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Seifert circles and knot polynomials

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In this paper I shall show how certain bounds on the possible diagrams presenting a given oriented knot or link K can be found from its two-variable polynomial P_K defined in [**3**]. The inequalities regarding exponent sum and braid index of possible representations of K by a closed braid which are proved in [**5**] and [**2**] follow as a special case.

Notation. In a diagram D for an oriented knot, write $c^+(D)$ and $c^-(D)$ for the number of positive and negative crossings, where  is a positive crossing.

The *crossing number*, $c(D)$, and the *algebraic crossing number*, $\tilde{c}(D)$, are defined by

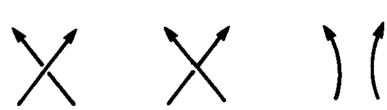
$$c(D) = c^+(D) + c^-(D)$$
$$\tilde{c}(D) = c^+(D) - c^-(D).$$

By cutting out each crossing, respecting the orientation, the diagram D is converted to a number of oriented simple closed curves in the plane, called the *Seifert circles* of D. Write $s(D)$ for the number of Seifert circles of D.

The two-variable polynomial, $P_K(v, z)$, of the oriented link K will be defined, as in [**5**], so that

$$\frac{1}{v} P_{K^+} - v P_{K^-} = z P_{K^0} \quad (*)$$

where K^+, K^- and K^0 have diagrams differing only by the change



D^+ $\qquad$ D^- $\qquad$ D^0

near one crossing.

Write $P_K(v, z) = \sum_{k=e}^{k=E} a_k(z) v^k$, with $a_e(z) \neq 0 \neq a_E(z)$, as a Laurent polynomial in v, to define its range, $[e, E]$, in v. Write also $P_K(v, z) = \sum_{r=m}^{r=M} b_r(v) z^r$, with $b_m(v) \neq 0 \neq b_M(v)$, to define its range in z.

Lickorish and Millett [**4**] show that $m = 1 - |K|$, where $|K|$ = number of components of K.

I shall show here that:

THEOREM 1. *For any diagram D of K,*

$$\tilde{c}(D) - (s(D) - 1) \leqslant e \leqslant E \leqslant \tilde{c}(D) + (s(D) - 1).$$

THEOREM 2. *For any diagram D of K,*

$$M \leqslant c(D)-(s(D)-1).$$

COROLLARY 1 [5], [2]. *If K is presented as the closure of a braid (β, n) on n strings, then $\tilde{c}(\beta)-(n-1) \leqslant e \leqslant E \leqslant \tilde{c}(\beta)+(n-1)$, where $\tilde{c}(\beta)$ is the exponent sum of β.*

COROLLARY 2. *Under the same conditions $M \leqslant$* length $(\beta)-(n-1)$.

Proof. The diagram presenting K as the closure of β has n Seifert circles following the braid strings.

An extension of the braid index bound for K to give a lower bound for $s(D)$ in terms of the 'spread' of v in P_K follows:

COROLLARY 3. *For any diagram D of K, $s(D) \geqslant \frac{1}{2}(E-e)+1$.*

COROLLARY 4 (Compare Bennequin [1]). *For any diagram D of the unknot, or any amphicheiral knot, we must have $|\tilde{c}(D)| < s(D)$.*

Proof. In the case of the unknot $e = E = 0$. For an amphicheiral knot $e = -E$, so that $e \leqslant 0 \leqslant E$.

Remarks 1. It is conceivable that $e \leqslant 1-\chi$ where χ is the Euler characteristic of a minimal genus spanning surface for K. This would give a sharp form of Bennequin's inequality for braid presentations of K.

2. The bound $c-(s-1)$ for M in Theorem 2 is just $1-\chi(D)$, where $\chi(D)$ is the Euler characteristic of the spanning surface for K constructed from D using the Seifert circles. It is worth noting that in general M is *not* bounded above by $1-\chi$ for the *minimal genus* spanning surface for K. For example, in the case of the untwisted double of a trefoil $M = 6$ while $1-\chi = 2$. This illustrates quite sharply the possible difference between M and the highest degree in z in $P_K(1,z)$, the Conway polynomial, a variant of the Alexander polynomial, which is well-known to be bounded above by the minimal $1-\chi$.

Proof of Theorem 1. It will be enough to prove the inequality $\tilde{c}(D)-(s(D)-1) \leqslant e$. For if the diagram is reflected to give a diagram $\bar{D}$ of the mirror image knot $\bar{K}$ then $s(\bar{D}) = s(D)$, $\tilde{c}(\bar{D}) = -\tilde{c}(D)$, and it is known that $P_{\bar{K}}(v,z) = P_K(-v^{-1},z)$, so that $E_K = -e_{\bar{K}}$. The inequality above, for $\bar{K}$, gives $-\tilde{c}(D)-(s(D)-1) \leqslant e_{\bar{K}}$, and so $E_K \leqslant \tilde{c}(D)+(s(D)-1)$.

Write $\phi(D) = \tilde{c}(D)-(s(D)-1)$ for a knot diagram D. The theorem will then follow by showing that $v^{-\phi(D)}P_K(v,z)$ is a *polynomial* in v (i.e. has no negative powers of v) for every diagram D of K.

The Seifert circles arising from any three related diagrams D^+, D^- and D^0 are the same, so that $\phi(D^+) = \phi(D^0)+1$, $\phi(D^-) = \phi(D^0)-1$, and the recurrence relation (*) then gives

$$v^{-\phi(D^+)}P_{K^+} - v^{-\phi(D^-)}P_{K^-} = zv^{-\phi(D^0)}P_{K^0}.$$

So if $v^{-\phi(D)}P_K$ is a polynomial in v for two of D^+, D^- and D^0 then it is also for the third.

Proceed by induction on $c(D)$, the number of crossings in D. The result is true when D has no crossings, since then K is the unlink with s components, and

$$P_K = ((v^{-1}-v)/z)^{s-1}.$$

Otherwise we can find a sequence of crossing changes on D which lead, as in [4], to an ascending diagram D' for an unlink. It is then enough to prove the result for D', since, for each crossing change in the sequence, the third diagram, D^0, in the recurrence formula given by cutting out the crossing has $v^{-\phi(D^0)}P_{K^0}$ a polynomial in v, by induction.

For an ascending diagram D' of the unlink with k components, say, we have $P = ((v^{-1}-v)/z)^{k-1}$, so we must prove that $-\phi(D') \geqslant k-1$.

In each component of an ascending diagram D' there is a base point; the component rises monotonically, relative to the direction of projection, until it lies vertically above the base point, when it returns to base by a vertical segment. Different components are stacked above each other in disjoint projection levels.

Case 1. Suppose first that one component of D' has a self-crossing point. We may then find the lowest self-crossing, p, in this component, i.e. the first one reached on starting from the base point. Because D' is ascending, the link whose diagram D'' is given by cutting out the crossing at p will be the unlink with $k+1$ components, for the component containing p will become a 2-component unlink lying between the levels of the other unchanged $k-1$ components. (In fact D'' will again be ascending, for the ascending arc from the undercrossing to the overcrossing at p will become a component lying entirely beneath the other arc of the component which is cut in two at p.) Now $\phi(D'') = \phi(D') \pm 1$ depending on the sign of the crossing at p. By induction, $-\phi(D'') \geqslant k$, giving $-\phi(D') \geqslant k \pm 1 \geqslant k-1$ as required.

Case 2. If no components of D' have self-crossings we may suppose that each lies in a single level. By changing the levels of two components with no crossings, if necessary, we can find two components in adjacent levels which cross each other. We can select a negative crossing of one with the other, since their algebraic crossing number is zero, and cut it out as before to get a new diagram D''. This time D'' (again an ascending diagram) represents the unlink with $k-1$ components. We have $\phi(D'') = \phi(D')+1$, and, by induction, $-\phi(D'') \geqslant k-2$, so that $-\phi(D') \geqslant k-1$, finishing the proof.

Proof of Theorem 2. Write $\psi(D) = c(D)-(s(D)-1)$ for a diagram D of K, and show, by a similar induction on c, that $z^{-\psi(D)}P_K(v,z)$ is a polynomial in z^{-1}. In this case $\psi(D^+) = \psi(D^-) = \psi(D^0)+1$. The recurrence relation (*) then gives

$$v^{-1}z^{-\psi(D+)}P_{K^+} - vz^{-\psi(D^-)}P_{K^-} = z^{-\psi(D^0)}P_{K^0}.$$

If any two are polynomials in z^{-1} then the third will be, so it is enough, as in the proof of Theorem 1, to prove for an ascending diagram D' of an unlink. If D' has k components then $P = ((v^{-1}-v)/z)^{k-1}$, so we must prove that $-\psi(D') \leqslant k-1$.

Select a new diagram D'' as before, with one fewer crossing, representing the unlink with either $k+1$ or $k-1$ components. In each case $\psi(D'') = \psi(D')-1$. By induction we have either $-\psi(D'') \leqslant k$ or $-\psi(D'') \leqslant k-2$. This ensures that $-\psi(D'') \leqslant k$, so that $-\psi(D') \leqslant k-1$, as required.

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