Plane curves, wave fronts and Legendrian knots

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Abstract

We survey some of the recent results on Legendrian knots and links in the standard contact 3-space and solid torus. These include the description of finite order invariants and estimates of the self-linking number coming from the classical polynomial link invariants. We also describe the combinatorial invariant introduced by Chekanov and Pushkar which allowed them to prove Arnold's conjecture on the necessity of four-cusp curves in generic eversions of a circular front in the plane.

1 Arnold's conjecture

A generic plane wave front is a curve whose only singularities are transversal double points and semi-cubical cusps. In this paper, all fronts will be co-oriented (hence, in particular, each of them will have an even number of cusps).

Figure 1 shows the family of equidistants of an ellipse. The fronts are naturally co-oriented by the direction of propagation. The family is an example of a generic eversion of a circular front: starting with a non-singular curve having the inward co-orientation we end up with a front co-oriented outwards.

The family of Figure 1 has two remarkable features:

(i) it does not contain fronts with so-called *dangerous self-tangencies* at which the co-orientations of the tangent branches would coincide, and

(ii) it contains curves with four cusps.

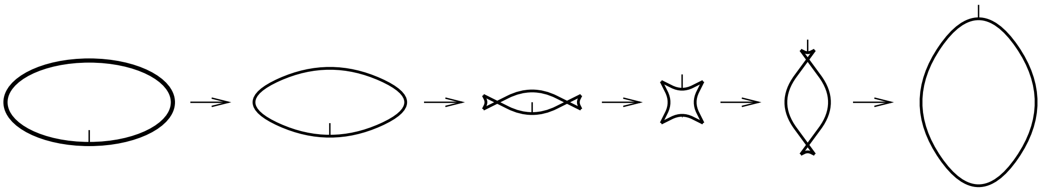


Figure 1: *Eversion of a circular front in the family of equidistants of an ellipse.*

Arnold attempted to construct a generic eversion of a circular front satisfying only condition (i) but not (ii), and failed. This led him to the following conjecture.

Conjecture 1.1 [2] *Any generic eversion of a smooth circular front in the Euclidean plane which does not contain curves with dangerous self-tangencies must contain curves with four cusps.*

Arnold's conjecture has been a considerable stimulus in the development of the theory of invariants of Legendrian knots and links over the last decade. In 1999 it was positively solved by Chekanov and Pushkar [9] who introduced a new, rather delicate invariant of Legendrian knots in the 3-space for doing this [7, 9]. In the present paper, we describe the invariant of Chekanov-Pushkar (see Section 5) as well as some of the recent results on Legendrian knots.

Remarks 1.2 (a) Prohibiting dangerous self-tangencies is a natural physical condition: speed of propagation of a front at a given point is defined by the direction of propagation.

(b) It is not so difficult to construct a generic eversion of a circular front involving only curves with at most two cusps once dangerous self-tangencies are allowed (see [2] for an example).

2 Legendrian links in $ST^*\mathbf{R}^2$ and $J^1(\mathbf{R},\mathbf{R})$

2.1 The standard contact spaces

To remind the reader of the relation between plane fronts and Legendrian links, we have first to recall a few basic notions.

A contact element at a point of a plane is a line in the tangent plane. Its *co-orientation* is a choice of one of two half-planes into which it divides the tangent plane. The manifold M of all co-oriented contact elements of the plane is the spherisation $ST^*\mathbf{R}^2$ of the cotangent bundle of the plane. It is diffeomorphic to the solid torus $\mathbf{R}^2 \times S^1$: the co-orienting normal vector is defined by the angle $\varphi \bmod 2\pi$ which it makes with a fixed direction in the plane. The manifold M has the standard contact structure defined as zeros of the 1-form $\alpha = (\cos\varphi)dx + (\sin\varphi)dy$, where (x, y) are co-ordinates on $\mathbf{R}^2$ with the positive direction of the x-axis being that fixed above (see Figure 2). We equip M with the orientation $dx \wedge dy \wedge d\varphi = -\alpha \wedge d\alpha$.

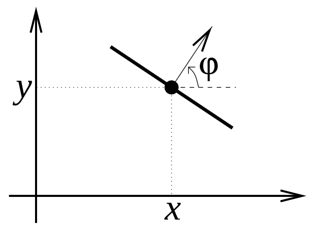


Figure 2: *Co-ordinates in the solid torus $ST^*\mathbf{R}^2$.*

The universal cover $\widetilde{M}$ of M is of one the realisations of the standard contact $\mathbf{R}^3$. However, we shall work with its isomorphic realisation which is the space $J^1(\mathbf{R},\mathbf{R})$ of 1-jets of functions on the line. The contact structure on it is $du - pdq$, where q and u parametrise the source and target lines respectively, and p is the derivative. The mapping

$$u = x\cos\varphi + y\sin\varphi \qquad p = -x\sin\varphi + y\cos\varphi \qquad q = \varphi$$

is a contactomorphism between $\widetilde{M}$ and the jet-space.

2.2 Fronts

Definition 2.1 A *Legendrian curve* in a contact 3-manifold is a mapping of a disjoint union of a finite number of circles for which the pull-back of the contact form vanishes. A *Legendrian link* is an embedded Legendrian curve.

The image F of the canonical projection of a Legendrian link L from M to the plane is called *the front of* L. An arbitrary small perturbation in the class of Legendrian links is able to bring a link in general position with respect to the canonical projection. The front of such a generic Legendrian link is exactly what we called above a generic wave front, that is, it has only transverse double points and semi-cubical cusps as its singularities.

There is a natural co-orientation at any point of a front by the co-orienting normal of the contact element $a \in L$ whose projection this point is.

A co-oriented multi-component generic plane front is the front of a unique Legendrian link in M.

A generic homotopy in the class of Legendrian *immersions* produces generic perestroikas of the front (Figure 3). Only dangerous self-tangencies of fronts correspond to topological changes in the links. A Legendrian link in the solid torus $ST^*\mathbf{R}^2$ acquires a double point and experiences the change-crossing at each of these instants (Figure 4).

For Legendrian links in $J^1(\mathbf{R}, \mathbf{R}) = \mathbf{R}^3_{q,u,p}$, the natural fronts are the images of their projections to the (q, u)-plane $J^0(\mathbf{R}, \mathbf{R})$. These are *fronts without vertical tangents* since their gradient p is always finite. The canonical co-orientation is that by the positive direction of the u-axis. Therefore, all self-tangencies of such fronts are dangerous.

Definition 2.2 An invariant of plane fronts is called a *J^+-type invariant* if it does not change under homotopies which involve no dangerous self-tangencies.

Our terminology follows the name of the first invariant of this type introduced by Arnold in [1, 2].

From the above discussion, we see that the theory of invariants of Legendrian links in $ST^*\mathbf{R}^2$ (respectively in $J^1(\mathbf{R}, \mathbf{R})$) is isomorphic to that of J^+-type invariants of fronts (respectively of fronts without vertical tangents). So in what follows we will make no distinction between invariants of Legendrian links in any of the two standard contact spaces and their lowerings to the J^+-type invariants of the relevant fronts.

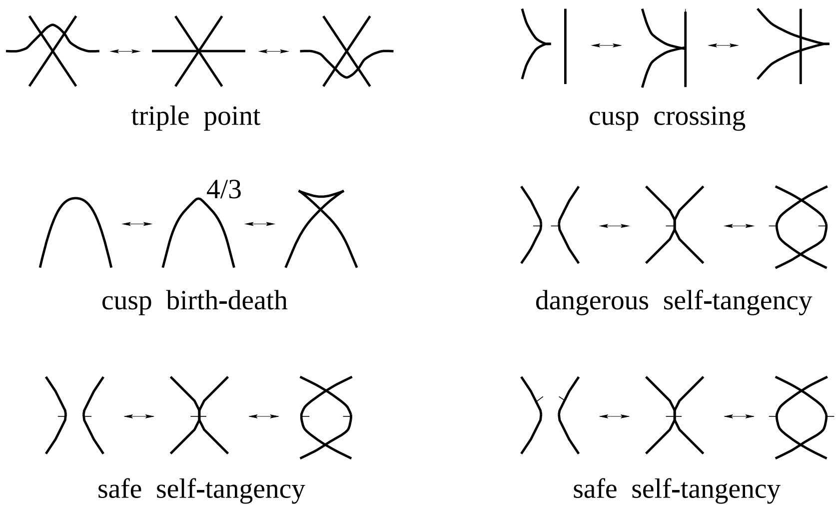


Figure 3: *Perestroikas of generic fronts.*

2.3 The Bennequin-Tabachnikov number

Any unframed link type in M and $J^1(\mathbf{R}, \mathbf{R})$ has a Legendrian representative (see, e.g. [18]). Moreover, according to [12], in the case of M it is possible to choose a representative whose front has no cusps. However, this does not hold in the framed setting.

A Legendrian link in a 3-manifold with a co-oriented contact structure has a canonical framing by the co-orienting vectors of the contact planes. In the cases of the standard 3-space and solid torus, this is isomorphic to the framing by the Legendrian lift of the front of the original link slightly shifted in the direction of its co-orientation (Figure 5).

Definition 2.3 The self-linking number β of a canonically framed oriented Legendrian link in $J^1(\mathbf{R}, \mathbf{R})$ is called the *Bennequin number* of the link.

The Bennequin number can be calculated in terms of singular points of the underlying generic oriented plane front in $J^0(\mathbf{R}, \mathbf{R}) = \mathbf{R}^2_{q,u}$. Every double point of the front, at which the q-components of the two velocities have different signs, contributes $+1$ in β. When the horizontal components

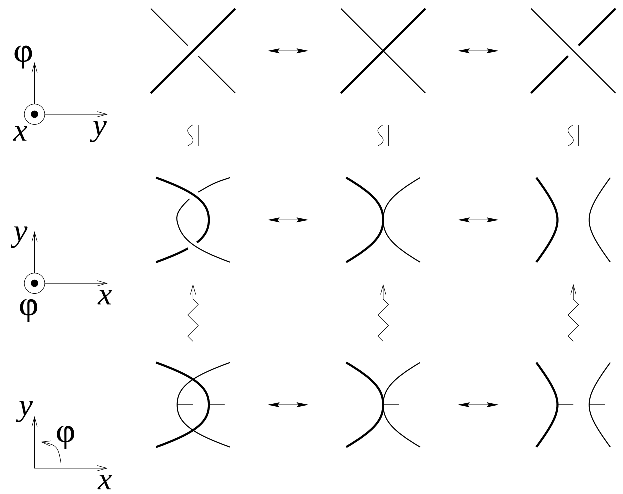


Figure 4: *A dangerous self-tangency of a front rises to the change-crossing of the Legendrian link.*

have the same sign, a double point contributes -1. The contribution from every cusp is $+1/2$.

One of the main results of [4] shows that the Bennequin number is bounded from below (for our choice of orientation) on the set of all Legendrian knots representing the same unframed knot type in $\mathbf{R}^3$:

Theorem 2.4 *For a Legendrian knot L in the standard contact $\mathbf{R}^3$,*

$$\beta(L) \geq 1 - 2g(L)\,,$$

where $g(L)$ is the genus of L.

Insertion of a two-cusp zig-zag (as in Figure 5) preserves the unframed topological type of a Legendrian knot and increases β by 1.

Examples 2.5 a) For an unknot in the standard contact $\mathbf{R}^3$, $\beta \geq 1$ [4]. This fact allowed Bennequin in [4] to show that a certain contact structure on $\mathbf{R}^3$ is not isomorphic to the standard one.

b) The estimate of the Theorem provides the same bounds for a knot and its mirror image. For example, for both the right- and left-handed trefoils it

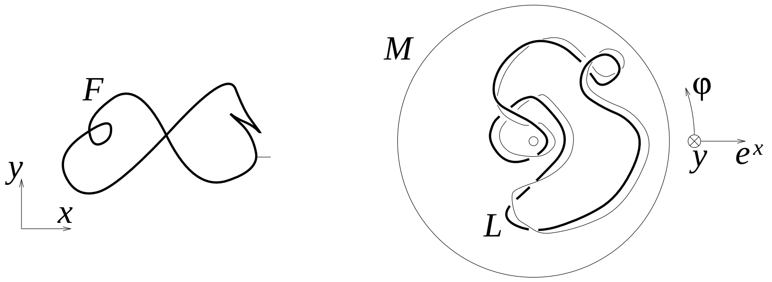


Figure 5: *The Legendrian lifting of a plane front to a canonically framed knot in the solid torus $M = ST^*\mathbf{R}^2$.*

gives $\beta \geq -1$. This bound is exact for the left-handed trefoil. In [20, 14] it was shown that for the right-handed trefoil the exact lower bound is 6.

An analog of the Bennequin number for an oriented Legendrian link L in $ST^*\mathbf{R}^2$ was introduced by Tabachnikov in [26]. He set it to be the index of intersection of the shift L' of L in the direction of the framing and a 2-film S realising the homology between L and a multiple of the fibre of the projection $ST^*\mathbf{R}^2 \to \mathbf{R}^2$ over a sufficiently distant point. The orientation of S is chosen so that L enters its boundary with a positive sign. For example, $\beta = 4$ in Figure 5.

Via the inclusion of the standard solid torus into the standard 3-space, the boundedness of the Bennequin numbers mentioned above implies a similar boundedness of the Tabachnikov numbers.

3 The Kauffman polynomial of a front

There exists a series of other estimates of the Bennequin-Tabachnikov number, coming from the classical polynomial link invariants. These estimates, in particular, differentiate between a link and its mirror image. Here we consider in detail the Kauffman polynomial.

3.1 The polynomial of Legendrian links in the standard solid torus

In [28] Turaev introduced the Kauffman polynomial of a framed non-oriented link in a solid torus. This is an element of $\mathbf{Z}[x^{\pm 1}, y^{\pm 1}, \xi_1, \xi_2, \ldots]$ uniquely defined by the relations and initial data of Figure 6. The links L_1 and L_2 there are mutually unlinked. As usual, all the links participating in each particular relation coincide except for the fragments shown. All the links are equipped with the framing which is blackboard with respect to a fixed projection of the solid torus to the annulus. The knot Ξ_3 provides a pattern for the whole series Ξ_i.

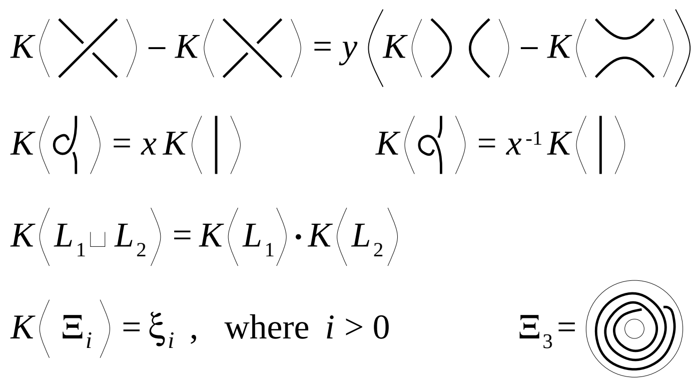


Figure 6: *Definition of the framed version of the Kauffman polynomial for links with the blackboard framing in a solid torus.*

Example 3.1 On an unknot with the trivial framing $K = \frac{x - x^{-1}}{y} + 1$.

The Legendrian lifting lowers the polynomial to generic plane fronts. Translation of the rules of Figure 6 to fronts gives rise to the rules of Figure 7. The fronts F_1 and F_2 of the third line are lying in disjoint half-planes. The relation between the Legendrian generators z_i and the blackboard generators ξ_i of Figure 6 is easily seen to be $z_i = x^i \xi_i$ [12].

Theorem 3.2 ([10]) *There exists a unique J^+-type invariant $K(F) \in \mathbf{Z}[x, y^{\pm 1}, z_1, z_2, \ldots]$ of a generic front F satisfying the relations and initial data of Figure 7.*

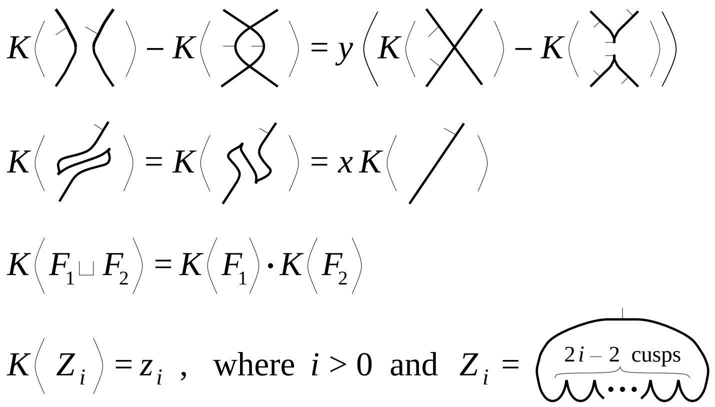
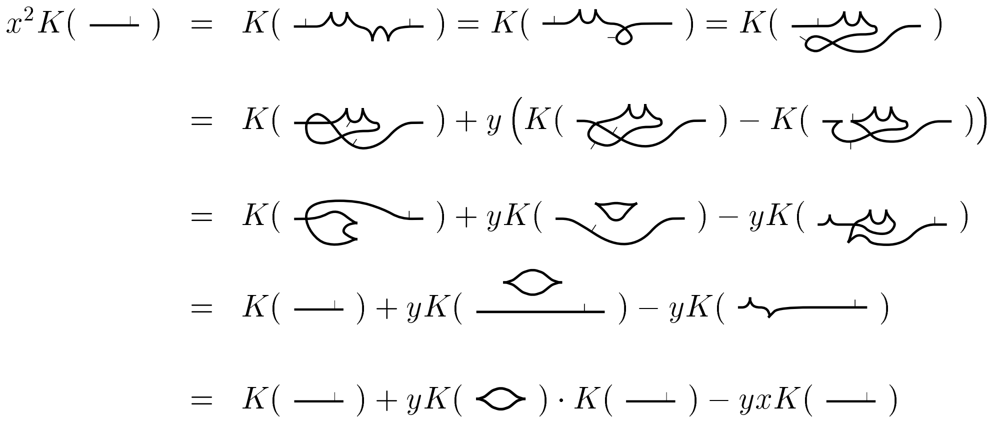


Figure 7: *Definition of the Kauffman polynomial for fronts.*

Note that there are no negative powers of the framing variable x now.

Example 3.3 Consider the lips front, with two cusps and no double points. To calculate its Kauffman polynomial, one can proceed as follows (the curves used in the second equality are connected by a homotopy involving one safe self-tangency and one cusp death):



So, $K($  $) = \frac{x^2-1}{y} + x$. Indeed, the lips front lifts to the Legendrian unknot in M with $\beta = 1$, so its Kauffman polynomial should be that of an unknot with the trivial framing times x.

3.2 The Bennequin-Tabachnikov number estimate

It follows from Theorem 3.2 that the Kauffman polynomial of a plane front is a genuine polynomial in x, not a Laurent one. This implies the following restriction on the values of the Bennequin-Tabachnikov numbers of oriented Legendrian links in the solid torus and 3-space.

For an oriented link L with the blackboard framing in a solid torus (like those considered in Turaev's definition of the Kauffman polynomial in Figure 6), we define the self-linking number $w(L)$ as the difference between the numbers of positive and negative crossing in its link diagram in the annulus (this just repeats the corresponding definition for a link in 3-space in terms of its plane diagram; the natural generalisation of this procedure to the case of non-blackboard framings gives the Tabachnikov number of a Legendrian knot). Now define the unframed version of the Kauffman polynomial as

$$K_u(L) = x^{-w(L)} K(L)\,.$$

Following [28, 21], this polynomial depends only on the unframed topological type of L. Thus we can speak about the polynomial K_u of unframed oriented links. In the case of knots the orientation does not matter.

Theorem 3.4 [10] *Let $\mathcal{L}$ be an unframed oriented link in the standard solid torus $M = ST^*\mathbf{R}^2$. Let x^k be the minimal power of the framing variable x in $K_u(\mathcal{L})$. Then the Bennequin-Tabachnikov number of any Legendrian representative of $\mathcal{L}$ is at least $-k$.*

Example 3.5 For an unknot (see Example 3.1) this coincides with the classical bound $\beta \geq 1$ [4].

Example 3.6 The Theorem implies that the minimal Bennequin-Tabachnikov number of a Legendrian representative of the basic knot Ξ_i in the solid torus (Figure 6) is that of the Legendrian lifting of the front Z_i, which is $2i - 1$. Note that the inclusion of the standard contact solid torus into the standard contact 3-sphere gives only $\beta \geq 1$ for any i: all the Ξ_i get unknotted in S^3.

An estimate for the Bennequin-Tabachnikov number similar to that of Theorem 3.4 but based on the HOMFLY polynomial for links in the solid torus also exists [10, 11]. Of course, there exist versions of both estimates for links in the standard $\mathbf{R}^3$ [14, 10, 11]. The one coming from the HOMFLY polynomial in $\mathbf{R}^3$ was easily derived [14] from one of the intermediate results of Bennequin's original paper [4] and from a result of Morton and Frank-Williams [23, 13].

The HOMFLY polynomial also provides similar estimates for the Bennequin-Tabachnikov number of *transverse* knots in $J^1(\mathbf{R},\mathbf{R})$ [14] and $ST^*\mathbf{R}^2$ [17]. Such knots are everywhere transversal to the contact structure. The Kauffman polynomial is not defined for them.

Remark 3.7 The estimate for the Bennequin number of Legendrian knots in $J^1(\mathbf{R},\mathbf{R})$ by the lowest degree of the framing variable in the mod 2 Kauffman polynomial was derived in [14] from the results of [25]. It is not known if the lowest degrees in the integer and mod 2 Kauffman polynomials for $\mathbf{R}^3$ may differ. In all the examples we know they coincide. The work [27] implies that for alternating knots they are equal. See also [24].

4 Finite order invariants

One of the most powerful tools of modern knot theory are Vassiliev's finite order invariants. Their definition is based on the inductive extension of invariants of ordinary knots to immersed spatial curves with a finite number of double points. In a similar way one constructs a theory of finite order invariants of Legendrian knots in our standard spaces or, equivalently, a theory of finite order J^+-type invariants of their fronts [19]. As an illustration, we shall consider now a slightly simpler case of fronts with no cusps at all, that is immersed plane curves (more details can be found in [16]). Our curves will be oriented and have just one component. We assume that the co-orientation used for the Legendrian lifting to $ST^*\mathbf{R}^2$ is obtained from the orientation by the clockwise rotation through $\pi/2$.

The list of generic perestroikas of Figure 3 now reduces to just three cases: triple point, *direct* self-tangency (when the two velocity vectors at the self-tangency point have the same direction) and *inverse* self-tangency (when the velocities are opposite). Each of the three degenerations defines a hypersurface in the space of all C^∞-immersions $S^1 \to \mathbf{R}^2$ (see Arnold's papers [1, 2] for details on order one invariants *Strangeness*, J^+ and J^-

dual to these hypersurfaces). J^+-type invariants which we are considering are invariants of plane curves without direct self-tangencies and do *not* change in homotopies crossing only the first and third hypersurfaces.

In the spirit of the Vassiliev theory, a J^+-type invariant extends inductively to curves with finitely many quadratic direct self-tangencies by setting

A J^+-type invariant is said to be of *order at most* n if its extension vanishes on any regular curve with n quadratic direct self-tangencies.

It is not very difficult to see that the extensions are subject to the 2- and 4-term relations of Figure 8.

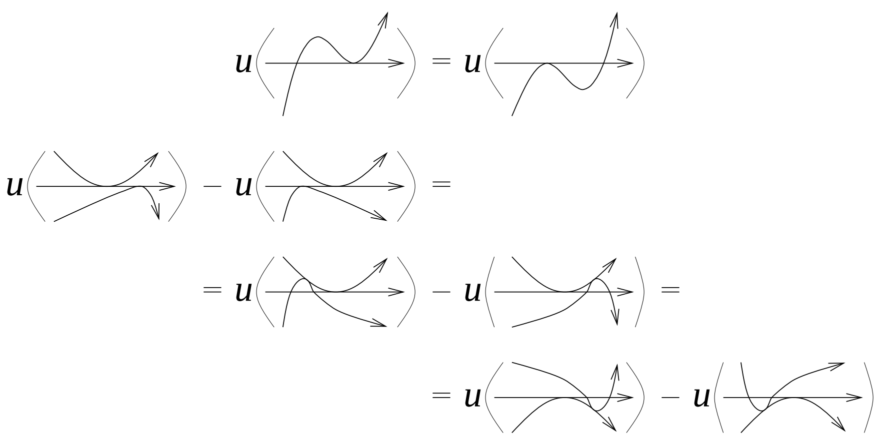


Figure 8: *Relations for extended invariants: 2-term (the top line) and 4-term.*

After raising plane curves to Legendrian curves in the solid torus $ST^*\mathbf{R}^2$, the 4-term relation becomes the 4-term relation of the theory of knots in a solid torus (ST, for short) [15]. Moreover, the following theorem is valid.

Theorem 4.1 [16] *The graded spaces of finite order complex-valued invariants of oriented framed knots in a solid torus and of oriented plane curves without direct self-tangencies are isomorphic. The isomorphism is provided by the Legendrian lift of plane curves to the solid torus* $ST^*\mathbf{R}^2$.

The grading here is that by the order of an invariant.

The graded space of complex-valued finite order invariants of oriented framed knots in ST was described in [15]. It turned out to be isomorphic to the graded **C**-linear space $\mathcal{M}$ spanned by all marked chord diagrams on a circle modulo the marked 4-term relation. $\mathcal{M}$ is graded by the number of chords in a diagram.

The above-mentioned diagram marking is actually the marking by the fundamental group **Z** of the solid torus. Namely, consider an immersion of an oriented circle into ST with a finite number of double points. As usual, on the oriented source circle, we connect the two preimages of a double point by the chord. We mark each chord on both sides by the classes of the images of the subtended halves of the circle in $\pi_1(\mathrm{ST})$. We mark the whole circle with the fundamental class of its image. The sum of the two markings on a chord is the marking of the circle. The space $\mathcal{M}$ is spanned by all the chord diagrams with this restriction on the marking.

For regular plane curves, the marking described is by the Whitney winding numbers of a curve and of the arcs into which the curve is cut by its direct self-tangencies (Figure 9). Every marked chord diagram is the diagram of a

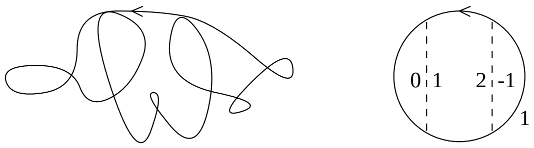
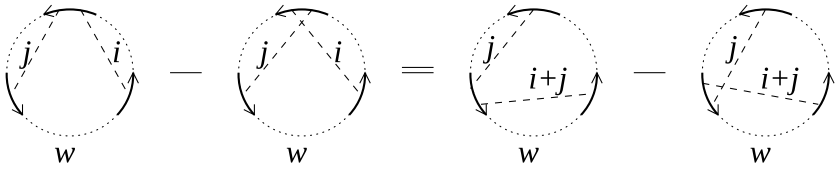


Figure 9: *A plane curve and its marked chord diagram.*

plane curve. Moreover, the set of immersed plane curves whose direct self-tangencies are subject to a given marked chord diagram is connected [16], due to the Whitney-Graustein theorem [30] and the '2-term-relation' move. So, the value of an order n J^+-type invariant on an immersed plane curve with n direct self-tangencies depends only the marked chord diagram of this curve. In this case, the 4-term relation of Figure 8 implies the marked 4-term relation on marked diagrams which is exactly the one in the definition of the graded space $\mathcal{M}$:



Here we mark the chords just on one side since the circle marking implies the rest of the data. With all the markings omitted, the relation is exactly the ordinary 4-term relation of the Vassiliev theory for knots in $\mathbf{R}^3$ [29, 3, 5, 22].

Remark 4.2 As was shown in [12] and independently by A. Shumakovich (unpublished), the Legendrian lift of regular curves provides representatives for all unframed topological types of knots in the solid torus $M = ST^*\mathbf{R}^2$ (the question of the existence of such realisations was formulated by Arnold in [2]). The same is of course valid for the lift of regular plane curves with zero winding number to $\widetilde{M} = \mathbf{R}^3$ and unframed knots in $\mathbf{R}^3$. However, the Bennequin-Tabachnikov number β of a lifted regular curve is always odd. Moreover, it seems that the lower bound for β in this case must be (as it may a priori be) higher than for lifted fronts with cusps. For example, for 'regular' representatives of right-handed trefoils in $\widetilde{M}$, β apparently cannot be less than 9 [12], while in the general case the minimum of 6 is achieved.

Remark 4.3 The 4-term relation for Vassiliev type invariants of knots in 3-manifolds comes from the bifurcations of a triple point on a singular knot. The 4-term relation for invariants of curves on surfaces comes from the bifurcations of a triple self-tangency point. Both relations are locally isomorphic. In Figure 10 we show the bifurcation diagrams (that is, germs of generic sections of the hypersurfaces of non-generic maps in the corresponding spaces of mappings) of the two degenerations. The diagrams are not diffeomorphic.

Now consider J^+-type finite order invariants of cusped co-oriented one-component plane fronts. Such a generalisation turns out to add only one more invariant, of order 0, to finite order invariants of framed knots in ST [19]. The extra invariant is the *Maslov index* μ of a front: μ is twice the number of rotations made in the contact plane by the velocity vector of the Legendrian knot (the contact structure on $ST^*\mathbf{R}^2$ is parallelisable, hence this number is well-defined). Thus finite type invariants cannot distinguish Legendrian knots in $ST^*\mathbf{R}^2$ which are topologically isotopic and have the same Bennequin-Tabachnikov number and Maslov index. A similar result for Legendrian knots in the standard contact 3-space was proved in [14].

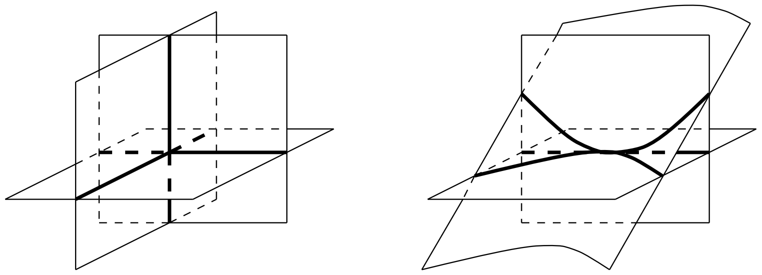


Figure 10: *Bifurcation diagrams of a triple point of a space curve and of a triple self-tangency of a plane curve.*

5 The Chekanov-Pushkar combinatorial invariant of Legendrian links in the standard contact three-space

The problem of finding invariants that could distinguish Legendrian knots which are not Legendrian isotopic but have the same topological type, Bennequin number and Maslov index is very hard. The first invariant that is able to do this in the standard 3-space $J^1(\mathbf{R},\mathbf{R}) = \mathbf{R}^3_{q,u,p}$ was introduced by Chekanov in 1997 [6, 7]. In 1999 Chekanov and Pushkar defined the second invariant of this kind [7, 8, 9]. This section concentrates on their construction.

Let $F \subset J^0(\mathbf{R},\mathbf{R}) = \mathbf{R}^2_{q,u}$ be a one-component front without vertical tangents, which is *q-generic* that is, all of its singular points (cusps and transversal self-intersections) have distinct q-coordinates.

Consider a *decomposition* of F,

$$F = X_1 \cup \ldots \cup \ldots X_n \,,$$

into a set of closed curves with a finite number of mutual intersections.

We call a decomposition *admissible* if the four conditions (a)-(d) below hold.

(a) *Each X_i bounds a topological disc: $X_i = \partial B_i$.*

(b) *Every vertical slice $B_{iq} = B_i \cap \{q = const\}$ of each disc is either an interval, or a single point (in which case we allow it to be only a cusp of F),* *or empty.*

Condition (b) in particular implies that the number n of the curves X_i in our decomposition is half the number of cusps of F. It also implies that double points of F are of two types shown in Figure 11, which we call *switching* and *non-switching*. The remaining two conditions provide constraints on switching double points.

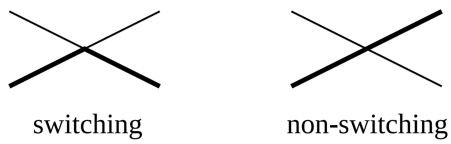


Figure 11: *Types of double points.*

(c) *Sufficiently close to a switching double point of curves X_i and X_j, the endpoints of the corresponding intervals B_{iq} and B_{jq} do not alternate, that is, the intersection of these intervals is either empty or coincides with one of them.*

To formulate the final condition, we have to introduce the notion of a *Maslov potential* of F. Recall that the Maslov index of F is equal to the difference between the numbers of its ascending and descending cusps. Consistently with this, a Maslov potential of F is a $\mathrm{mod}\mu(F)$ integer marking of smooth branches of F joining its cusps, such that at each cusp the marking of the locally upper branch is the marking of the locally lower branch plus 1. Such a marking is uniquely defined up to addition of a constant.

We call a double point of F *Maslov* if the values of the potential on the two branches of F meeting at this point are equal.

(d) *Each switching double point is Maslov.*

Example 5.1 The first three fronts in Figure 12 have one admissible decomposition each. The forth has none (like any front with the zig-zag fragment). The last front has three admissible decompositions: two decompositions with one switching double point and one decomposition with three switches.

Consider a function Θ on the set $Adm(F)$ of admissible decompositions of a front F, which is the difference between the number n of the curves in a decomposition and the number of switching double points in it.

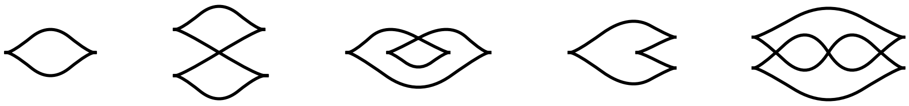


Figure 12: *Sample fronts with respectively 1, 1, 1, 0 and 3 admissible decompositions.*

Theorem 5.2 [7, 9] *Let L and L' be two Legendrian knots in $J^1(\mathbf{R}, \mathbf{R})$, which are Legendrian isotopic. Assume that the corresponding fronts, F and F' are q-generic. Then there exists a one-to-one mapping from $Adm(F)$ to $Adm(F')$ preserving Θ.*

Example 5.3 [7, 9] The two Legendrian knots whose fronts are shown in Figure 13 have the same topological type (both are 5_2-knots), the same Maslov index $\mu = 0$ and the same Bennequin number $\beta = -1$. However, the knots are not Legendrian isotopic: the first front has only one admissible decomposition (with all its four Maslov double points switching) while the second front has two (one with all six double points switching and one with four switches).

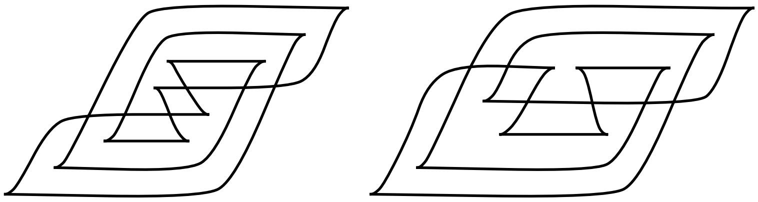


Figure 13: *Fronts of two Legendrian representatives of the 5_2-knot having the same $\mu = 0$ and $\beta = -1$, but which are not Legendrian isotopic due to Theorem 5.2.*

Chekanov and Pushkar have also proved a version of Theorem 5.2 for fronts of Legendrian knots in $ST^*\mathbf{R}^2$. It is that version which implies a positive solution of Arnold's Conjecture 1.1. The details will appear in [9].

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