MINIMAL ARC-PRESENTATIONS OF SOME NON-ALTERNATING KNOTS

ELISABETTA BELTRAMI

Dipartimento di Matematica, Università di Pisa, Via F. Buonarroti 2, 56127 Pisa. Italy.

PETER R. CROMWELL

Department of Pure Mathematics, University of Liverpool, PO Box 147, Liverpool. L69 3BX. England.

ABSTRACT

We construct minimal arc-presentations of some non-alternating links such as the Kinoshita-Terasaka and Conway mutants. Our construction is unaffected by the mutation operation.

Keywords: knot, tangle, mutation, Kauffman polynomial, Hamiltonian graph.

Birman and Menasco used arc-presentations of links in their study of the braid index of satellite links [2]. The binding circle of an arc-presentation of the companion knot can be used as a braid axis for the satellite, and it is hoped that the braid index of the satellite can be expressed in terms of the framing, the arc index of the companion, and some properties of the pattern.

In [6] an algorithm was presented for constructing an arc-presentation of a link from a link diagram. The algorithm can be applied to all except a clearly defined set of diagrams. The construction produces an upper bound on the arc index in terms of the crossing number of the link: $\alpha(L) \leqslant c(L)+2$. Cromwell and Nutt conjectured that equality holds for alternating links. This is known to be the case for 2-bridge links, for example. Further evidence to support this followed in [5] where it is shown that $\alpha(L) \geqslant c(L)+2$ for all alternating knots. However, a simple algorithmic construction to prove the reverse inequality is not possible in the general case [1].

In this paper we extend the ideas in [6] to construct minimal arc-presentations of some non-alternating links. These links have adequate diagrams which can be

1991 Mathematics Subject Classification: 57M25

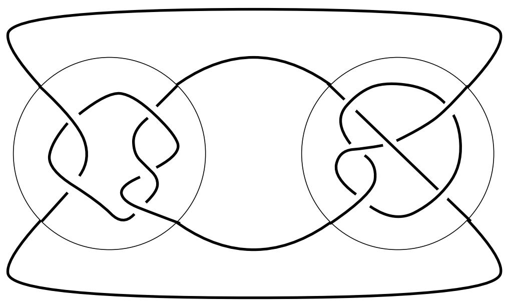


Figure 1. Tangle decomposition of the Kinoshita-Terasaka knot.

expressed as the sum of strongly alternating tangle diagrams. The presentations satisfy $\alpha(L) = c(L)$. A typical example is the Kinoshita-Terasaka 11-crossing knot shown in Figure 1. This tangle decomposition also allows us to consider the behaviour of arc index under the knot operation of mutation. The Kinoshita-Terasaka and Conway mutant pair were both shown to have arc index 11 by a computer enumeration of arc-presentations of small knots [9]. The arc-presentations given there are not related in any obvious way. Using our construction it becomes apparent that the arc indices of these mutants must be equal.

1 Preliminaries.

This paper continues a series [4, 6, 9, 10, 11] in which the properties of arc-presentations of knots and links are studied. See [5] for an introduction to the subject and a survey of results. The basic idea is as follows. Constructing an *arc-presentation* of a link involves placing it in a special knot-holder: the link is embedded in a sheaf of half-planes so that it meets each half-plane in a single simple arc. The minimum number of planes required to present a given link, L, in this manner is a knot invariant called the *arc index* of L. It is denoted by $\alpha(L)$.

We recap the following terminology from knot theory. A 2-string tangle is *strongly alternating* if both the closures shown on the left of Figure 2 are irreducible, alternating diagrams. The figure also shows how a closed link is formed as the *sum* of two tangles. The non-alternating sum of two strongly alternating tangles is called a *semi-alternating* diagram [7]. *Mutation* is an operation on link diagrams which can change the link type: a 2-string tangle is removed, rotated by 180 degrees in any of the three possible ways, and re-inserted to form a new link diagram.

We also need some basic terminology from graph theory. All our graphs will be planar graphs embedded in a plane. Let V be the set of vertices of a connected

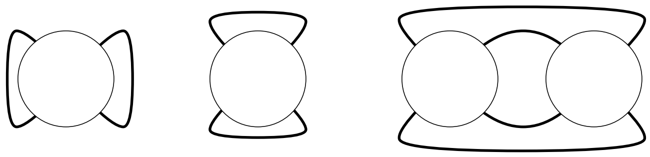


Figure 2. Closures of tangles, and the sum of two tangles.

graph G. An *edge* of G is a pair of vertices $[u, v]$. If $U \subset V$ then $\mathrm{span}(U)$ is the subgraph of G whose edges connect vertices in U:

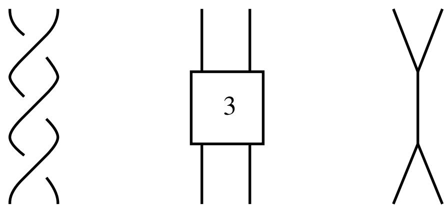
$$\mathrm{span}(U) \;=\; \{\, [u_i, u_j] \mid \text{ for all pairs } u_i,\, u_j \text{ in } U \,\}.$$

If $U \subset V$ then $G - U$ denotes the subgraph of G spanned by $V - U$. We say $G - U$ is obtained by *deleting* the vertices in U. The graph is said to be *r-connected* if $r < |V|$ and at least r vertices must be deleted from G to disconnect it. A *path* is a simple curve which follows the edges of G. A *cycle* is a closed path. A *Hamiltonian circuit* (*path*) of G is a cycle (path) which contains every vertex in V.

2 Arc-presentations of tangles

Given a 2-string tangle diagram, our goal is to construct an arc-presentation of the tangle like the one shown in Figure 4(e). The thin horizontal line represents the binding axis; the semicircular parts of the diagram are arcs that can be layered and embedded in pages around the binding; the four other pieces of the tangle can also be thought of as arcs but they are attached to the axis at only one end. Figures 4(a)–(d) illustrate the construction process described below.

Let D be a 2-string tangle diagram. As in [6], sequences of half-twists in the diagram are replaced by rectangular boxes known as *twist-boxes* as shown in



(a) (b) (c)

Figure 3.

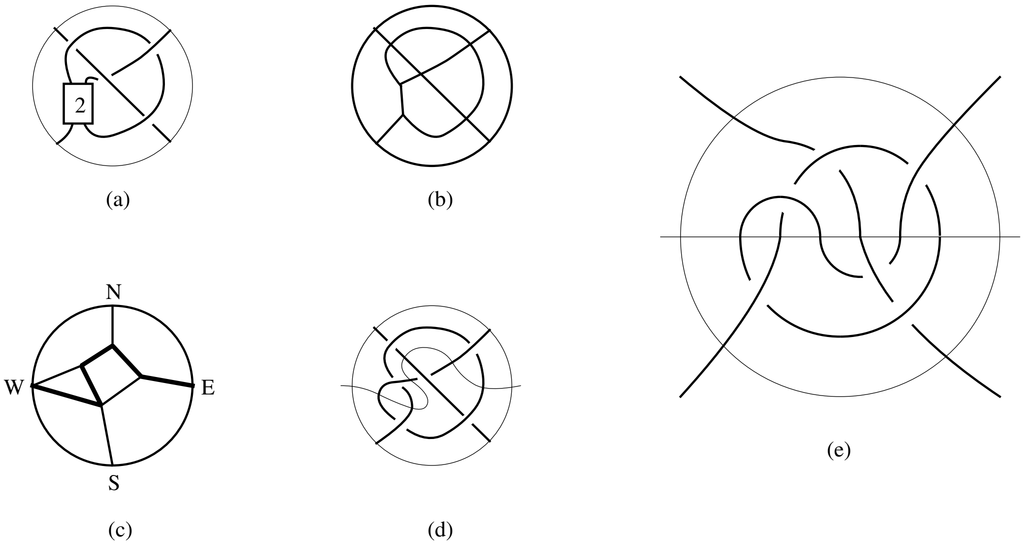


Figure 4. Constructing an arc-presentation of a tangle.

Figures 3(a) and (b). The diagram now consists of a collection of twist-boxes connected by a set of disjoint simple arcs. We form a graph G from this diagram as follows: if a twist-box contains only one crossing we replace it by a 4-valent vertex, otherwise we replace the twist-box by the graph shown in Figure 3(c). Thus all vertices in G are 3-valent or 4-valent. The edges of G can be divided into two sets: D-type edges which contain part of the original diagram, and T-type edges which are contained in a twist-box. The four D-type edges which contain the free ends of the tangle are called *leaves*.

Following the approach of [6] we take the topological dual graph G^* of G and try to construct the binding axis as a path in G^*. Here, however, we have an incomplete link diagram with free ends and we need to be careful about the construction of G^*.

Firstly, we complete the graph G by adding four extra edges that follow the boundary circle of the tangle so that they connect the four leaves — see Figure 4(b). We call these *peripheral edges*. A bounded region of G which is adjacent to a peripheral edge is called a *peripheral face*.

The dual graph G^* is formed in the usual way (place a vertex in each finite region and connect vertices that lie in adjacent regions). We label the four vertices in the peripheral faces N, E, S and W as shown in Figure 4(c).

Take π to be a Hamiltonian path in $G^* - \{N, S\}$ which connects E to W — see the thickened edges in Figure 4(c). (The existence of such paths is briefly discussed in §5.) The path π passes through all the regions of G except the upper

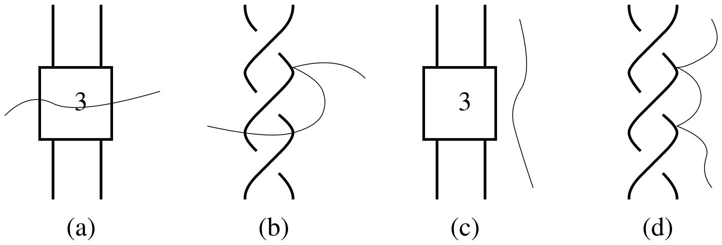


Figure 5. Deforming the axis near a twist-box.

and lower ones. We convert π into the axis as follows. If π crosses a T-type edge of G then it passes through a twist-box. The path π and the m half-twists represented by the box can be arranged as shown in Figures 5(a) and (b). Any twist-box that does not meet π is treated as shown in Figures 5(c) and (d). This is always possible since π can be deformed to lie along one side of the twist-box.

Lastly, we can isotop $D \cup \pi$ in the plane keeping the boundary circle fixed so that π is straight and horizontal.

To show that this construction produces what we want, we make use of the same colouring procedure as in [6].

Proposition. The path π just constructed represents a binding axis for the tangle diagram D.

Proof. Let L be an embedding of two line segments in $\mathbb{R}^3$ such that there is a projection ϕ which carries L onto D. Notice that all the bounded regions of D have contact with π: either π passes through the region or π has point-contact with its boundary.

Let $P = \{p_1, \cdots, p_r\}$ be the set of points where π meets D. The points $\phi^{-1}(p_i)$ divide L into a set of arcs $A = \{a_1, \cdots, a_{r+2}\}$ each of which starts and ends at points in $\phi^{-1}(P)$ or at the four free ends, and which contains no such points in its interior. Each of these arcs will be embedded in a half-plane. To each arc a_i in L there is a corresponding segment $\phi(a_i)$ of the diagram D which we will also refer to as an arc.

We will colour the arcs, and the order in which they are coloured will determine the position of the half-planes in which they are embedded. The plane is separated by π into two regions. We will colour the arcs in D by taking each region separately. We start with the upper one.

The colouring process proceeds under the following rules: an uncoloured arc can be coloured if

(a) any arc crossing over it is coloured, and

(b) any arc crossing under it is uncoloured.

The first arc to be coloured is labelled 1. Successive arcs are numbered in the order they are coloured. The proof that this colouring procedure stops only when all the arcs in the upper region have been coloured is identical to that used in [6].

The arcs on the lower side of π are coloured using the same process except that the first arc to be coloured is labelled $r+2$, the next $r+1$, and so on until every arc has been labelled.

If we now arrange $r+2$ half-planes around π at angles of $(360/(r+2))^\circ$ and label them from 1 to $(r+2)$ in order, the arc labelled i can be embedded in the corresponding half-plane.

Remark. Note that the rotations of 180 degrees used in mutation operations preserve the horizontal axis.

Proposition. The number of times the binding axis meets the tangle equals number of crossings in D.

Proof. Suppose that D required n twist-boxes of which r contain a single crossing. Then G has $2(n-r)$ 3-valent vertices, r 4-valent vertices and 4 peripheral vertices making $2n-r+4$ in total. Similarly, it has $2n+6$ D-type edges and $(n-r)$ T-type edges. From Euler's formula we deduce that it has $n+3$ bounded regions. A Hamiltonian path through $G^*-\{N,S\}$ passes through $n+1$ vertices and therefore contains n edges. Thus π crosses n edges of G. When a T-type edge is replaced by m half-twists as in Figure 5(b), a single intersection of the axis and the diagram is increased to m intersections. (Recall that $m>1$ for the existence of a T-type edge.) Running π alongside a twist-box containing m twists as in Figure 5(d) also increases the number of intersections of π with D by $m-1$. Therefore the total number of intersections is

$$\begin{aligned} & n + \sum_{i=1}^{n} (m_i - 1) \\ = \ & \text{the number of crossings in } D. \end{aligned}$$

3 Sums of tangles

Given arc-presentations of two tangles with horizontal binding axes it is clear how we would like to join them together to form an arc-presentation of the sum of the tangles. We want to join the two axes end to end and then rotate the pages in one presentation around the axis so that the pages containing the free ends in one tangle line up with those in the other. The free ends can then be joined to produce a link without introducing any extra pages. The question is: 'When can this be done?'.

Proposition. Let $\theta(X)$ denote the page number (or angle) of the page containing the point X. The arc-presentations of the two tangles shown in Figure 6(a)

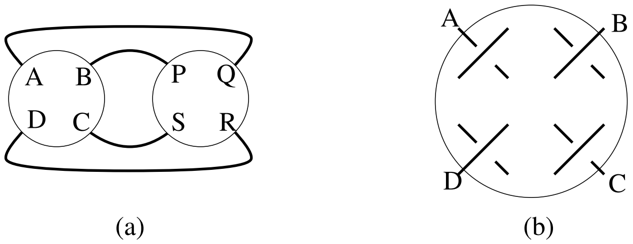


Figure 6.

can be joined in this way if the orderings $\theta(A)$, $\theta(B)$, $\theta(C)$, $\theta(D)$ and $\theta(Q)$, $\theta(P)$, $\theta(S)$, $\theta(R)$ are equivalent.

Proposition. Given an alternating tangle diagram of the form shown in Figure 6(b) for which we can construct an arc-presentation as described above, the ordering of the free ends is

$$\theta(B) \ < \ \theta(A) \ < \ \theta(C) \ < \ \theta(D).$$

Proof. In the colouring procedure described above, the arcs in the region above the axis are coloured before those below it. This means that both $\theta(A)$ and $\theta(B)$ must come before $\theta(C)$ and $\theta(D)$ in the ordering.

To check the ordering of $\theta(A)$ and $\theta(B)$ consider the peripheral face at the top of the tangle. The segments of the tangle diagram D which bound this region are part of an alternating diagram and the colouring procedure ensures that this natural layering is preserved in the arc-presentation. Since the crossing nearest B is an over-crossing, this arc will be coloured before the arc containing A and hence $\theta(B) < \theta(A)$. The same argument shows that $\theta(C) < \theta(D)$ — recall that these arcs are coloured in the reverse order.

From this we can deduce the following corollary.

Theorem. Suppose we have two arc-presentations of alternating tangles in which the binding axes are horizontal. Then the link formed as the non-alternating sum of these tangles (as in Figure 6(a)) has an arc-presentation with at most n arcs where n is the total number of crossings in the two tangle diagrams.

Corollary. Let $(p, q, -r)$ denote the non-alternating pretzel knot with p, q and r all positive integers greater than one. Then

$$\alpha(p, q, -r) \ \leqslant \ p + q + r.$$

Remark. There are cases where this construction is known not to be minimal. For example, the $(3,3,-2)$ pretzel knot (8_{19} in [12]) is the (3,4) torus knot. Also $(5,3,-2) = 10_{124}$ is the (3,5) torus knot. It is known [4] that the arc index of a (p,q) torus knot is at most $p+q$.

Remark. The method does not work for the alternating sum of the tangles as the ordering principle is invalid. We know that we must get at least $c(D)+2$ arcs for alternating links [5, 8].

Remark. Arc-presentations of mutants constructed in this way will use the same number of arcs. It would be interesting to know whether arc index is preserved by mutation in general.

4 Minimality of the arc-presentations

We will now show that this construction gives the minimal arc index if the tangles are strongly alternating. (The pretzel knots are not semi-alternating.)

A relationship between the arc index of a link and its two-variable Kauffman polynomial was discovered by Nutt [9, 11]. In this original form, the relationship was valid only for knots and with the coefficients of the polynomial reduced modulo 2. The full generality of the following statement was recently established in [8].

Theorem. Let L be a link and let $F_L(a,x)$ be its Kauffman polynomial. Then

$$\alpha(L) \;\geqslant\; \mathrm{breadth}_a(F_L) + 2.$$

We will show that $\mathrm{breadth}_a(F_L) \geqslant c(L) - 2$ for the links we are interested in. This can be deduced from a result of Thistlethwaite [13]. He works from a graph theoretical viewpoint and we first need to recall his notation.

Let G be a graph with $V(G)$ vertices and $E(G)$ edges. Suppose the edges are labelled '+' or '−'. Then G_+ denotes the subgraph of G formed by deleting all the edges labelled '−', and $\overline{G_+}$ denotes the quotient graph formed by collapsing all the edges labelled '−'. The graphs G_- and $\overline{G_-}$ are defined similarly.

Now take G to be the graph derived from the chessboard shading of a diagram D by placing a vertex in each shaded region and connecting them through the crossings in the usual way. The edges of such a graph can be labelled '+' or '−' according to the sense of the crossings.

The Kauffman polynomial $F_L(a,x)$ also comes in a framed version which is usually denoted $\Lambda_D(a,x)$. Since we are interested in the breadth in a of the polynomial, and the two versions differ only by multiplication by a power of a, we can consider the framed version. Now

$$\Lambda_D(a,x) \;=\; \sum_{i,j} \alpha_{i,j}\, a^i\, x^j$$

and $\alpha_{i,j}$ is non-zero only when $|i|+j \leqslant c(D)$. Thistlethwaite calls the coefficients for which $|i|+j=c(D)$ *outermost* and encapsulates them in two single variable polynomials:

$$\phi_D^+(t) \;=\; \sum_i \alpha_{i,n-i}\, t^i \qquad \text{and} \qquad \phi_D^-(t) \;=\; \sum_i \alpha_{-i,n-i}\, t^i \,.$$

From Corollary 1.1 of [13] we get the following.

Proposition. Suppose that D is a semi-alternating link diagram, and that G is its associated signed chessboard graph. Then the highest degree terms in ϕ_D^+ and ϕ_D^- are monic and have degrees

$$\begin{aligned} \operatorname{degree}(\phi_D^+) &= \operatorname{rank}(G_+) + V(\overline{G_-}) - 1 \\ \operatorname{degree}(\phi_D^-) &= \operatorname{rank}(G_-) + V(\overline{G_+}) - 1. \end{aligned}$$

We can now easily establish the desired bound. For a semi-alternating diagram G is a 2-connected graph and the separating pair of vertices divide the graph into G_+ and G_-. Hence we have $V(\overline{G_+}) = V(G_+) - 1$ and $V(\overline{G_-}) = V(G_-) - 1$, and also $\operatorname{rank}(G) = \operatorname{rank}(G_+) + \operatorname{rank}(G_-) + 1$. Therefore

$$\begin{aligned} \operatorname{breadth}_a(F_L) &\geqslant \operatorname{rank}(G_+) + V(\overline{G_+}) - 1 \quad + \quad \operatorname{rank}(G_-) + V(\overline{G_-}) - 1 \\ &= (\operatorname{rank}(G) - 1) + (V(G_+) - 1) + (V(G_-) - 1) - 2 \\ &= (E(G) - V(G) + 1) - 1 + V(G) - 2 \\ &= E(G) - 2 \\ &= c(D) - 2. \end{aligned}$$

The result follows because semi-alternating diagrams have minimal crossing number [7].

Corollary. Let $(p,q,-r,-s)$ denote the non-alternating generalised pretzel knot with p, q, r and s all positive integers greater than one. Then

$$\alpha(p,q,-r,-s) \;=\; p+q+r+s.$$

5 The existence of Hamiltonian paths

The construction described in [6] relied on the existence of Hamiltonian circuits. In that case, it was possible to find conditions on the link diagram which ensured that the construction was applicable. These were deduced from the fact that all 4-connected graphs and all 'small' 3-connected graphs contain a Hamiltonian circuit.

The construction we have used here relies on the existence of Hamiltonian paths. It is clear that such paths will not always exist. For example, if the tangle used in Figure 4 is rotated by 90 degrees then the construction fails. It would be nice, therefore, to be able to state conditions on a tangle which imply that our construction is applicable. We had hoped to be able to derive conditions analogous to those in [6] from a theorem like the following: In a 4-connected graph, there is a Hamiltonian path connecting any two vertices [14, 3]. However, an analysis of cases similar to that in [6] produced such a long list of exclusions that it seems pointless to include it here. The main reason for the explosion of cases is that we no longer have a condition on the whole graph, G^*, but only on $G^* - \{N, S\}$. The construction can be applied to many more cases than those which pass through this particular net.

References

1. E. Beltrami and P. R. Cromwell, *A limitation on algorithms for producing arc-presentations from link diagrams*, Preprint 1996. (University of Liverpool).

2. J. S. Birman and W. W. Menasco, 'Special positions for essential tori in link complements', *Topology* 33 (1994) pp525–556.

3. N. Chiba and T. Nishizeki, 'A theorem on paths in planar graphs', *J. Graph Theory* 10 (1986) pp449–450.

4. P. R. Cromwell, 'Embedding knots and links in an open book I: Basic properties', *Topology and its Applications* 64 (1995) pp37–58.

5. P. R. Cromwell, 'Arc presentations of knots and links', *Proc. 1995 Warsaw Knot Theory Conference*. (to appear).

6. P. R. Cromwell and I. J. Nutt, 'Embedding knots and links in an open book II: Bounds on arc index', *Math. Proc. Cambridge Philos. Soc.* 119 (1996) pp309–319.

7. W. B. R. Lickorish and M. B. Thistlethwaite, 'Some links with non-trivial polynomials and their crossing-numbers', *Comment. Math. Helv.* 63 (1988) pp527–539.

8. H. R. Morton and E. Beltrami, 'Arc index and the Kauffman polynomial', *Math. Proc. Cambridge Philos. Soc.* (to appear).

9. I. J. Nutt, *Braid index of satellite links*, Ph.D. thesis, University of Liverpool. 1995.

10. I. J. Nutt, *On the braid index of satellite links*, Preprint 1995. (University of Liverpool).

11. I. J. Nutt, 'Arc index and the Kauffman polynomial', *J. Knot theory and its Ramifications* (to appear).

12. D. Rolfsen, *Knots and Links*, Publish or Perish Inc. 1976.

13. M. B. Thistlethwaite, 'On the Kauffman polynomial of an adequate link', *Invent. Math.* 93 (1988) pp285–296.

14. C. Thomassen, 'A theorem on paths in planar graphs', *J. Graph Theory* 7 (1983) pp169–176.