $\gamma_0 = 1$, $\gamma_r = \sigma_{n-1}\sigma_{n-2}\ldots\sigma_{n-r}$. In γ_r the string ending at position n crosses exactly r others, while no other strings cross each other. The braids $w_\pi\gamma_r$ with $\pi(n) = n$ are then positive permutation braids for all such π and for all $r = 0, \ldots, n-1$. All permutations of strings arise exactly once on this list, so all positive permutation braids are counted exactly once as

$$E_n = \sum_{\pi \in S_n} w_\pi = \left(\sum_{\pi(n)=n} w_\pi\right)\left(\sum_{r=0}^{n-1} \gamma_r\right).$$

Replace σ_i by $A^3\sigma_i$ to get

$$(*)_n \qquad f_n = i(f_{n-1})\left(\sum_{r=0}^{n-1} A^{3r}\gamma_r\right).$$

We can calculate $\varphi_A(f_n)$ inductively, using $(*)_n$, since $\varphi_A(\gamma_r) = A^r$. For we have

$$\varphi_A(f_n) = \varphi_A(f_{n-1})\left(\sum_{r=0}^{n-1} A^{4r}\right) = [n]_q\varphi_A(f_{n-1}),$$

where $[n]_q = 1 + q + \cdots + q^{n-1}$ ($= n$ when $q = 1$) and $q = A^4$. Consequently,

$$\varphi_A(f_n) = [n]_q[n-1]_q \ldots [1]_q = [n]_q!.$$

To complete the proof of the theorem it will be enough to establish the relation

$$\widehat{f}_n = [n]_q\alpha\widehat{f}_{n-1} - [n]_q[n-1]_q\widehat{f}_{n-2},$$

as the right hand side is then, by the induction hypothesis, $\varphi_A(f_n)(\alpha w_n - w_{n-1}) = \varphi_A(f_n)w_{n+1}$.

We now use $(*)_n$ to find the closure $\widehat{f}_n$. For any elements T_1 and T_2 in TL_n the products T_1T_2 and T_2T_1 have the same closure in $\mathcal{B}$. We can then replace f_n by the product $P_n = \left(\displaystyle\sum_{r=0}^{n-1} A^{3r}\gamma_r\right) i(f_{n-1})$. Now $\sigma_j\, i(f_{n-1}) = A\, i(f_{n-1})$, for $j < n-1$, by the multiplicative property of f_{n-1}, so $\gamma_r i(f_{n-1}) = A^{r-1}\sigma_{n-1}i(f_{n-1})$, for $r > 0$.

Then

$$
\begin{aligned}
P_n &= i(f_{n-1}) + \left(\sum_{r=1}^{n-1} A^{4r-1}\right) \sigma_{n-1} i(f_{n-1}) \\
&= \left(\sum_{r=0}^{n-1} A^{4r}\right) i(f_{n-1}) + \left(\sum_{r=1}^{n-1} A^{4r-2}\right) h_{n-1} i(f_{n-1}) \\
&= [n]_q i(f_{n-1}) + A^2[n-1]_q h_{n-1} i(f_{n-1}),
\end{aligned}
$$

since $\sigma_{n-1} = A + A^{-1}h_{n-1}$ in TL_n. Hence $\widehat{f}_n = \widehat{P}_n = [n]_q \alpha \widehat{f}_{n-1} + A^2[n-1]_q \widehat{Q}_n$, where $Q_n = h_{n-1} i(f_{n-1}) \in TL_n$.

We complete the proof by showing that $\widehat{Q}_n = -A^{-2}[n]_q \widehat{f}_{n-2}$.

By $(*)_{n-1}$ we have

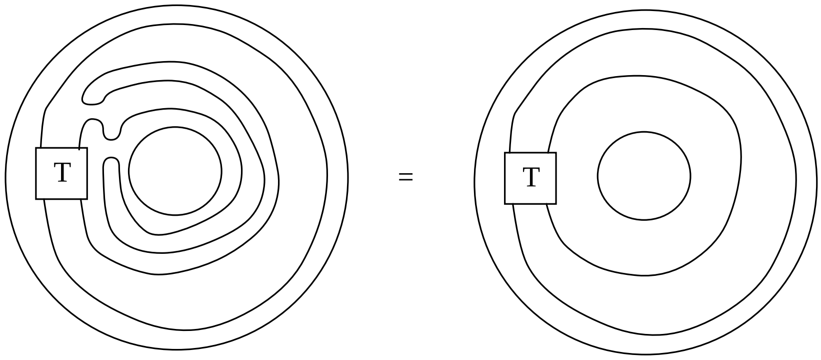
$$
Q_n = h_{n-1} i(f_{n-1}) = h_{n-1} i(f_{n-2}) \left(\sum_{r=0}^{n-2} A^{3r} \gamma'_r\right),
$$

where $\gamma'_r = \sigma_{n-2} \ldots \sigma_{n-r-1}$.

Then Q_n has the same closure as $\left(\sum_{r=0}^{n-2} A^{3r} \gamma'_r\right) h_{n-1} i(f_{n-2}) = R_n$, say. Now σ_j commutes with h_{n-1} and $\sigma_j i(f_{n-2}) = A\, i(f_{n-2})$, for $j < n-2$, so as above we get

$$
\begin{aligned}
R_n &= h_{n-1} i(f_{n-2}) + \left(\sum_{r=1}^{n-2} A^{4r-1}\right) \sigma_{n-2} h_{n-1} i(f_{n-2}) \\
&= \left(\sum_{r=0}^{n-2} A^{4r}\right) h_{n-1} i(f_{n-2}) + \left(\sum_{r=1}^{n-2} A^{4r-2}\right) h_{n-2} h_{n-1} i(f_{n-2}) \\
&= [n-1]_q h_{n-1} i(f_{n-2}) + A^2[n-2]_q h_{n-2} h_{n-1} i(f_{n-2}).
\end{aligned}
$$

Now for $T \in TL_{n-2}$ the closures of the elements $h_{n-1}\, i(T$



Thus

$$\begin{aligned}\widehat{Q}_n = \widehat{R}_n &= ([n-1]_q\delta + A^2[n-2]_q)\widehat{f}_{n-2}\\ &= -A^{-2}\left((1+A^4)[n-1]_q - A^4[n-2]_q\right)\widehat{f}_{n-2}\\ &= -A^{-2}(A^4[n-1]_q+1)\widehat{f}_{n-2} = -A^{-2}[n]_q\widehat{f}_{n-2}.\end{aligned}$$

This completes the last step in the proof. □

As remarked earlier, this result establishes that the elements w_i are eigenvectors of the twist-induced map $\mathcal{F}$, with eigenvalue $\lambda_i = (-1)^{i-1}A^{i^2-1}$.

3. Invariants of 3-manifolds.

3.1 Surgery on framed links.

A description of closed orientable 3-manifolds has been known for some time in terms of surgery on framed links in S^3.

Given a framed link L in S^3, the technique of surgery produces a manifold $M(L)$ by removing a solid torus neighbourhood V_i of each link component L_i from S^3, leaving the 'exterior' of L, a compact 3-manifold whose boundary consists of k tori. The closed manifold $M(L)$ is built up from this piece and k solid tori, by gluing each solid torus to one of the boundary components. On the boundary of each solid torus there is a distinguished family of closed curves, the meridians, which bound discs in the solid torus. To specify $M(L)$ we must say which curves on the boundary of the exterior of L are to be matched with the meridians by the gluing.

We use the framing of L to determine this match. The framing of the component L_i specifies a choice of curves parallel to L_i which determines a distinguished family of curves on the corresponding boundary component of the exterior of L; the surgery is defined by matching these curves with the meridians.

We may think of the link L as giving us a view in S^3 of a large part of the manifold $M(L)$, namely the exterior of L. All that remains unseen are the added solid tori, and the picture provides a good indirect knowledge of these as well. Of course there can be other views of the same 3-manifold, based on a different link L' say, in other words we may find links L and L' for which $M(L) \cong M(L')$.

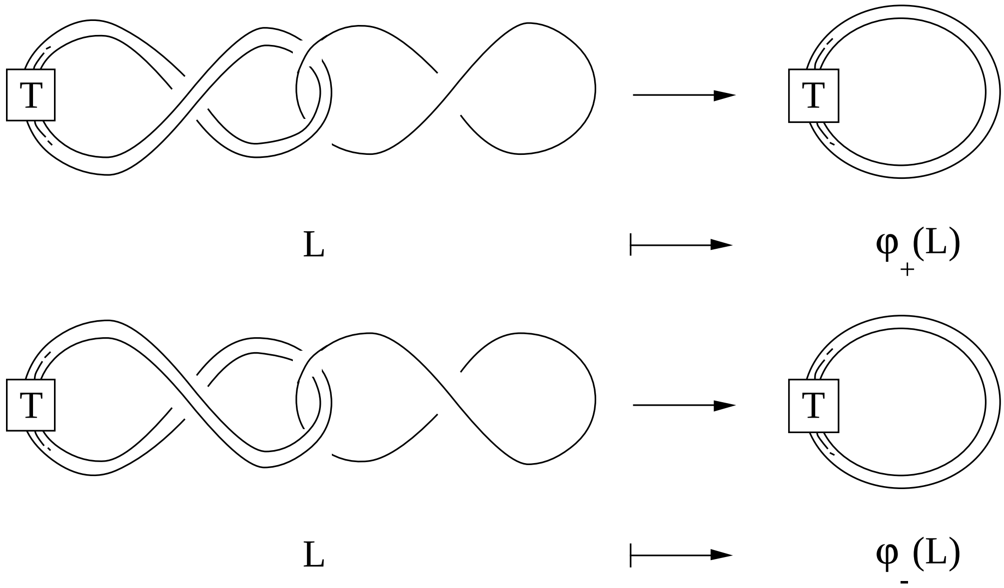
The study of 3-manifolds by means of framed links is greatly simplified by the results of Kirby [11] and Fenn and Rourke [5].

Theorem 3.1 (Kirby, Fenn-Rourke).

(1) *Every closed oriented 3-manifold arises as $M(L)$ for some framed link L.*

(2) *There is an orientation preserving homeomorphism* $M(L) \cong M(L')$ *if and only if* L *and* L' *are related by a finite sequence of Kirby moves.*

Kirby moves are of two types, shown below.



As in the earlier sections we assume that each diagram specifies a framed link using the diagrammatic framing.

These moves have been used by Reshetikhin and Turaev [32], and subsequently several other authors, as a means of approaching the family of 3-manifold invariants described by Witten [W]. The central idea is to look for an element $\Omega \in \mathcal{B}$ with the property that the value $J_L(\Omega,\ldots,\Omega) \in \Lambda$, possibly normalised in some way, is unaltered when L is changed by Kirby moves. If such an Ω were to exist, then $J_L(\Omega,\ldots,\Omega)$ would depend only on the manifold $M(L)$, and so would give an element of Λ which is an invariant of $M(L)$. Unfortunately this does not prove to be possible without some modification, even allowing Ω to be a formal power series in α rather than a polynomial.

The modification which works is to decide initially on a 'level' l, or equivalently to select a $4r$-th root of unity, with $r = l+2$, which is to be substituted for the variable A in Λ. Having decided on l, it is *then* possible to choose $\Omega \in \mathcal{B}$, (depending on l), so that the complex number given by substituting a $4r$-th root of unity in $J_L(\Omega,\ldots,\Omega)$ is, after suitable normalisation, unaltered by the Kirby moves, and is thus an invariant of $M(L)$.

In keeping with Segal's view of Witten's invariantsum an invarΩan J

3.2 Evaluations of the total invariant at roots of unity.

I shall start by discussing the evaluation of J_L on the ideal in $\mathcal{B}$ generated by one of the elements w_r. For any given component of the diagram of L it is possible, using moves R_{II} and R_{III}, to draw it in an annulus as the closure of some $(1,1)$-tangle T so that the chosen component, L_1 say, is the single arc in T while the remaining components $L_2, \ldots, L_k$ lie entirely in T. This diagram $\widehat{T}$ in the annulus induces a multilinear map $J_{\widehat{T}} : \mathcal{B} \times \ldots \times \mathcal{B} \to \mathcal{B}$ and J_L is the composite of this with the evaluation map $< \cdot >: \mathcal{B} \to \Lambda$.

Theorem 3.2. *For any n, and any $\beta_2, \ldots, \beta_k \in \mathcal{B}$ we have*

$$J_{\widehat{T}}(w_{n+1}, \beta_2, \ldots, \beta_k) = \lambda w_{n+1}, \text{ for some } \lambda \in \Lambda.$$

Proof: From the tangle T construct an (n,n)-tangle $T^{(n)}$ with n parallel arcs in place of the single arc. Decorate the $k-1$ closed curves by $\beta_2, \ldots, \beta_k$ to give an element, G say, in TL_n. The closure of $f_n G$ will then be $J_{\widehat{T}}(\widehat{f}_n, \beta_2, \ldots, \beta_k) \in \mathcal{B}$. Now the multiplicative property of f_n allows us to write $f_n G = \lambda f_n$, where $\lambda = \varphi_A(G) \in \Lambda$. Thus $J_{\widehat{T}}(\widehat{f}_n, \beta_2, \ldots, \beta_k) = \lambda \widehat{f}_n$, and the result follows since $\widehat{f}_n$ is a multiple of w_{n+1}. □

Theorem 3.3. *Let L be a link diagram. Then*

(a) $J_L(w_r \times \mathcal{B} \times \ldots \times \mathcal{B}) \subset \Lambda < w_r >$, the ideal generated by $< w_r >$ in Λ, and

(b) $J_L(\mathcal{I}_r \times \mathcal{B} \times \ldots \times \mathcal{B}) \subset \Lambda < w_r >$, where $\mathcal{I}_r \subset \mathcal{B}$ is the ideal of $\mathcal{B}$ generated by w_r.

Proof: Part (a) is an immediate corollary of the previous theorem, with $r = n+1$, on drawing L to lie appropriately in the annulus. The result holds when any of the components is decorated by w_r.

To prove part (b) it is enough to deal with the element $w_r \beta \in \mathcal{I}_r$ for any $\beta \in \mathcal{B}$. We can use the multiplication in $\mathcal{B}$ to write

$$J_L(w_r \beta, \beta_2, \ldots, \beta_k) = J_{L'}(w_r, \beta, \beta_2, \ldots, \beta_k),$$

where L' is the link with two parallel components in place of the first component of L but otherwise identical to L. The result now follows from (a) applied to L'. □

The evaluation map $< \cdot > : \mathcal{B} \to \Lambda$ is a ring homomorphism, and hence

$$< w_{n+1} > = < \alpha >< w_n > - < w_{n-1} >.$$

Starting from $< w_1 > = 1$ and $< w_2 > = < \alpha > = \delta = -(A^2 + A^{-2})$ it follows readily that $< w_r > = (-1)^{r-1} \dfrac{A^{2r} - A^{-2r}}{A^2 - A^{-2}} \in \Lambda$. Then $< w_r > = 0$ when $A^{4r} = 1,\ A^4 \neq 0$.

Notation. Write $\Lambda_r \subset \mathbf{C}$ for the image of Λ when A is mapped to a primitive $4r$-th root of unity, e.g. $A = e^{\pi i/2r}$.

Equivalently, take Λ_r to be the quotient of Λ by the ideal generated by Euler's polynomial $\varphi_{4r}(A)$.

Write also $\mathcal{B}_r$ for the finite-dimensional Λ_r-module $(\mathcal{B}/\mathcal{I}_r) \otimes \Lambda_r$, where, as noted above, the coefficient ring has been changed from Λ to Λ_r by substitution for A. Theorem 3.3 can then be reformulated.

THEOREM 3.4. *For any link diagram L the invariant $J_L : \mathcal{B} \times \ldots \times \mathcal{B} \to \Lambda$ induces a multilinear map $J_L^{(r)} : \mathcal{B}_r \times \ldots \times \mathcal{B}_r \to \Lambda_r$.*

Proof: The value of $J_L^{(r)}$ can be calculated by choosing decorations $(\beta_1, \ldots, \beta_k)$ in $\mathcal{B} \times \ldots \times \mathcal{B}$, and substituting the chosen $4r$-th root of unity for A in $J_L(\beta_1, \ldots, \beta_k)$. The previous theorem shows that this number in Λ_r is unchanged when an element of the ideal $\mathcal{I}_r$ is added to any β_i, by multilinearity of J_L, and so the result depends only on the elements represented by β_i in $\mathcal{B}_r$. □

3.3 STRUCTURE OF THE ALGEBRAS $\mathcal{B}$ AND $\mathcal{B}_r$.

The product $w_j w_k$ of two basis elements in $\mathcal{B}$ can be written as a sum $\sum_i n_{ijk} w_i$, with structure constants $n_{ijk} \in \Lambda$. It can be established inductively that $n_{ijk} \in \mathbf{N}$, and that $n_{1jk} = \delta_{jk}$; in fact

$$n_{ijk} = \begin{cases} 1, & \text{if } i+j+k = 1 \bmod 2 \ \text{ and } |j-k| < i < j+k, \\ 0, & \text{otherwise.} \end{cases}$$

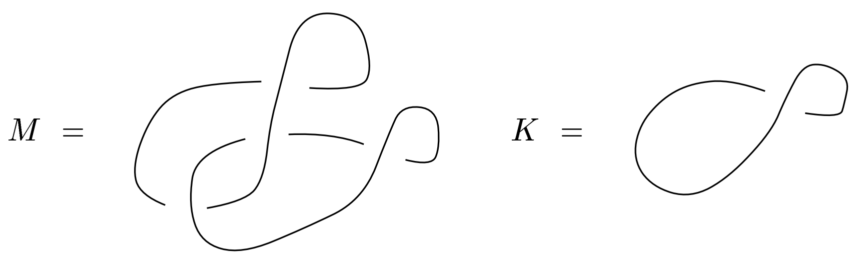
This is more obvious once we are able to identify $\mathcal{B}$ with the representation ring of $SU(2)$ and w_i with the irreducible representations.

Then n_{ijk} is the coefficient of w_1 in the product $w_i w_j w_k$. Since $\mathcal{B}$ is commutative, n_{ijk} is unchanged by permutation of i, j and k.

The algebra $\mathcal{B}_r$ has a basis $w_1, \ldots, w_{r-1}$, or properly speaking the images of these elements. Each $w_i \in \mathcal{B}$ represents some integer linear combination of $w_1, \ldots, w_{r-1}$ in $\mathcal{B}_r$, and we can write $w_j w_k = \sum_{i=1}^{r-1} m_{ijk} w_i$ in $\mathcal{B}_r$, for some integers m_{ijk}. It can be shown [24] that m_{ijk} is also symmetric in i, j and k when $j, k < r$ and only takes the values 0 or 1.

3.4 THE 3-MANIFOLD INVARIANTS.

Let us now compare the invariants of two links related by a positive Kirby move. Suppose that the two links are as shown in the Kirby move diagram, and that the second link $\varphi_+(L)$ has k components, corresponding to the first k components of the link L. Regard the closure $\widehat{T}$ as a diagram in the annulus, determining $J_{\widehat{T}} : \mathcal{B} \times \ldots \times \mathcal{B} \to \mathcal{B}$. Choose any decoration $\beta_1, \ldots, \beta_k$ of $\widehat{T}$ and write $X = J_{\widehat{T}}(\beta_1, \ldots, \beta_k) \in \mathcal{B}$. The satellite formula shows that $J_L(\beta_1, \ldots, \beta_k, \Omega) = J_M(X, \Omega)$, where M is the link shown below.



Since M itself is two parallel copies of the diagram K, which in turn is the unknot with a positive curl, we can write $J_M(X, \Omega) = J_K(X\Omega) = < F(X\Omega) >$. We want to compare this with the invariant after the Kirby move, namely $J_{\varphi_+(L)}(\beta_1, \ldots, \beta_k) = < X >$.

THEOREM 3.5. *Given* r *there exists* $\Omega \in \mathcal{B}$ *and* $c_\pm \neq 0 \in \Lambda_r$ *depending only on* r *such that, for any choice of* L *and decorations* $\beta_1, \ldots, \beta_k$,

$$J_L(\beta_1, \ldots, \beta_k, \Omega) = c_\pm J_{\varphi_\pm(L)}(\beta_1, \ldots, \beta_k)$$

when evaluated in Λ_r.

Proof: Choose $\Omega = \sum_{k=1}^{r-1} a_k w_k$ with $a_k = < w_k >$. By the calculations above, it is enough to find c_+ so that $J_M(X, \Omega) = c_+ < X >$ in Λ_r for all $X \in \mathcal{B}$. Since we are evaluating in Λ_r it is enough to check for X in a spanning set of $\mathcal{B}_r$, e.g. $X = w_j,\ j = 1, \ldots, r-1$.

Now $J_M(w_j, \Omega) = J_K(w_j \Omega)$. Again it is enough to work with $w_j \Omega$ as an element of $\mathcal{B}_r$, since we are only concerned with the evaluation in Λ_r, so that in Λ_r we have

$$\begin{aligned} J_K(w_j \Omega) &= J_K(\sum_{k=1}^{r-1} a_k w_j w_k) \\ &= J_K(\sum_{k=1}^{r-1} \sum_{i=1}^{r-1} m_{ijk} a_k w_i) \\ &= < \sum_{k=1}^{r-1} \sum_{i=1}^{r-1} m_{ijk} a_k \lambda_i w_i > \\ &= \sum_{i=1}^{r-1} \lambda_i < w_i > \sum_{k=1}^{r-1} m_{ijk} a_k . \end{aligned}$$

On the other hand, $< w_i >< w_j > = < w_i w_j > = \sum_{k=1}^{r-1} m_{kij} < w_k > = \sum_{k=1}^{r-1} m_{ijk} a_k$ by symmetry of the coefficients m_{ijk}. Thus

$$\begin{aligned} J_M(w_j, \Omega) &= \sum_{i=1}^{r-1} \lambda_i < w_i >< w_i >< w_j > \\ &= c_+ < w_j >, \end{aligned}$$

where $c_+ = \sum_{i=1}^{r-1} \lambda_i < w_i >^2$.

The assignment $c_- = \sum_{i=1}^{r-1} \lambda_i^{-1} < w_i >^2$ will handle the negative Kirby move similarly, as the only difference is in the use of F^{-1} in place of F to deal with the left-handed curl. Noting that $c_\pm$ are complex conjugates in Λ_r since $|A| = 1$ we can write $c_\pm = \rho c^{\pm 1}$ in polar form, with $\rho > 0$ and $|c| = 1$. It is possible to calculate c, ρ in terms of the root of unity A, and check also that $\rho \neq 0$. □

Assignment of Ω to each component then gives an element of Λ_r which is invariant under the Kirby moves, except for the appearances of $c_\pm$. It is not difficult to introduce a normalising factor to correct for this, as follows.

To a framed *oriented* link $L = L_1 \cup L_2 \cup \ldots \cup L_k$ we can associate a quadratic form with $k \times k$ matrix (l_{ij}) where

$$l_{ij} = \mathrm{lk}(L_i, L_j),\ i \neq j, \quad l_{ii} = \text{framing on } L_i.$$

Write $\mathrm{sig}(L)$ for the signature of this form.

Then $\mathrm{sig}(L)$ is independent of the choice of orientation of L, and

$$\mathrm{sig}\,\varphi_\pm(L) = \mathrm{sig}\,L \mp 1.$$

COROLLARY 3.6. *When $M(L)$ is given by surgery on the framed link L with k components the complex number*

$$\rho^{-k} c^{-\mathrm{sig}\,L} J_L(\Omega, \ldots, \Omega),$$

evaluated at the given root of unity, is an invariant of the 3-manifold $M(L)$.

Proof: It is enough to show that the number is unaltered by a Kirby move on L. Consider the case of the positive Kirby move, giving $\varphi_+(L)$ with $k-1$ components. Then

$$\begin{aligned} \rho^{-k} c^{-\mathrm{sig}\,L} J_L(\Omega, \ldots, \Omega) &= c_+ \rho^{-k} c^{-\mathrm{sig}\,L} J_{\varphi_+(L)}(\Omega, \ldots, \Omega) \\ &= \rho^{-(k-1)} c^{-(\mathrm{sig}\,L - 1)} J_{\varphi_+(L)}(\Omega, \ldots, \Omega) \\ &= \rho^{-(k-1)} c^{-\mathrm{sig}\,\varphi_+(L)} J_{\varphi_+(L)}(\Omega, \ldots, \Omega), \end{aligned}$$

which is the corresponding number for $\varphi_+(L)$. The negative Kirby move works similarly, with c_- in place of c_+ covered by the alteration in signature. □

Remarks. There has only been a limited amount of calculation of these invariants. A recent tabulation of known evaluations is given in [27]. Kirby and Melvin have been able to give a closed formula for the invariants for Lens spaces as r varies, and also show how the value for $r = 2, 3, 4$ or 6 can be related in general to known topological invariants. Strickland has also developed programs to compute for Lens spaces, using knowledge of J for torus knots. The difficulty in general comes in calculating $J_L(\Omega)$ for larger values of r, as this requires knowledge of $J_L(w_k)$, at least in Λ_r, for all $k < r$. This in turn is equivalent to knowing $J_L(\alpha^j)$ for $j < r-1$, in other words, the bracket invariant of the j-fold parallels of L. As a computational exercise this rapidly becomes impractical with increasing j, even when L has a braid presentation on as few as 3 strings.

4. The quantum group approach.

In this section I shall discuss the alternative view of the invariants J_L of a framed link which was pioneered by Reshetikhin [29] and Turaev [34].

The starting point here is a quantum group $\mathcal{G}_q$, most conveniently one which is associated to a classical Lie group G; in the present context it is enough to consider $G = SU(2)$. The quantum group is an algebra over a ring Λ which includes a parameter q. Many of the constructions involve polynomials in $q^{\pm\frac{1}{4}}$ at the worst, and with care the ring can be regarded as $\mathbf{Z}[q^{\pm\frac{1}{4}}]$.

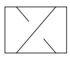
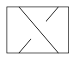
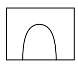
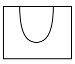
Finite-dimensional representations of the quantum group $\mathcal{G}$ (i.e. $\mathcal{G}$-modules) play a central role in the definition of link invariants. The most important property of $\mathcal{G}$

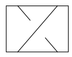
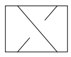
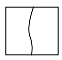
is that it is a Hopf algebra, in other words it admits a comultiplication $\Delta : \mathcal{G} \to \mathcal{G} \times \mathcal{G}$ which has a sufficiently natural interaction with the algebra multiplication to allow tensor products of $\mathcal{G}$-modules to be themselves regarded as $\mathcal{G}$-modules.

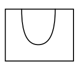
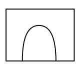
The map Δ is not symmetric, in the sense that $\tau \circ \Delta \neq \Delta$ where $\tau : \mathcal{G} \otimes \mathcal{G} \to \mathcal{G} \otimes \mathcal{G}$ is induced by $\tau(g \otimes h) = h \otimes g$. Consequently, when V and W are two $\mathcal{G}$-modules, the simple interchange map $\tau : V \otimes W \to W \otimes V$ need not be an isomorphism of $\mathcal{G}$-modules, since Δ is used in the definition of $V \otimes W$ as a $\mathcal{G}$-module. There is, however, as part of the definition of a quantum group, an element R in a suitable extension of $\mathcal{G} \otimes \mathcal{G}$ which relates $\tau \circ \Delta$ and Δ. From this 'universal R-matrix' R there arises a $\mathcal{G}$-module isomorphism $R_{VW} : V \otimes W \to W \otimes V$ for all modules V and W, which is not the simple interchange map; thus in general $R_{VW}^{-1} \neq R_{WV}$.

4.1 Construction of link invariants.

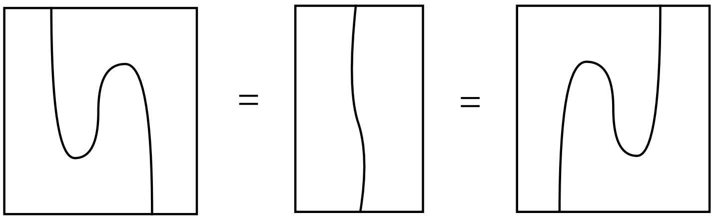
The aim is to start with any (m,n)-tangle T and choose a 'colouring' of its components by finite-dimensional $\mathcal{G}$-modules, in other words, select a $\mathcal{G}$-module for each component. Then try to represent coloured tangles by $\mathcal{G}$-module homomorphisms in such a way that when the strings at the bottom of the tangle T have been coloured by modules $V_1, \ldots, V_n$ and the strings at the top by $W_1, \ldots, W_m$ then the coloured tangle is represented by a module homomorphism $V_1 \otimes \ldots \otimes V_n \to W_1 \otimes \ldots \otimes W_m$, while the composite of two consistently coloured tangles placed one above the other is represented by the composite of the two homomorphisms.

Every tangle can be built up as the composite of a number of elementary tangles which are either a simple crossing  or  or a local maximum  or minimum  , alongside a number of parallel straight strings. Once it is decided how to assign a homomorphism to each of these elementary tangles, with colouring, the homomorphism for the whole tangle will be determined as a composite. To show that the homomorphism defined in this way for a coloured tangle is independent of how the tangle is drawn, up to say moves R_{II} and R_{III}, it is sufficient to show that certain combinations of the elementary tangles determine the same homomorphism.

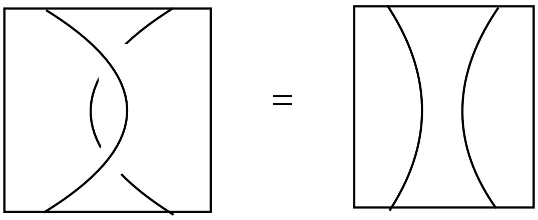
To make the assignments for the elementary coloured tangles we require homomorphisms $V \otimes W \to W \otimes V$ for each of the $(2,2)$-tangles  and  , for which we use R_{VW} and R_{WV}^{-1} respectively. The identity $(1,1)$-tangle,  , is represented by 1_V; when placed alongside other elementary tangles a number of parallel straight strings are represented by taking the tensor product with the appropriate identity homomorphism.

When a tangle has no points at the top or bottom, the appropriate $\mathcal{G}$-module to use as domain or target is the trivial module, in other words the coefficient ring Λ. Thus the local minimum $(2,0)$-tangle, $U =$  , requires a homomorphism $\Lambda \to V \otimes V$, while the local maximum tangle, $V =$  , requires a homomorphism $V \otimes V \to \Lambda$. Turaev observed that only a small number of checks on these are needed to ensure invariance of the homomorphism when the strings of the tangle are moved. These are shown pictorially below, and should be read as the equality of the composites of the homomorphisms determined when the tangle is coloured arbitrarily, and regarded as the product of elementary tangles.

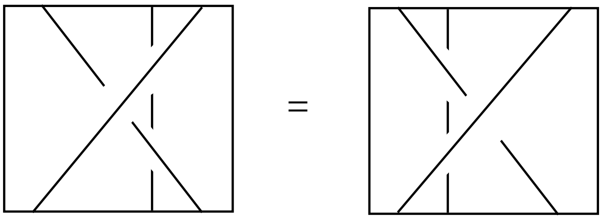
R_0



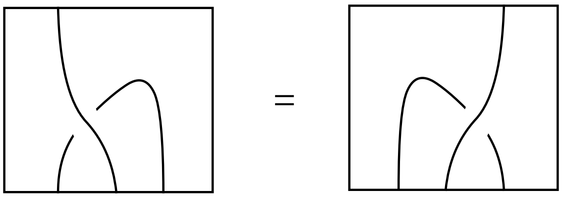
R_{II}



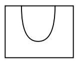
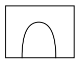
R_{III}



R_{IV}



A little care is needed in defining the homomorphisms to represent the local maximum and minimum coloured by the general module V. Reshetikhin and Turaev [31] give details in a wider context; for irreducible V and the quantum group $SU(2)_q$ there is an almost canonical choice, and having made this choice to satisfy R_0 the other relations are guaranteed by the nature of the universal R-matrix. The consequence of the definition is that a link diagram L, regarded as a $(0,0)$-tangle, determines a homomorphism $\Lambda \to \Lambda$ for each assignment of modules $V_1, \ldots, V_k$ to its components. This homomorphism is simply multiplication by some scalar $J(L; V_1, \ldots, V_k)$ which depends only on L up to moves R_{II} and R_{III} and so gives an invariant of the framed link L.

Whatever definition of the homomorphisms representing  and  is used, a little care can be taken to ensure that

(1) $J(L)$ is multilinear on sums of modules,

(2) when one component of L, say the first, is coloured with the tensor product $V \otimes W$ then

$$J(L; V \otimes W, V_2, \ldots, V_k) = J(L'; V, W, V_2, \ldots, V_k),$$

where the link L' has two parallel components in place of the first component of L, coloured with V and W separately.

A fuller account is given in [23], in which condition (1) is forced by working primarily with irreducible representations, and then (2) has to be proved. In [31] the definitions guarantee property (2) immediately, while (1) then needs a little proof. Rosso [33] has shown that in the general case, where $\mathcal{G}$ is regarded as an algebra over the field of rational functions in an indeterminate $q^{\pm\frac{1}{4}}$, finite dimensional $\mathcal{G}$-modules are completely reducible, (i.e. isomorphic to a direct sum of irreducible modules). In this generic case write $\mathcal{R}$ for the representation ring of $\mathcal{G}$, as an algebra over Λ. An element of $\mathcal{R}$ is then a finite Λ-linear combination of finite dimensional irreducible $\mathcal{G}$-modules, and every $\mathcal{G}$-module can be written in $\mathcal{R}$ as a positive-integer combination of irreducible modules. Tensor product of modules makes $\mathcal{R}$ into a ring.

4.2 The total quantum invariant.

The multilinear property (1) of $J(L)$ means that it can be extended to give a multilinear map $J(L) : \mathcal{R} \times \ldots \times \mathcal{R} \to \Lambda$. By definition, $\mathcal{R}$ has a basis consisting of irreducible representations of $\mathcal{G}$; in this case with $\mathcal{G} = SU(2)_q$ we know that $\mathcal{R}$ is isomorphic to the representation ring of $SU(2)$ having one irreducible module W_i in each dimension $i \geq 1$. Details of these modules and the corresponding R-matrices are given in [13]; an account following the universal R-matrix prescription of Drinfeld is given in [12].

The generic case, where the parameter q is treated as an indeterminate, has the advantage that the representation ring is isomorphic to the representation of the corresponding classical Lie group, and so its structure is understood. Construction of link invariants can also be done when $\mathcal{G}$ is replaced by a finite dimensional algebra, and the coefficient ring is altered by specialising q to a root of unity. In this case the representation theory becomes more complicated, as modules are not always completely reducible, so that a direct interpretation of the link invariant as a function on the representation ring is no longer possible, and more detailed work is needed to handle the invariant comfortably, as in [32] and [12].

Returning to the generic case, it is straightforward to use properties (1) and (2) for $J(L)$, and knowledge of the ring $\mathcal{R}$, to identify $\mathcal{R}$ with the ring $\mathcal{B}$ and $J(L)$ with the total bracket invariant J_L.

Theorem 4.1. *The Λ-linear map $h : \mathcal{R} \to \mathcal{B}$ defined by $h(W_i) = w_i$ is a ring isomorphism, where $\Lambda = \mathbf{Z}[A^{\pm 1}]$, $A^4 = q$. For a framed link L the invariants $J(L)$ and J_L can be identified by*

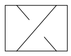
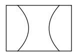
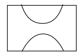
$$J(L; V_1, \ldots, V_k) = J_L(h(V_1), \ldots, h(V_k)).$$

Proof: It is a classical result that the representation ring of $SU(2)$ is a polynomial ring generated by the fundamental 2-dimensional irreducible representation, so that $\mathcal{R}$ is the polynomial ring generated by W_2. Hence there is an isomorphism from $\mathcal{R}$ to $\mathcal{B}$ carrying W_2 to $\alpha = w_2 \in \mathcal{B}$. To establish that this is the map h it is enough to show that the elements W_i satisfy the recurrence relation $W_{n+1} = W_2 W_n - W_{n-1}$ in $\mathcal{R}$. Now it is readily established from the representation theory of $SU(2)$ that the tensor product $W_2 \otimes W_n$ decomposes as the direct sum of irreducibles $W_{n-1} \oplus W_{n+1}$ so that in $\mathcal{R}$ we have $W_2 W_n = W_{n-1} + W_{n+1}$.

Using the fact that $\mathcal{R}$ is spanned by the powers of W_2 we may evaluate the invariant $J(L)$ by evaluating it simply on modules $V_j = W_2^j$, for varying j. When the invariant $J(L)$ is evaluated at W_2^j on one component of L we may use property (2) to replace this by the link L' with j components in place of the one component, each coloured by W_2. In this way comparison of $J(L)$ and J_L reduces to showing that for each link L

$$J(L; W_2, \ldots, W_2) = J_L(w_2, \ldots, w_2).$$

Now $J_L(w_2, \ldots, w_2) = < L >$ so it remains to identify $J(L; W_2, \ldots, W_2)$ with the bracket polynomial of L. It is enough to show that the three linear maps from $W_2 \otimes W_2$ to itself representing the diagrams

$\sigma =$  , Id $=$  and $H =$ 

satisfy the relation $\sigma = A\,\mathrm{Id} + A^{-1}H$, and that the invariant of the simple unknot, as a $(0,0)$-tangle, is $\delta = -A^2 - A^{-2}$.

When all strings are coloured by W_2 the $(2,2)$-tangles σ, Id and H are each represented by an endomorphism of the module $W_2 \otimes W_2$. These endomorphisms are $R_{W_2 W_2}$, $1_{W_2 \otimes W_2}$ and the composite of the local minimum and local maximum maps for W_2 respectively. It is possible, given the detailed information from the quantum group, to calculate these maps explicitly and confirm that they satisfy the linear relation corresponding to the equation $\sigma = A\,\mathrm{Id} + A^{-1}H$. We can also confirm from the explicit maps that the composite of the local maximum and local minimum maps when coloured with W_2 represents the simple unknotted circle by the map from Λ to Λ which is multiplication by $\delta = -A^{-2} - A^2$. Consequently the linear map $\mathcal{D}(\mathbf{R}^2) \to \Lambda$ defined on the diagram L by $J(L; W_2, \ldots, W_2)$ respects the defining relations for $\mathcal{S}(\mathbf{R}^2)$ and hence factors through $\mathcal{S}(\mathbf{R}^2)$. Thus, applied to the diagram L, we have

$$J(L; W_2, \ldots, W_2) = < L > J(\phi;) = < L >,$$

since $L = < L > \phi$ in $\mathcal{S}(\mathbf{R}^2)$. □

Remark. It is in fact more accurate to take the isomorphism determined by $W_2 \mapsto -w_2$, and the identification of A with $-e^{-h/4}$, where $q = e^h$. The quantum group homomorphism $R_{W_2 W_2}$ is then given directly by Drinfeld's universal R-matrix for $SU(2)_q$.

We may thus use either the bracket invariant approach or the quantum group approach to determine the same multilinear invariant $J(L)$ in terms of $\mathcal{B}$, the skein of the annulus, or equally of $\mathcal{R}$, the representation ring of $SU(2)$. In this second guise some of the properties of the invariant which we have already discussed appear quite naturally, in particular that w_i is an eigenvector of the map $\mathcal{F} : \mathcal{B} \to \mathcal{B}$. The framing change in the quantum view requires the insertion of a curl on the component of a link, to which some element of $\mathcal{R}$ has been attached. Suppose that this element is one of the irreducibles, W_i. We may draw the diagram after the framing change so that the extra curl is viewed as a $(1,1)$-tangle coloured with W_i, inserted at some point in the original diagram. This $(1,1)$-tangle is represented by a module homomorphism from W_i to W_i. Since W_i is irreducible, such a map must, by Schur's lemma, be a scalar multiple, λ_i say, of the identity. Hence the curl can be removed at the expense of multiplying $J(L)$ by λ_i without any other change.

Having made the identification of the two descriptions for the generic link invariant it is possible to move on to discuss the 3-manifold invariant, as in the previous section, via the quotient ring $\mathcal{B}_r$ (or $\mathcal{R}_r$) without having to consider the actual representations of $SU(2)_q$ at the root of unity.

4.3 The Temperley-Lieb algebra again.

One further link between the two viewpoints arises when we apply the quantum group viewpoint to tangles in which every component is coloured with the fundamental representation $W_2 = V$, say. Each $(m.n)$-tangle then determines a linear map from $V^{\otimes n}$ to $V^{\otimes m}$, which is a $\mathcal{G}$-module homomorphism, while composition of tangles induces composition of maps. Because the skein relations are satisfied when W_2 is used on all strings there is an induced map from the skein $\mathcal{S}(R_n^n) = TL_n$ to the linear endomorphisms of $V^{\otimes n}$. This gives a representation, which is in fact faithful, of the Temperley-Lieb algebra TL_n as an algebra of $2^n \times 2^n$ matrices, with coefficients in Λ. Since $V^{\otimes n}$ is a $\mathcal{G}$-module, and the tangles are all represented by module endomorphisms, we can see further that TL_n is represented as a subalgebra of all $\mathcal{G}$-module endomorphisms of $V^{\otimes n}$. Indeed, if the coefficient ring Λ is extended to include sufficient denominators then the image of TL_n can be shown to be the algebra of all $\mathcal{G}$-module endomorphisms of $V^{\otimes n}$.

There is just one submodule of $V^{\otimes n}$ which is isomorphic to the irreducible W_{n+1}. Projection to this submodule determines a $\mathcal{G}$-module endomorphism of $V^{\otimes n}$, and hence an element of TL_n. This element of TL_n is in fact the element f_n discussed earlier, divided by $\varphi_A(f_n)$. The multiplicative property of f_n is seen in this context from the fact that $V^{\otimes k}$ with $k < n$ has no summands isomorphic to W_{n+1}, so that the composition of the projection with the map representing any (n,k)-tangle, $k < n$, must be zero. Now each generator h_i of TL_n is the composite of an $(n, n-2)$-tangle with an $(n-2, n)$-tangle, so that the projection when composed with any of these must be zero. This leads to the equation $f_n h_i = 0$, and thus to the multiplicative property, given that φ_A can also be recognised by the property that $\varphi_A(h_i) = 0$.

The representation of TL_n on $V^{\otimes n}$ can be quickly recovered from the two maps representing the local maximum and minimum. These can be chosen to have matrices

$$\begin{pmatrix} 0 & A & -A^{-1} & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & -A & A^{-1} & 0 \end{pmatrix}^T,$$

representing the linear maps $\mathrm{Max} : V \otimes V \to \Lambda$ and $\mathrm{Min} : \Lambda \to V \otimes V$ respectively, where V has a basis v_1, v_2 and the basis elements of $V \otimes V$ are written in the order $v_1 \otimes v_1,\ v_1 \otimes v_2,\ v_2 \otimes v_1,\ v_2 \otimes v_2$. These maps satisfy the condition R_0 and can be combined as $\mathrm{Max}.\mathrm{Min}$ to represent H. The matrix representing σ is then given by $\sigma = A + A^{-1}H$, while the value of δ can be checked by calculating the product $\mathrm{Min}.\mathrm{Max}$.

This representation of TL_n can be used as a means of calculating explicitly the bracket polynomial of the closure of any (n,n)-tangle. It also provides a representation of the braid group B_n on $V^{\otimes n}$ in which the generators σ_i satisfy a quadratic relation, and so have only two eigenvalues. This representation preserves each $\mathcal{G}$-submodule of $V^{\otimes n}$ which consists of the sum of all submodules isomorphic to a given irreducible W_i, and hence it breaks up into a number of lower dimensional representations of B_n and indeed of TL_n. Details of this are discussed in Reshetikhin's papers [29]. Other representations of the braid group arise in a similar way, with higher degree minimal polynomial for σ_i, using (n,n)-tangles coloured by one of the other irreducible modules W_j in place of W_2.

5. A geometric view of the invariants.

In defining the 'generic' type of link invariant J_L, taking values in a ring Λ containing an indeterminate $A = q^{\frac{1}{4}}$, I have described two different approaches which arrive at essentially the same end result. The interpretations of the parameter space $\mathcal{B} = \mathcal{R}$ in terms of 'decorations' or 'colourings' which can be applied to the link components highlight different properties, depending on whether the view as quantum group representations or as bracket invariants of satellites is uppermost in the mind.

Either of these views constitutes a first stage for the invariants. The second stage arises when they are used to build invariants of general 3-manifolds, typically in terms of evaluations of the generic invariants, where the indeterminate is replaced by a specified root of unity. The account given so far has made use of some features which are special to $SU(2)_q$, or equally to the bracket invariant, but there is much which will work readily in a wider context. In the final section I shall give a brief account of the generic stage in constructing invariants, using the quantum groups $SU(k)_q$ on one hand, and linear skein theory based on the Homfly polynomial on the other. Similar work relates Kauffman's Dubrovnik polynomial with the quantum groups of the B, C and D series, coming from the orthogonal and symplectic groups, [42]. The corresponding second stage can be pursued, with a little care, following the general lines of section 3.

In the remainder of this section I look first at the generic invariant from the 3-dimensional point of view, and then note how this and the second stage invariants fit in to the framework of Witten.

5.1 The generic invariant and modular functors.

Both approaches, from linear skein theory and from the representation theory of $SU(2)_q$, lead to a framed link invariant $J_L : \mathcal{B}^k \to \Lambda$, and a satellite formula relating J_K for a satellite K to J_C for its companion C and $J_P : \mathcal{B}^k \to \mathcal{B}$ for the pattern P, viewed as a diagram in the annulus.

There are two alternative views of the pattern

(1) as a k-component diagram in the annulus, and

(2) as a $k+1$-component link P' consisting of P together with one distinguished unknotted component which determines the annulus.

View (1) determines a multilinear map $J_P : \mathcal{B}^k \to \mathcal{B}$, while view (2) gives a map $J_{P'} : \mathcal{B}^{k+1} \to \Lambda$. These can be related by regarding P' as a satellite of the Hopf link H using the pattern P, so that $J_{P'} = J_H \circ (J_P \times 1)$ as maps from $\mathcal{B}^k \times \mathcal{B}$ to Λ. The Hopf link invariant $J_H : \mathcal{B}^2 \to \Lambda$ thus provides a bilinear form which plays a central role in comparing the two views.

The remaining feature of the generic invariant is the linear automorphism $\mathcal{F} : \mathcal{B} \to \mathcal{B}$ describing the framing change, and the basis of $\mathcal{B}$ consisting of its eigenvectors.

When we move to a more 3-dimensional view one characteristic feature is the behaviour of the invariants when pieces of 3-manifold with boundary are glued together. In Witten's framework, once the choice of a quantum group $\mathcal{G}$ and a level k have been made there should then be determined a 'modular functor' from the category of cobordisms of surfaces to the category of complex vector spaces and linear maps.

Definition. We say that the boundary of a 3-manifold has been *marked* if for each boundary component of genus g there is an explicit choice of homeomorphism from a standard copy of the surface of genus g to that boundary component. We refer to the homeomorphism as a *marking.*

A full description of the required ingredients is given for example in [38]. The central idea is that every oriented 3-manifold M with boundary ∂M can be regarded as a cobordism when its boundary is marked and is partitioned into two parts, each consisting of a union of closed surfaces. In categorical terms, the objects of the category are unions of oriented surfaces, and the morphisms are oriented 3-manifolds with marked partitioned boundary, so that M with boundary $\partial M^- \cup \partial M^+$ is regarded as a morphism from the incoming boundary ∂M^- to the outgoing boundary ∂M^+. Morphisms (cobordisms) are composed by gluing the outgoing boundary of one manifold to the incoming boundary of the other, using the marking of each component to determine the gluing.

A *modular functor* is a functor from this category to the category of vector spaces and linear maps. It associates a vector space to each surface of genus g, and the tensor product of such spaces to a disjoint union of surfaces. The marked cobordism M provides a linear map from the space for ∂M^- to the space for ∂M^+. This map is assumed to be unchanged when the marking of a component is altered by isotopy. The functorial property ensures that composition of cobordisms translates into composition of linear maps. The marking of a boundary component may be altered by composing the original cobordism with another of the form surface $\times I$, in which different choices of marking are made at the two ends. Such cobordisms determine ine companion to the exterior of P translates exactly to the appropriate composition of linear maps, provided that the marking of the boundary of the solid torus is suitably chosen. Further comments on this point of view are made in [23].

We could try to base a limited cobordism functor on these definitions, with the restriction that the only boundary components allowed should be unions of tori. We do not however have enough freedom to do this; the most serious problem is that we are in general unable to change the linear map appropriately when we change the assignment of boundary components on a link exterior from incoming to outgoing. The case of the pattern link P' is a special case in which the component to be switched is unknotted in S^3; in this case the marking to be used on the outgoing component differs from that of the incoming components by switching the factors in $S^1 \times S^1$. The two maps $J_{P'} : \mathcal{B}^{\otimes k+1} \to \Lambda$ and $J_P : \mathcal{B}^{\otimes k} \to \mathcal{B}$ exhibit the sort of change that we would like to use generally when switching components from incoming to outgoing. They are related by the invariant $J_H : \mathcal{B} \otimes \mathcal{B} \to \Lambda$ of the Hopf link. This represents the exterior of H, in which both components are incoming. The same 3-manifold is homeomorphic to the product $(S^1 \times S^1) \times I$, represented by the identity $\mathcal{B} \to \mathcal{B}$ when viewed as a cobordism with one incoming and one outgoing component. The expected procedure for altering the map when a component is switched from incoming to outgoing would be to change a copy of $\mathcal{B}$ in the domain of the map to a copy of its dual $\mathcal{B}^*$ in the target, and then use the bilinear form J_H to identify $\mathcal{B}^*$ with $\mathcal{B}$. This would at least agree with the case of a pattern link P' and its unknotted component.

The problem with doing this in general is that $\mathcal{B}$ is infinite dimensional, so that J_H does not provide a good identification. The other missing ingredient is the ability to alter the marking of a boundary component, so as to allow freedom to glue boundaries together in different ways. The change of framing, which corresponds to certain changes of marking, can indeed be represented by use of the automorphism $\mathcal{F}$ on the vector space $\mathcal{B}$, but there is no immediate analogue available to account for the other homeomorphisms in the mapping class group of the torus.

5.2 The finite-dimensional invariants.

Both of these problems disappear when we fix the level l, and thus $r = l - 2$, and pass to the corresponding quotient ring $\mathcal{B}_r$ in place of $\mathcal{B}$ as the linear space to use for each boundary torus. The exterior of a link L can now be represented by the map $J_L^{(r)}$, regarded either as a multilinear map from $(\mathcal{B}_r)^k$ to $\Lambda_r \subset \mathbf{C}$ or equivalently as a linear map on the tensor product $(\mathcal{B}_r)^{\otimes k}$. This map is determined by the full polynomial invariant J_L after replacement of the variable A by a $4r$-th root of unity.

The complex vector space $\mathcal{B}_r$ is finite dimensional, and can be readily identified with its dual, using the non-degenerate bilinear form $J_H^{(r)}$. This permits link exteriors to be used in defining cobordism invariants, where any selection of boundary components may be taken as the incoming part of the boundary. With these as basic ingredients, a coherent assignment of linear maps can be made to cover the case of compact 3-manifolds with torus boundary components, up to a power of the number c (depending on r) mentioned in section 3. For example, the trivial knot, whose exterior is a solid torus, determines the invariant $< \cdot > : \mathcal{B}_r \to \Lambda$ when regarded as a cobordism from the torus to the empty set. As a cobordism from the empty set to the torus, it gives the element $w_1 \in \mathcal{B}_r$, regarded as a map from Λ to $\mathcal{B}_r$. In this setting, the torus is marked in such a way that composing this cobordism with a link exterior has the effect of gluing the solid torus to the boundary of the neighbourhood of one component of the link L so as to replace the neighbourhood exactly. The new cobordism is just the exterior of the link given by deleting the chosen component of L, and its invariant is given by decorating that component of L by w_1, i.e. by the empty decoration, as expected.

To perform surgery on the link exterior we must reglue the solid torus in a different way, or equivalently we must choose a different marking of its boundary torus, switching the two factors $S^1 \times S^1$. When working with $\mathcal{B}_r$ it is possible to represent the full mapping class group of the torus on $\mathcal{B}_r$, (up to a power of c), and in particular to represent the switching homeomorphism. The image of w_1 under the switch is $\rho^{-1}\Omega$, and so the solid torus glued in to one boundary component of a link exterior by surgery is a cobordism which is represented by the map $\Lambda_r \to \mathcal{B}_r$ which takes 1 to $\rho^{-1}\Omega$. The cobordism invariant of the new manifold is then given from that of the manifold before gluing by evaluation at $\rho^{-1}\Omega$ on the appropriate component. So we anticipate in this view that we might get an invariant of the manifold given by surgery on a framed link L by regarding the manifold as a composite of cobordisms, starting with k solid tori, and attaching them to the exterior of L. The resulting invariant would then be $J_L^{(r)}(\rho^{-1}\Omega, \ldots, \rho^{-1}\Omega)$ up to a power of c, which is indeed the form of the invariant discussed in section 3.

The invariant of a manifold constructed by general Dehn surgery from a framed link L, where solid tori are glued in to the link exterior using other markings of the boundaries, can similarly be found by evaluation of $J_L^{(r)}$ on suitably chosen elements of $\mathcal{B}_r$, depending on the nature of the marking for each individual boundary component. The determination of these elements is a matter of finding the image of w_1 under the automorphism of $\mathcal{B}_r$ corresponding to the self-homeomorphism of the torus which alters the chosen marking to the marking determined by the framing of L. They can be found once the action of the mapping class group of the torus on $\mathcal{B}_r$ has been established. The powers of c mentioned as an indeterminacy can be handled as in [24], or they can be incorporated into the cobordism invariant by regarding the marked 3-manifolds as also carrying a framing, adjustments to which account for multiplication by powers of c.

It is possible to extend the invariant from a similar point of view to handle general cobordisms in which the boundary components need not be tori. An account of the linear space related to the surface of genus g can be given in terms of the skein of a planar surface with g holes, just as $\mathcal{B}_r$ is described in terms of the skein of the annulus. See for example the recent account by Lickorish [18], following work of Vogel, or an earlier account by Kohno from the quantum group viewpoint [14].

6. Unitary invariants and the Hecke algebras.

In this final section I shall give a brief indication of the similarities and modifications to the previous work which are needed in considering the invariants related to the Homfly polynomial [6] by satellite constructions, or equivalently to the unitary quantum groups $SU(k)_q$, for different values of k. There is a similar relation between the orthogonal/symplectic quantum groups and Kauffman's 2-variable invariant. Wenzl [41] gives an account of this in which the quantum group approach, and the appropriate algebra, is much to the fore. He continues, with Turaev [37], to develop this to the second stage when a root of unity is involved, so as to discuss 3-manifold invariants based on modifications to the quantum group. This in turn entails a separate study of the representation theory for the modified quantum group, rather than using the classical representation theory based on the generic case. Although I will not attempt to move to this stage for the general quantum group, it is possible to reach the 3-manifold invariants in a similar way to the discussions above by dealing with invariants defined on what is in effect a natural quotient ring of the representation ring of the quantum group being used, or equivalently of the corresponding classical group.

6.1 The Homfly polynomial.

The Homfly polynomial $P_L(v,z) \in \mathbf{Z}[v^{\pm 1}, z^{\pm 1}]$ was developed independently by several groups shortly after the discovery of the Jones polynomial [6, 28]. It is an invariant of an *oriented* link, characterised by the Homfly skein relation

$$v^{-1}\,P(\quad) - v\,P(\quad) = z\,P(\quad)$$

between oriented link diagrams differing only where shown. It is invariant under all three Reidemeister moves, and so $P_{L \amalg O} = \delta P_L$, where $\delta = (v^{-1} - v)/z$, and $L \amalg O$ consists of the diagram L together with a disjoint simple closed curve.

It provides a simultaneous generalisation of the Alexander polynomial and Jones' polynomial by

$$P_L(v,z) = \begin{cases} \Delta_K(t), & \text{the Alexander polynomial, when } v = 1, z = s - s^{-1}, t = s^2 \\ \nabla_K(z), & \text{Conway's version of the Alexander polynomial, when } v = 1, \\ V_K(t), & \text{the Jones polynomial, when } v = s^2 = t, z = s - s^{-1}. \end{cases}$$

In this original form P is normalised so that the unknot O has invariant 1; it is more convenient in work which relates to quantum groups to normalise so that the empty knot ϕ has invariant 1 and the unknot has invariant δ. I shall adopt this convention in the present work.

We may construct close relatives of the Homfly polynomial which are invariants of an oriented diagram D only up to R_{II} and R_{III} for any scalar λ by setting

$$X_D = \lambda^{w(D)} P_D(v,z),$$

where $w(D)$ is the writhe of the diagram D. Then X can be recognised by the properties

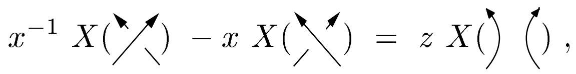
$$X(\quad) = \lambda X(\quad)$$

and the skein relation

$$\lambda^{-1} v^{-1}\,X(\quad) - \lambda v\,X(\quad) = z\,X(\quad),$$

up to normalisation. In this way we can identify any invariant of oriented diagrams which satisfies a skein relation between , and with such a variant of the Homfly polynomial, provided that it multiplies by a fixed scalar λ under R_I. The bracket polynomial, for example, arises with $\lambda = -A^3, z = A^{-2} - A^2$ and $v = A^{-4}$.

In general, when we write the relation as

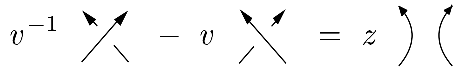


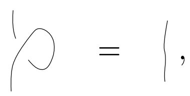
we have $X=\lambda^{w(D)}P_D(x\lambda^{-1},z)=(xv^{-1})^{w(D)}P_D(v,z)$.

6.2 Skein theory.

We can use the Homfly skein relation to define skeins based on the Homfly polynomial, following the methods used in the first section for the bracket invariant. We shall consider diagrams, up to moves R_{II} and R_{III} in a planar surface F whose boundary contains a finite set of distinguished points. We insist that each boundary point is given an orientation either as an input or an output, and we consider *oriented* diagrams in F whose string orientation matches the orientation of the boundary points.

Definition. For a planar surface F the *Homfly skein* $\mathcal{S}_P(F)$ is the set of linear combinations of *oriented* diagrams in F subject to the relations

(1) 

(2) 

for diagrams which differ as shown.

The existence and uniqueness theorem for the Homfly polynomial shows that $\mathcal{S}_P(\mathbf{R}^2)$ is isomorphic to the scalars, and the diagram L represents the multiple $P_L(v,z)\times\phi$ of the empty diagram ϕ, given our convention that $P_\phi=1$.

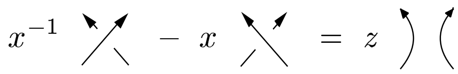
As an example, if we take F to be the rectangle R_n^n with n *inputs* at the bottom and n *outputs* at the top then the skein $\mathcal{S}_P(R_n^n)$, constructed from oriented (n,n)-tangles, forms an algebra with composition induced by putting rectangles one below the other, as for the Temperley-Lieb algebra. This algebra is spanned by $n!$ elements, represented by the positive permutation braids $w_\pi, \pi\in S_n$ discussed above. It is generated as an algebra by the elementary braids σ_i, oriented with all strings upwards, and it is known to be isomorphic to the n-th Hecke algebra H_n, as shown in [26].

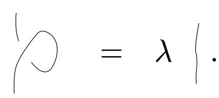
A presentation for this algebra is given by generators σ_i satisfying the braid relations

$$\sigma_i\sigma_j=\sigma_j\sigma_i,\ |i-j|>1,$$
$$\sigma_i\sigma_{i+1}\sigma_i=\sigma_{i+1}\sigma_i\sigma_{i+1}$$

and the skein relation $v^{-1}\sigma_i-v\sigma_i^{-1}=z$.

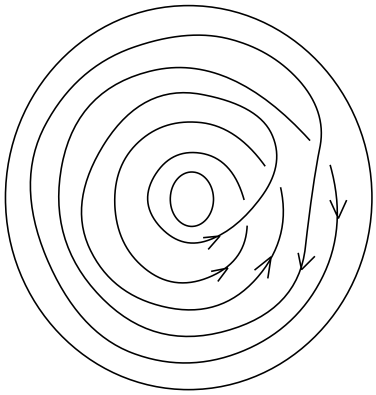
Variants on the skein definitions can be adopted, by use of a scalar λ as in the invariant X above, with $x=\lambda v$, from which we can define a variant skein $\mathcal{S}'_P(F)$ by the relations

(1′) 

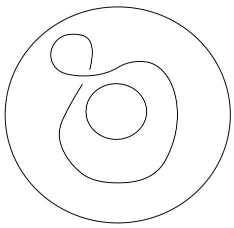
(2) 

There is a linear isomorphism $\mathcal{S}_P(F) \to \mathcal{S}'_P(F)$ defined on each diagram D in F by $D \mapsto \lambda^{-w(D)} D$, where $w(D)$ is the writhe of the diagram. In the case when $F = \mathbf{R}^2$ the link diagram L, which represents $P_L(v,z) \times \phi$ in $\mathcal{S}_P(F)$, will represent $\lambda^{w(L)} P_L(v,z) \times \phi$ in $\mathcal{S}'_P(F)$.

One frequent choice for this variant is $x\beta ned$



Oriented wiring diagrams can be used as before to induce linear maps between skeins. We may also decorate oriented link diagrams by elements of $\mathcal{C}$ and thus determine a multilinear map $P_D : \mathcal{C}^k \to \mathcal{S}_P(F)$ for any diagram D in F with k closed components. This map will be independent of D up to moves R_{II} and R_{III}, while changes of framing on a component of D, in other words alteration by moves R_I, can be accounted for by use of a framing change map $\mathcal{F} : \mathcal{C} \to \mathcal{C}$ defined as before by decorating the simple curl



regarded as a diagram in the annulus.

In this way we can extend the Homfly polynomial to define invariants P_L of a framed oriented link L by decorating its components with elements of $\mathcal{C}$, so as to give a multilinear map $P_L : \mathcal{C} \times \ldots \mathcal{C} \to \mathcal{S}_P(\mathbf{R}^2)$, the 'total Homfly invariant' of L. The Homfly polynomial itself is recovered by evaluating the map P_L at $(\alpha_1, \ldots, \alpha_1)$, when $\mathcal{S}_P(\mathbf{R}^2)$ is identified with the ring of scalars. Other decorations give rise to further invariants of L, which I shall term 'satellite Homfly invariants' of L, as they are constructed from the Homfly polynomials of satellites of L.

6.4 Representing the Hecke algebra.

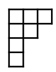
The closure wiring of a rectangle into the annulus induces a linear map $H_n \to \mathcal{C}$ for each n, with image $\mathcal{C}_n$ say. Every diagram in the annulus can be viewed as the closure of some tangle, but we cannot assume that the string orientations at the top of the tangle are all inputs, so the skein $\mathcal{C}$ is not necessarily the union of the subspaces $\mathcal{C}_n$. We can certainly recover the whole of $\mathcal{C}$ by considering tangles in which the boundary points at the bottom are divided into n inputs and p outputs, with the matching points at the top forming n outputs and p inputs, for varying n and p.

The algebra $\mathcal{C}$ is the product $\mathcal{C}_+ \times \mathcal{C}_-$ of the subalgebras generated respectively by $\{\alpha_i\}$ alone and by $\{\alpha_i^*\}$ alone. The image $\mathcal{C}_n$ of H_n lies in $\mathcal{C}_+$ for each n; it has a basis consisting of monomials in $\{\alpha_i\}$ of total weight n, where α_i has weight i. Its dimension is thus $\lambda(n)$, the number of partitions of n.

An alternative basis for $\mathcal{C}_n$ is suggested by the representation theory of H_n, which is a deformation of the group algebra $\mathbf{C}[S_n]$ of the symmetric group. For generic values of the parameter $z = s - s^{-1}$ (in fact for $s^{2r} \neq 1,\ r \leq n$) the algebra H_n is known to decompose as the direct sum of $\lambda(n)$ subalgebras, $\bigoplus M_\lambda$, each isomorphic to the algebra of $d_\lambda \times d_\lambda$ matrices for some d_λ. This decomposition is similar to the classical case of $\mathbf{C}[S_n]$; the subalgebras M_λ are traditionally indexed by the Young diagrams λ with n cells. Any such Young diagram is determined by a sequence of non-negative integers

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0$$

with $\lambda_1 + \ldots + \lambda_n = n$, and is commonly drawn diagramatically as an array of n cells with λ_i cells in row i. For example, the diagram



corresponds to the partition $3 \geq 2 \geq 1 \geq 1 \geq 0 \geq 0 \geq 0$ with $n = 7$.

Given the structure of H_n as a direct sum there will be a central idempotent $e_\lambda \in H_n$ for each λ, corresponding to the identity element of the subalgebra M_λ. These are orthogonal, in the sense that $e_\lambda e_\mu = 0$ if $\lambda \neq \mu$, while $e_\lambda^2 = e_\lambda$. The algebra H_n decomposes in this way, provided that the coefficient ring allows denominators $s^r - s^{-r}$ for $r \leq n$. The idempotents can be found explicitly, for example in [41]. The simplest of these are multiples of the elements a_n and b_n given above. They correspond to the two Young diagrams, each with $d_\lambda = 1$, which have n cells and just one row or just one column.

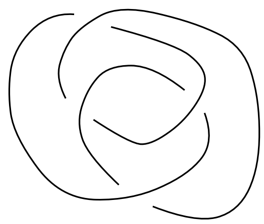
The closure $\widehat{e}_\lambda$ of the idempotents provide between them an alternative basis for $\mathcal{C}_n$ consisting of $\lambda(n)$ elements. They have the merit of all being eigenvectors of the framing change map $\mathcal{F} : \mathcal{C} \to \mathcal{C}$. This follows since any central element of H_n can be written as a linear combination of the idempotents $\sum c_\mu e_\mu$. The n-string curl Q_n, with appropriate orientation, which commutes up to R_{II} and R_{III} with all (n,n)-tangles, can then be written as $Q_n = \sum c_\mu e_\mu$. Orthogonality of the idempotents shows that $Q_n e_\lambda = c_\lambda e_\lambda$ and hence $\mathcal{F}(\widehat{e}_\lambda) = c_\lambda \widehat{e}_\lambda$. The elements $\widehat{e}_\lambda$ thus behave rather like the elements $w_i \in \mathcal{B}$.

Example. When $n = 2$ there are just two Young diagrams $\square\!\square$ and $\begin{smallmatrix}\square\\\square\end{smallmatrix}$, with corresponding idempotents

$$e_{\square\!\square} = (sv^{-1}\sigma_1 + 1)/(1 + s^2), \quad e_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} = (-s^{-1}v^{-1}\sigma_1 + 1)/(1 + s^{-2}) .$$

When $s = v = 1$ these are the symmetriser and skew-symmetriser respectively for the symmetric group S_2. It is easy to express each $\widehat{e}_\lambda$ in terms of the basis of monomials for $\mathcal{C}$, noting that for $(2,2)$-tangles the closure of the identity braid 1 is α_1^2 and the closure of σ_1 is α_2.

Thus when K is the figure-eight knot with framing as shown



we have its satellite Homfly invariant

$$P_K(\widehat{e}_{\square\!\square}) = 1/(1+s^2)\left(v^{-1}s\, P_K(\alpha_2) + P_K(\alpha_1^2)\right)$$

given by calculating the Homfly polynomials of two 2-string satellites of K. The invariant can be written as

$$\frac{\delta}{s^2-s^{-2}}(v^{-5}s^5 - v^{-3}(s^7+s^5+s^{-1}) + v^{-1}(s^7+s^3+2s+s^{-5})$$
$$- v(s^{-7}+s^{-3}+2s^{-1}+s^5) + v^3(s^{-7}+s^{-5}+s) - v^5 s^{-5}),$$

where $\delta = \dfrac{v^{-1}-v}{s-s^{-1}}$ is the Homfly invariant of the unknot. For comparison the standard Homfly invariant of the figure-eight knot is $P_K(\alpha_1) = \delta(v^{-2}-s^{-2}+1-s^2+v^2)$.

Similarly when $n=3$ we can write down the two idempotents $e_{\square\!\square\!\square}$ and $e_{\substack{\square\\\square\\\square}}$ as above, using a_n and b_n. The remaining idempotent $e_{\substack{\square\!\square\\\square}}$ can be found from the equation $1=\sum e_\lambda$ in H_n. The closures of all three can be calculated in terms of the monomial basis, giving for instance

$$\widehat{e}_{\substack{\square\!\square\\\square}} = \frac{2}{s^{-2}+1+s^2}\left(\alpha_1^3 + v^{-1}(s-s^{-1})\alpha_1\alpha_2 - v^{-2}\alpha_3\right).$$

This can also be written as $\dfrac{2}{s^{-2}+1+s^2}\widehat{d}$ where $d = 1-\sigma_1^{-1}\sigma_2$.

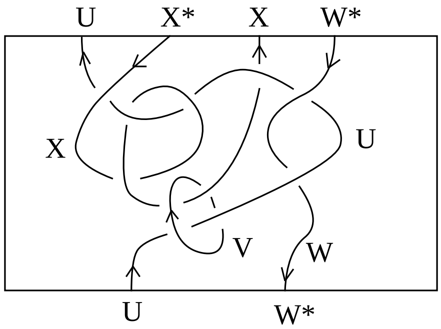
Transition between the monomial basis and the basis $\{\widehat{e}_\lambda\}$ is not so convenient as in the case of $\mathcal{B}$, where the two bases of interest, $\{\alpha^j\}$ and $\{w_i\}$, are integrally related. In $\mathcal{C}_n$ we need a limited set of denominators of the form $s^r - s^{-r}$ and $vs^r - v^{-1}s^{-r}$ with $|r|\le n$ to perform a complete transition. In principle, though, the information available from a link by taking its Homfly polynomial after decoration by elements of $\mathcal{C}_+$ is equivalent to knowing, on the one hand, its satellite Homfly polynomials when decorated by all possible monomials in the α_i and, on the other hand, the invariants when decorated by all possible $\widehat{e}_\lambda$ for Young diagrams λ. The connection with the quantum group $SU(k)_q$ invariants of the framed link L comes about through an identification of the quantum group invariants with the invariants above which use $\widehat{e}_\lambda$, as λ varies through Young diagrams restricted according to the value of k.

6.5 Unitary quantum groups.

The methods of Reshetikhin and Turaev [31] allow the quantum groups $\mathcal{G} = SU(k)_q$ to be used to represent oriented tangles whose components are coloured by $\mathcal{G}$-modules as $\mathcal{G}$-module homomorphisms. The scheme and necessary ingredients are similar to those outlined in section 4, with one additional feature, namely the use of the dual module V^* defined by means of the antipode in $\mathcal{G}$, (an antiautomorphism of $\mathcal{G}$ which is part of its structure as a Hopf algebra). When the components of the tangle are coloured by modules the tangle itself is represented by a homomorphism from the tensor product of the modules which colour the strings at the bottom to the tensor product of the modules which colour the strings at the top, provided that the string orientations are inwards at the bottom and outwards at the top. The dual module

V^* comes into play in place of V when an arc of the tangle coloured by V has an output at the bottom or an input at the top.

For example, the $(4,2)$-tangle below, when coloured as shown, is represented by a homomorphism $U \otimes W^* \to U \otimes X^* \otimes X \otimes W^*$.



As in the earlier case it is possible [31] to build up the definition so that consistently coloured tangles are represented by the appropriate composite homomorphisms, starting from a definition of the homomorphisms for the elementary oriented tangles. Two cases, depending on the orientation, must be considered for both the local maximum and the local minimum, and a little care is needed here to ensure consistency. The final result is a definition of a homomorphism which is invariant when the coloured tangle is altered by R_{II} and R_{III}. When applied to an oriented k-component link diagram L regarded as an oriented $(0,0)$-tangle it gives an element $J(L; V_1, \ldots, V_k) \in \Lambda = \mathbf{Q}[[h]]$ for each colouring of the components of L by $\mathcal{G}$-modules, which is an invariant of the framed oriented link L. This element, apart from a simple factor, is an integer polynomial in $q = e^h$. A categorical account of the appropriate features needed to define an invariant in this way is given in [44].

As in section 4, this invariant $J(L)$ (for a fixed quantum group $\mathcal{G}$) is

(1) multilinear under direct sums of modules, and

(2) multiplicative on parallels.

We can use (1) to extend the definition of $J(L)$ to allow colouring by linear combinations of modules, and thus determine a multilinear map $J(L) : \mathcal{R} \times \ldots \times \mathcal{R} \to \Lambda$, where $\mathcal{R}$ is the representation ring of $\mathcal{G}$.

Definition. Refer to the map $J(L)$ as the *coloured invariants* of L, where the choice of quantum group $\mathcal{G}$ is clear. A *colouring* of L will mean a choice of an element of $\mathcal{R}$, (in other words, a linear combination of modules,) for each component of the link, and will determine an element of Λ by evaluation of $J(L)$.

Notation. Write $\mathcal{R}^{(k)}$ for the representation ring in the case when $\mathcal{G} = SU(k)_q$.

For generic q this ring is shown in [33] to be isomorphic to the classical representation ring of $SU(k)$. The irreducible modules of $SU(k)$ and hence of $SU(k)_q$ are also indexed by Young diagrams. There is an irreducible $SU(k)_q$-module V_λ for every Young diagram λ provided that λ is either the diagram with k rows and 1 column or otherwise has at most $k-1$ rows. Such Young diagrams are referred to later as 'admissible' for k. Among these modules there is a 'fundamental' irreducible module of dimension k, which is indexed by the Young diagram $\Box$. Write $V_\Box$ for this module. Each module V_λ whose Young diagram has n cells occurs as a summand of $V_\Box^{\otimes n}$.

An early relation between the Homfly polynomial and the quantum invariants of a link was discovered by Jones and Turaev [34], when considering the invariant given by colouring all components with the fundamental module $V_\Box$.

THEOREM 6.1 (Turaev, Jones). *For the quantum group $SU(k)_q$ the invariant $J(L;V_\Box,\ldots,V_\Box)$ of the framed oriented link L is, up to normalisation, the Homfly polynomial $P_L(v,z)$ with $z=s-s^{-1}$, $v=s^{-k}$ and $s=\sqrt{q}=e^{h/2}$. Assuming that $P_\phi=1$ we have*

$$J(L;V_\Box,\ldots,V_\Box)=(xv^{-1})^{w(L)}P_L(v,z),$$

where $w(L)$ is the writhe of a correctly framed diagram of L, evaluated at $z=s-s^{-1}$, $v=s^{-k}$ and $x=s^{-1/k}=e^{-h/2k}$.

Proof: It is enough to show that $J(L;V_\Box,\ldots,V_\Box)$ satisfies a quadratic skein relation, and multiplies by a scalar under R_I, to identify it with some specialisation of P_L as at the beginning of this section. Turaev represents the $(2,2)$-tangle σ when coloured with the fundamental representation $V_\Box$ by a map $R: V_\Box\otimes V_\Box\to V_\Box\otimes V_\Box$ which satisfies the quadratic relation

$$R-R^{-1}=(s-s^{-1})\,\mathrm{Id}\ .$$

It is possible to deduce the existence of *some* quadratic relation for R from the fact that $V_\Box\otimes V_\Box$ is the sum of just two irreducible modules.

The $(1,1)$-tangle  when coloured with any irreducible must be represented by a multiple of the identity, by Schur's lemma. Turaev shows that this multiple is s^{-k} when the k-dimensional fundamental module $V_\Box$ of $SU(k)_q$ is used. This would lead to the result of theorem 6.1, without the factor x. It appears, however, that a scalar multiple of Turaev's endomorphism is more appropriate, to permit a more consistent behaviour of the family of invariants $J(L)$ when evaluated on different modules. In the general construction of $J(L)$ this behaviour is ensured by the use of the universal R-matrix to determine the representation of the elementary tangle σ under each colouring. Since the universal R-matrix satisfies a non-homogeneous equation it is not possible to replace it by a scalar multiple of itself without losing the multiplicative behaviour of $J(L)$ on parallels. The endomorphism R used by Turaev is a non-trivial multiple of the one which arises from Drinfeld's universal R-matrix. The appropriate endomorphism R as given in [3] satisfies instead the relation

$$(*)\qquad x^{-1}R-xR^{-1}=(s-s^{-1})\,\mathrm{Id}\ ,$$

with $x=s^{-1/k}$.

Assuming that we use this endomorphism R to represent σ, equation $(*)$ enables us to define a function $\mathcal{S}'(\mathbf{R}^2)\to\Lambda$ from the variant skein $\mathcal{S}'(\mathbf{R}^2)$ with $z=s-s^{-1}$, $v=s^{-k}$ and $x=s^{-1/k}$ by taking the diagram L to $J(L;V_\Box,\ldots,V_\Box)$. Since $L=(xv^{-1})^{w(D)}P_L(v,z)\times\phi$ in $\mathcal{S}'(\mathbf{R}^2)$ and the value of J on the empty diagram ϕ is 1 we have the equation

$$J(L;V_\Box,\ldots,V_\Box)=s^{(k-1/k)w(L)}P_L(s^{-k},s-s^{-1})\ ,$$

where the Homfly polynomial P_L is normalised to have value 1 on the empty diagram, and $w(L)$ is the writhe of any diagram of L which realises the chosen framing. □

Given a Young diagram λ there is a corresponding $SU(k)_q$-module V_λ for each k, which should properly be distinguished from each other as k varies. It is, however, possible to organise things systematically so as to handle all the unitary quantum group invariants with colouring V_λ at once, by finding a 2-variable function of v and s depending on C and λ, from which the substitution $v = s^{-k}$ allows us to recover the invariant $J(C; V_\lambda)$ for the quantum group $SU(k)_q$, as shown in [41]. In the case when $\lambda = \Box$ the theorem above shows that the Homfly polynomial of C itself provides a suitable function. For general λ we use a satellite Homfly polynomial of C; in fact we can use the closure $\widehat{e}_\lambda$ derived from the Hecke algebra idempotent for the same Young diagram λ as the element of $\mathcal{C}$ to provide the satellite decoration.

We shall see that it is possible to realise all possible colourings of C as linear combinations of invariants which arise by varying the decoration P while restricting the colouring of P to the fundamental module $V_\Box$. Thus all coloured invariants of C for the unitary quantum groups will arise, by the theorem of Jones and Turaev, as linear combinations of the Homfly polynomials of satellites of C, in which the variable v has been specialised to $v = s^{-k}$ for $SU(k)_q$.

There is a satellite theorem for the quantum invariants $J(L)$ of a satellite link L. This allows us to express the invariants of the link given when a companion knot C is decorated by some pattern P in the annulus in terms of the invariants of C and of the pattern P. From the point of view of constructing invariants of C we may choose the decorating pattern, and then choose a colouring of $P = \widehat{T}$ to determine a coloured invariant of the satellite; this is an invariant of the original C, and the satellite theorem shows how to realise this as a coloured invariant of C itself, in other words as the value of $J(C)$ for some colouring of C.

Suppose that the pattern P, and hence the satellite, has r components, which we colour by modules $U_1, \ldots, U_r$. The tangle T, forming a subdiagram of P will then itself be coloured by these modules so that the top and bottom endpoints are represented by the same tensor product of modules, W say, drawn from $\{U_i, U_i^*\}$. The tangle T is represented by an endomorphism $T(\mathbf{U})$ of the module W. Write W as a direct sum $\bigoplus V_{\lambda_i}$ of irreducible modules, and choose $v_i \neq 0$ in V_{λ_i} for each i. The endomorphism $T(\mathbf{U})$ then determines $a_{ij} \in \Lambda$ with $T(\mathbf{U})(v_j) = \sum a_{ij} v_i$. Define a weighted trace $\mathrm{Tr}(T(\mathbf{U})) \in \mathcal{R}^{(k)}$ by setting

$$\mathrm{Tr}(T(\mathbf{U})) = \sum b_\lambda V_\lambda \ , \ \text{where } b_\lambda = \sum_{V_{\lambda_i} \cong V_\lambda} a_{ii}.$$

Satellite theorem 6.2. *Let L be the framed oriented satellite of C with pattern $P = \widehat{T}$ and let $\mathbf{U} = (U_1, \ldots, U_r)$ denote a colouring of its components. Then*

$$J(L; \mathbf{U}) = J(C; \mathrm{Tr}(T(\mathbf{U}))).$$

The proof can be constructed with care from [31]. Notice that $\mathrm{Tr}(T(\mathbf{U}))$ depends only on P and the colouring, and not on the companion C. It provides a multilinear map $J(P) : \mathcal{R}^{(k)} \times \ldots \times \mathcal{R}^{(k)} \to \mathcal{R}^{(k)}$ whose value on $(U_1, \ldots, U_r)$ is $\mathrm{Tr}(T(\mathbf{U}))$.

Theorem 6.3. *Let C be an oriented framed knot, let λ be any Young diagram and let V_λ be the corresponding irreducible $SU(k)_q$-module. Then, with the convention that $J(C; V_\lambda) = 0$ if λ is not an admissible shape for $SU(k)_q$, we have*

$$d_\lambda J(C; V_\lambda) = (xv^{-1})^{|\lambda|^2 w(D)} P_C(\widehat{e}_\lambda),$$

as functions of $s=\sqrt{q}$*, when the variable* v *on the right-hand side is replaced by* s^{-k} *and* x *by* $s^{-1/k}$*. Here* d_λ*, independent of* k*, is the degree of the matrix algebra* M_λ *in the appropriate Hecke algebra,* $|\lambda|$ *is the number of cells in the Young diagram* λ *and* $w(D)$ *is the writhe of a diagram for* C *with the chosen framing.*

A corresponding result holds for oriented framed links, dealing with each component independently.

Proof: An outline of the proof follows. Apart from the normalising factor this result is given in [41]; some further discussion will be found in [19]. Suppose that the given Young diagram λ has n cells, so that $|\lambda|=n$. We shall make use of a representation of H_n on $W=V_\square^{\otimes n}$ which carries the idempotent e_λ to the projection of W to the 'isotypic' submodule for V_λ, namely the submodule isomorphic to $\bigoplus V_{\lambda_i}$ for which $V_{\lambda_i}\cong V_\lambda$.

Any oriented (n,n)-tangle T determines an endomorphism $f(T)$ of W by colouring each of its components with the module $V_\square$. Because of the relation $(*)$ among the endomorphisms $f(T)$ as T varies, the map f induces a representation of the variant skein $\mathcal{S}'_P(R_n^n)$, with $v=s^{-k}$, $x=s^{-1/k}$, on W. Using the isomorphism of $H_n=\mathcal{S}_P(R_n^n)$ with this variant skein gives an explicit homomorphism

$$\varphi_k : H_n=\mathcal{S}_P(R_n^n)\to \mathrm{End}(W)$$

induced by $\varphi_k(T)=(x^{-1}v)^{w(T)}f(T)$, where again v and x are replaced appropriately when dealing with $SU(k)_q$.

Now decorate the diagram of C with the pattern $\widehat{T}$ to form a link diagram L, and colour all components of L with $V_\square$. By the satellite theorem we can calculate

$$J(L;V_\square,\ldots,V_\square)=J(C;\,\mathrm{Tr}\,f(T))=(xv^{-1})^{w(T)}J(C;\,\mathrm{Tr}\,\varphi_k(T))\ .$$

On the other hand, theorem 6.1 shows that

$$J(L;V_\square,\ldots,V_\square)=(xv^{-1})^{w(L)}P_L=(xv^{-1})^{w(L)}P_C(\widehat{T})\ ,$$

where $v=s^{-k}$. Now the writhe of the decorated diagram L can readily be given as $w(L)=w(T)+n^2w(C)$, since each crossing in C will give n^2 crossings of the same sign in L where the groups of n parallel strings cross. We can then write

$$J(C;\,\mathrm{Tr}\,\varphi_k(T))=(xv^{-1})^{n^2w(C)}P_C(\widehat{T})\ ,\ \text{ with } v=s^{-k}.$$

We may now replace T by any linear combination of (n,n)-tangles to get a similar result. In particular the idempotent e_λ in H_n can be written in this way, and then we have

$$(xv^{-1})^{|\lambda|^2w(C)}P_C(\widehat{e}_\lambda)=J(C;\,\mathrm{Tr}\,\varphi_k(e_\lambda))\ ,\ \text{ with } v=s^{-k},\ x=s^{-1/k}.$$

The proof of theorem 6.3 can then be completed by showing that $\varphi_k(e_\lambda)$ is the projection of W to the isotypic submodule for V_λ which is isomorphic to d_λ copies of V_λ. The trace of this projection is $d_\lambda V_\lambda$ so that the right-hand side in the equation above becomes $d_\lambda J(C;V_\lambda)$ as claimed. □

In the proof above the identification of $\varphi_k(e_\lambda)$ with the projection to one of the isotypic submodules of W remains to be established. A deeper understanding of the structure both of H_n and of the modules $W=V_\square$ for different k can be achieved by use of the representation φ_k. This representation gives a direct analogue of the setting for classical invariant theory of the symmetric group, where the Hecke algebra corresponds to the group algebra of the symmetric group S_n and the quantum groups to the special linear groups. By drawing on work of Wassermann [39] and Wenzl [41] it can be shown that the following generalisations of the classical results hold in this context.

THEOREM 6.4. *The homomorphism* $\varphi_k : H_n \to \mathrm{End}_{SU(k)_q} V_{\Box}^{\otimes n}$ *is*

(1) *surjective for all* k,

(2) *injective when* $k \geq n$.

The first part shows that every module endomorphism of W can be represented as the linear combination of some tangles coloured with $V_{\Box}$. In particular the projection to any submodule of W must be representable in this way; the choice of the element e_λ is then simply one explicit way to realise $J(C;V_\lambda)$ by means of a satellite Homfly polynomial. Indeed the element e_λ is generally rather complicated and it is usually possible to find a simpler combination with the same closure in $\mathcal{C}$.

The isomorphism of H_n with the endomorphism ring for large enough k permits us to extend the classical correspondence between the idempotent e_λ and the projection to the corresponding isotypic submodule in this case as well. It is also possible to describe readily the kernel of φ_k when $k < n$ as the ideal generated by those idempotents e_λ whose Young diagram has too many rows to be admissible for k, again exactly as in the classical case.

The most striking consequence of the approach using the skein of the annulus is the existence of the 2-variable invariant of C indexed by λ whose specialisations at $v = s^{-k}$ provide the quantum invariants $J(C;V_\lambda)$ for all $SU(k)_q$ at once. Links L can be treated in essentially the same way, taking the satellite Homfly polynomial when each component is decorated independently by some $\widehat{e}_\lambda$, multiplied by a suitable power of v, to specialise to the corresponding quantum invariant $J(L)$. It is interesting to note that when the orientation of one component is reversed the quantum invariant of the new link can be recovered from that of the old link by replacing the module on that component with its dual. The dual of the irreducible module V_λ is again irreducible, but its Young diagram λ^* depends on k as well as λ so it is not possible to give a similar universal treatment to handle string reversals for satellite Homfly invariants.

By way of example, the dual of the fundamental module $V_{\Box}$ has Young diagram λ^* with a single column and $k-1$ cells. In the case of $SU(2)_q$ the fundamental module is then self-dual, as are all the other irreducibles, which accounts for the insensitivity of the bracket invariant to string orientation. For $SU(3)_q$ the calculation $J(C;V_{\substack{\Box\\\Box}})$ will then give $J(\overline{C};V_{\Box}) = (xv^{-1})^{w(C)}P_{\overline{C}}$ with $v = s^{-3}$, where $\overline{C}$ is C with the opposite orientation. The Homfly polynomial of a knot is unchanged by string reversal, so we see that

$$(xv^{-1})^{4w(C)}P_C(\widehat{e}_{\substack{\Box\\\Box}}) = J(C;V_{\substack{\Box\\\Box}}) = J(\overline{C};V_{\Box}) = (xv^{-1})^{w(C)}P_{\overline{C}},$$

and so $P_C(\widehat{e}_{\substack{\Box\\\Box}}) = (x^{-1}v)^{3w(C)}P_C$, the standard Homfly polynomial, with $v = s^{-3}$ and $x = s^{-1/3}$. This gives $P_C(\widehat{e}_{\substack{\Box\\\Box}}) = s^{-8w(C)}P_C$, when $v = s^{-3}$.

It is also possible to identify the module $V_{\substack{\Box\\\Box}}$ for $SU(4)_q$ with the fundamental module for $SO(6)_q$ and so relate $P_C(\widehat{e}_{\substack{\square\\\square}})$ with $v = s^{-4}$ to an evaluation of Kauffman's Dubrovnik polynomial, [20].

6.6 REMARKS.

The satellite theorem provides a multiplicative homomorphism from $\mathcal{C}$ to $\mathcal{R}^{(k)}$ for each k, which is most readily defined on the variant skein of the annulus by taking each pattern $P = \widehat{T}$ to the trace of T when coloured entirely with $V_\square$. On $\mathcal{C}_+$ this description is independent of k and can be given on the basis $\{\widehat{e}_\lambda\}$ by $\widehat{e}_\lambda \mapsto d_\lambda V_\lambda$, so that after suitable writhe adjustment the functions J_L and P_L agree. The map carries the element α_1 to $V_\square$ and α_2 to $vsV_{\square\square} - vs^{-1}V_{\substack{\square\\\square}}$, while on the other hand α_1^* is mapped to $V_\square^*$, which will depend on k as noted above.

The skein map P_L on the algebra $\mathcal{C}$, or even its restriction to the subalgebra $\mathcal{C}_+$, carries the information for all the total invariants J_L as k varies. Unlike the case earlier where we compared the algebra $\mathcal{B}$ for the bracket invariant and the representation ring of $SU(2)$, we have here a single algebra $\mathcal{C}$ arising from the Homfly polynomials and a whole series of non-trivial quotients $\mathcal{R}^{(k)}$ of $\mathcal{C}$ which organise the quantum invariants.

In fact the ring $\mathcal{R}^{(k)}$ is the quotient of $\mathcal{C}_+$ by the ideal generated by $X_n = \widehat{e}_\lambda$ for $n > k$, where λ is the Young diagram with one column and n cells. The corresponding module V_λ is the n-th exterior power of the fundamental module $V_\square$. It is possible to draw on classical knowledge of the representation rings $\mathcal{R}^{(k)}$ as polynomial rings in the exterior powers of the fundamental module to give alternative constructions for the general basis element $\widehat{e}_\lambda$ in $\mathcal{C}_+$ as a polynomial in the elements $\{X_n\}$. The element X_n is noted above to be $X_n = (\varphi_b(a_n))^{-1}\widehat{a}_n$. Equally the elements $Y_n = (\varphi_a(b_n))^{-1}\widehat{b}_n$, corresponding to the symmetric powers of $V_\square$, can be used to generate $\mathcal{C}_+$ as a polynomial ring.

An attempt to deal with 3-manifold invariants by means of $\mathcal{C}_+$, on the lines of the treatment in section 3, has the corresponding feature that when calculating with $v = s^{-k}$ and $s^{2(k+l)} = 1$, the invariant $P_C(Y_n) = 0$ for $n = l, l+1, \ldots, l+k-1$. When the ideal generated by the k elements corresponding to Y_n, $n = l, \ldots, l+k-1$ is factored out from $\mathcal{R}^{(k)}$, the quotient is a finite-dimensional algebra (a Verlinde algebra), which gives an analogue to $\mathcal{R}_r$ in the case of the $SU(2)$ invariant, with $r = k + l$. It corresponds closely with the ingredients used by Turaev and Wenzl [37] in their construction of a 3-manifold invariant of level l based on $SU(k)_q$. It would be interesting to consider this approach via $\mathcal{C}$ in more detail, with enough care about the denominators in the ring of scalars to ensure that the substitutions of variables cause no problems.

Acknowledgements.

These notes have evolved from a series of seminars given in Liverpool during 1991, and represent one of many ways to choose a path through the wealth of material available. I am grateful to my colleagues and co-workers for providing a willing and critical audience, and particularly to Peter Cromwell and Ian Nutt for helpful criticism of the manuscript.

I must acknowledge the considerable influence of others, notably Raymond Lickorish and Lou Kauffman from conversations and lectures as well as from the articles cited here. I must also thank CIMAT in Guanajuato for the opportunity to lecture on this material in December 1991, and the organisers of the NATO ASI in Erzurum in September 1992 for the incentive to assemble it in its present form.

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Version 1.4 Dec.1992 (transferred to eps pictures Jan 1997)