The Homfly polynomial of the decorated Hopf link

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Abstract

The main goal is to find the Homfly polynomial of a link formed by decorating each component of the Hopf link with the closure of a directly oriented tangle. Such decorations are spanned in the Homfly skein of the annulus by elements Q_λ, depending on partitions λ. We show how the 2-variable Homfly invariant $\langle \lambda, \mu \rangle$ of the Hopf link arising from decorations Q_λ and Q_μ can be found from the Schur symmetric function s_μ of an explicit power series depending on λ. We show also that the quantum invariant of the Hopf link coloured by irreducible $sl(N)_q$ modules V_λ and V_μ, which is a 1-variable specialisation of $\langle \lambda, \mu \rangle$, can be expressed in terms of an $N \times N$ minor of the Vandermonde matrix (q^{ij}).

Keywords: skein theory; Hopf link; Homfly polynomial; quantum $sl(N)$ invariants; symmetric functions; Schur functions; annulus; Hecke algebras.

Introduction.

The roots of this paper lie in the work of Morton and Strickland, [12], where the central role of the Hopf link in studying the invariants of satellite knots became clear in the context of the coloured Jones invariants. The Hopf link invariants also play a crucial part in the construction of 3-manifold invariants based on the Jones polynomial. As noted in [12], the behaviour of the Hopf link invariants for quantum groups such as $sl(N)_q$ was anticipated to be at the heart of constructions of corresponding 3-manifold invariants for other values of $N > 2$, as borne out by subsequent work such as that of Kohno and Takata [5].

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It is of considerable interest to find a general formula for the Homfly polynomial of the Hopf link in which each component is decorated by, for example, a closed braid. This will be a Laurent polynomial in the parameters v and z, as used in [11], whose evaluation at the '$sl(N)$ specialisation', with $v = s^{-N}$ and $z = s - s^{-1}$, gives a Laurent polynomial in $q = s^2$ which can be viewed in terms of $sl(N)_q$ invariants of the Hopf link.

In this paper we give a formula in theorem 4.5 to determine the Homfly polynomial $\langle \lambda, \mu \rangle$ of the Hopf link with components decorated by the closures Q_λ, Q_μ of the Gyoja-Aiston idempotents, [2, 1], corresponding to any two partitions λ, μ. Invariants determined using these decorations behave simply under change of framing. The set $\{Q_\lambda\}$ spans a large subspace in the Homfly skein of the annulus, including all elements represented by closed braids or more general closed tangles without reverse strings. Our calculations therefore will find in principle the Homfly polynomial of satellites of the Hopf link with any such decorations and any framing.

As a key step in establishing the formula we present an attractive expression in theorem 4.2 for the $sl(N)$ specialisation of $\langle \lambda, \mu \rangle$ as a Laurent polynomial in the single variable q in terms of an $N \times N$ minor of the Vandermonde matrix (q^{ij}), when λ and μ each have at most N parts. This Laurent polynomial in q also determines, up to an explicit power of q, the $sl(N)_q$ invariant of the Hopf link when its components are coloured by the irreducible $sl(N)_q$ modules V_λ and V_μ.

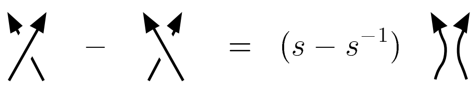
Much of our work was originally inspired by a similar expression for the further specialisation of this determinant to $q = \exp(2\pi i/r)$ used by Kohno and Takata [5]. Our proofs for generic v and s do not draw on their work and hence give an alternative skein theoretic way to interpret their formulae.

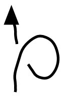
To describe and analyse the invariants under consideration we apply Homfly skein theory to the skein of the annulus and the Hecke algebras, coupled with methods from the classical theory of symmetric functions. We start with a brief review of this material before applying it to the Hopf link.

Much of the material in this paper appeared originally in the thesis of Lukac, [6].

1 The skein models.

The account here largely follows those of [11], [6] and [10]. The framed Homfly skein relations, in their simplest form, are



and  $= v^{-1}$  .

The coefficient ring can be taken as $\Lambda = \mathbf{Z}[v^{\pm 1}, s^{\pm 1}]$ with monomials in $\{s^k - s^{-k} : k \geq 0\}$ admitted as denominators. The framed Homfly skein $\mathcal{S}(F)$ of a planar surface F, with some designated input and output boundary points, is defined to be Λ-linear combinations of oriented tangles in F, modulo these two local relations, and Reidemeister moves II and III.

The empty tangle is admitted when F has no boundary points. We include the local relation which allows the removal of a null-homotopic closed curve without crossings, at the expense of multiplication by the scalar $\delta = \dfrac{v^{-1} - v}{s - s^{-1}}$. This is in fact a consequence of the main relations, except when removing the curve leaves only the empty diagram.

1.1 The plane

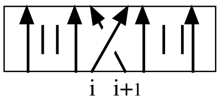
When $F = \mathbf{R}^2$ every element can be represented uniquely as a scalar multiple of the empty diagram. For a diagram D the resulting scalar $\chi(D) \in \Lambda$ is the framed Homfly polynomial of D. Taking $\chi^u(D) = v^{wr(D)}\chi(D)$, where $wr(D)$ is the writhe of the diagram, gives a scalar which is invariant under all Reidemeister moves. It is the Homfly polynomial of D, as a function of v and s, defined by the local relation

v^{-1}  $- \; v$  $= (s - s^{-1})$  ,

and normalised to have the value 1 on the empty diagram. Then $\chi^u(D) = \delta P(D)$, except when D is empty, where P is the more traditional Homfly polynomial defined as the ambient isotopy invariant which satisfies the local relation above and takes the value 1 on the unknot .

1.2 The Hecke algebras

Write R_n^n for the skein $\mathcal{S}(F)$ of n-tangles, where F is a rectangle with n inputs at the bottom and n outputs at the top. Composing n-tangles induces a product which makes R_n^n into an algebra. It has a linear basis of $n!$ elements, and is isomorphic to the Hecke algebra $H_n(z)$, with coefficients extended to the ring Λ. This algebra has a presentation generated by the elementary braids

$$\sigma_i \ = $$ 

subject to the braid relations

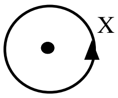
$$\begin{aligned} \sigma_i \sigma_j &= \sigma_j \sigma_i, \quad |i-j| > 1, \\ \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1}, \end{aligned}$$

and the quadratic relations $\sigma_i^2 = z\sigma_i + 1$, with $z = s - s^{-1}$, giving the alternative form $(\sigma_i - s)(\sigma_i + s^{-1}) = 0$.

A simple adjustment of the skein relations, as in [11], allows for a skein model of H_n whose parameters can be readily adapted to match any of the different appearances of the algebra, [10].

1.3 The annulus.

The Homfly skein of the annulus, $\mathcal{C}$, as discussed in [9] and originally in [14], is defined to be linear combinations of diagrams in the annulus, modulo the framed Homfly skein relations. An element $X \in \mathcal{C}$ will be indicated on a diagram as



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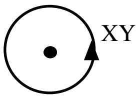
The skein $\mathcal{C}$ has a product induced by placing one annulus outside another, under which $\mathcal{C}$ becomes a commutative algebra;



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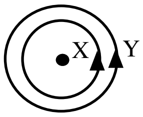

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There is an evaluation map $\langle\ \rangle : \mathcal{C} \to \Lambda$, induced by the inclusion of a standard annulus in the plane, in which $\langle X \rangle$ is defined as the framed Homfly polynomial of X when regarded as a diagram in the plane. Since the framed Homfly polynomial with our normalisation is multiplicative on split diagrams, the evaluation map $\langle\ \rangle$ is a ring homomorphism.

1.4 Partitions and Young diagrams

The skeins H_n and $\mathcal{C}$ contain important elements determined by partitions λ. We use here the terminology of Macdonald [8], where extensive algebraic and combinatorial properties of partitions can be found.

Definition. A *partition* λ is a non-increasing sequence of positive integers, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$, called the *parts* of λ.

Write $l(\lambda) = k$ for the number of parts of λ and $|\lambda| = \lambda_1 + \cdots + \lambda_k$ for the sum of its parts. A partition λ with $|\lambda| = n$ is called a *partition of* n.

Partitions are often pictured by Young diagrams. The diagram of λ is an array of cells arranged in $l(\lambda)$ rows, with λ_i adjacent cells in row i, starting always with a cell in column 1.

Each partition λ determines a *conjugate* partition λ' with λ_1 parts whose diagram is given by interchanging the rows and columns of the diagram of λ. The diagram of λ thus has λ'_j cells in column j.

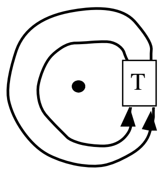
Definition. The *content* $c(x)$ of the cell x in row i and column j in the diagram of λ is defined by $c(x) = j - i$.

The *hook-length* $h(x)$ of x is the total number of cells in the diagram of λ which lie to the right of x in row i, or below x in column j, including x itself.

Thus $h(x) = \lambda_i + \lambda'_j - i - j + 1$.

1.5 Interrelations

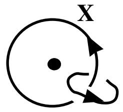
The best known relation of $\mathcal{C}$ with the Hecke algebra H_n is the closure map $R^n_n \to \mathcal{C}$, induced by taking a tangle T to its closure $\hat{T}$ in the annulus, defined by

$\hat{T} =$  .

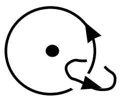
This is a linear map, whose image we call $\mathcal{C}_n$.

Elements of the skein $\mathcal{C}$ can be used to decorate closed curves in a framed diagram in any F to produce new skein elements of $\mathcal{S}(F)$. We shall restrict the decorations in our work to elements drawn from $\mathcal{C}_n$ for some n. We write $\mathcal{C}_+ \subset \mathcal{C}$ for the subspace spanned by $\cup_{n\geq 0}\mathcal{C}_n$.

A simple skein theory construction determines a natural linear map $\varphi : \mathcal{C} \to \mathcal{C}$, induced by taking a diagram X in the annulus and linking it once with a simple loop to get

$\varphi(X) =$  .

This construction can be interpreted as an example of the decoration technique above, applied to one loop of the diagram



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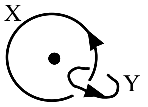
We can extend this by decorating the other loop with any $Y \in \mathcal{C}$ to give a linear map $\varphi_Y : \mathcal{C} \to \mathcal{C}$. Then

$$\varphi_Y(X) \ =$$



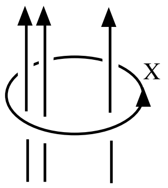
$$\in \mathcal{C}.$$

The map φ is then φ_{c_1} where $c_1 \in \mathcal{C}$ is represented by a single closed curve following the oriented core of the annulus.

Evaluation of $\varphi_Y(X)$ gives a scalar which we denote by $\langle X, Y\rangle$, and which forms the main focus of this paper.

The map $\psi_n : \mathcal{C} \to H_n$ induced by decorating the closed curve in the diagram below by an element $X \in \mathcal{C}$ is readily seen to be a ring homomorphism, whose image

$$\psi_n(X) \ =$$



$$\in H_n$$

lies in the centre of H_n.

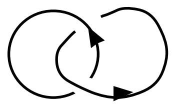
There is a set of quasi-idempotent elements, $e_\lambda \in H_n$, one for each partition λ of n, known as the Gyoja-Aiston idempotents. They were originally described algebraically by Gyoja [2], while skein pictures of these based on the Young diagram for λ can be found in [1] or [11]. The skein theory version displays a nice 'internal stability' under multiplication which shows readily that Ze_λ is a scalar multiple of e_λ for any *central* element $Z \in H_n$. We can then define a ring homomorphism $t_\lambda : \mathcal{C} \to \Lambda$ by the formula $\psi_n(Y)e_\lambda = t_\lambda(Y)e_\lambda$, using the homomorphism ψ_n from $\mathcal{C}$ to the centre of H_n.

Define $Q_\lambda \in \mathcal{C}$ to be the closure of the genuine idempotent $\frac{1}{\alpha_\lambda}e_\lambda$, where $e_\lambda^2 = \alpha_\lambda e_\lambda$, as in [11] or [6]. Then clearly Q_λ is an eigenvector of φ_Y with eigenvalue $t_\lambda(Y)$ for every $Y \in \mathcal{C}$. We make extensive use of the elements Q_λ, especially where the diagram of λ is a single column or row. We write c_i for Q_λ where λ is the partition with i parts all equal to 1, and d_j where

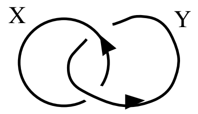
λ is the partition with one part j, and take $c_0 = d_0 = 1 \in \mathcal{C}$ represented by the empty diagram. It is shown in [1] that $\mathcal{C}_+$ is the polynomial algebra generated by the column elements $\{c_i\}$, or equally by the row elements $\{d_j\}$.

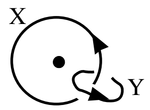
1.6 The Hopf link

We consider the Hopf link with linking number 1 as shown here.



Let X and Y be any elements of the skein $\mathcal{C}$ of the annulus. We write $\langle X, Y\rangle$ for the framed Homfly polynomial of the Hopf link with decorations X and Y on its components, giving

$\langle X, Y\rangle = \chi\Bigg($ 

$\Bigg) = \Big\langle$ 

$\Big\rangle = \langle \varphi_Y(X)\rangle\,.$

Clearly $\langle Y, X\rangle = \langle X, Y\rangle$.

Our aim is to determine $\langle X, Y\rangle$ for any $X, Y \in \mathcal{C}_+$. Now $\mathcal{C}_n$ is spanned by the elements Q_λ with $|\lambda| = n$, for each $n \geq 0$. It is then enough to find $\langle Q_\lambda, Q_\mu\rangle$ for all λ, μ. We make use of the homomorphism t_λ of the previous section.

Lemma 1.1 *We can write*

$$t_\lambda(Y) = \langle Q_\lambda, Y\rangle / \langle Q_\lambda\rangle\,.$$

Proof: Suppose that λ has n cells. Then $\psi_n(Y)e_\lambda = t_\lambda(Y)e_\lambda$. Apply the closure map $\hat{}: H_n \to \mathcal{C}$ to both sides. The left-hand side becomes $\varphi_Y(\hat{e}_\lambda)$. Now apply the evaluation map $\langle\ \rangle$ to both sides, using the fact that $\hat{e}_\lambda = \alpha_\lambda Q_\lambda$, to get

$$\langle \alpha_\lambda Q_\lambda, Y\rangle = t_\lambda(Y)\,\langle \alpha_\lambda Q_\lambda\rangle\,.$$

This gives the expression for $t_\lambda(Y)$, since $\alpha_\lambda \neq 0$. □

Corollary 1.2 *For any elements X and Y of $\mathcal{C}$ and any partition λ we have*

$$\langle Q_\lambda, X\rangle\langle Q_\lambda, Y\rangle = \langle Q_\lambda\rangle\,\langle Q_\lambda, XY\rangle.$$

Proof: Use the formula of lemma 1.1 and the fact that t_λ is a homomorphism. $\square$

Since any $Y \in \mathcal{C}_+$ can be written as a polynomial in the skein elements $\{c_i\}$ we only need to know the values of $\langle Q_\lambda, c_i\rangle$ for integers $i \geq 0$ in order to compute $\langle Q_\lambda, Y\rangle$ for all $Y \in \mathcal{C}_+$. In particular we can then compute all invariants $\langle Q_\lambda, Q_\mu\rangle$. Hence, it is useful to define a formal power series

$$E_\lambda(t) = \sum_{i\geq 0} t_\lambda(c_i)t^i = \frac{1}{\langle Q_\lambda\rangle} \sum_{i\geq 0} \langle Q_\lambda, c_i\rangle t^i \tag{1}$$

for any partition λ, which we study further in the next section.

2 Symmetric functions and the skein of the annulus.

In this section we recall some explicit results about elements in the Hecke algebras and their closure in $\mathcal{C}$, and their interpretation in the context of symmetric functions, following the methods of Macdonald [8].

2.1 Formal Schur functions

We follow Macdonald in defining the Schur functions of a formal power series $E(t) = 1 + \sum_{i=1}^{\infty} e_i t^i$ with coefficients e_i in a commutative ring. When $E(t)$ is a polynomial with a formal factorisation as $E(t) = \prod_{j=1}^{N}(1 + x_j t)$ then any partition λ of n determines the classical Schur symmetric function $s_\lambda(x_1, \ldots, x_N)$ in terms of determinants whose entries are powers of $\{x_j\}$, given explicitly by equation (14). This can be written as a polynomial in the elementary symmetric functions $\{e_i\}$ of $x_1, \ldots, x_N$, which are the coefficients of $E(t)$, by means of the Jacobi-Trudy formula. So long as $N \geq n$ this polynomial in $\{e_i\}$ is independent of N. Macdonald defines the Schur function $s_\lambda(E(t))$ for a formal power series to be this polynomial in the coefficients $\{e_i\}$, described explicitly in section 4.2.

The coefficient e_i itself is the Schur function $s_\lambda(E(t))$ for the partition $\lambda = 1^i = (1, \ldots, 1)$, represented by the Young diagram consisting of a single column with i cells. Equally, when the diagram of λ is a single row with j cells then $s_\lambda(E(t)) = h_j$, where h_j is the complete symmetric function of degree j in $x_1, \ldots, x_N$ in the case of the polynomial $E(t)$ above. In any case, the power series $H(t) = 1 + \sum h_j t^j$ satisfies the power series equation $E(-t)H(t) = 1$.

Since $s_\lambda(E(t))$ is a polynomial in $\{e_i\}$ depending only on λ it is clear that if R and S are any commutative rings and we apply a ring homomorphism $\rho : R \to S$ to the coefficients $\{e_i\} \in R$ to get the power series $\rho(E(t))$ with coefficients $\{\rho(e_i)\} \in S$ then

$$s_\lambda(\rho(E(t))) = \rho(s_\lambda(E(t))),$$

for every λ.

We make use later of the homogeneity of the Schur function s_λ in the form

$$s_\lambda(E(\alpha t)) = \alpha^{|\lambda|} s_\lambda(E(t)), \tag{2}$$

where λ has $|\lambda|$ cells.

2.2 Applications in the skein of the annulus

The elements Q_λ have a very nice interpretation as Schur functions in $\mathcal{C}$. Lukac showed in [7] that Q_λ can be identified with the Schur function

$$s_\lambda(\sum_{i\geq 0} c_i t^i)$$

of the series $\sum c_i t^i$ whose coefficients $c_i \in \mathcal{C}$ are the closures Q_λ of the single column idempotents where λ has i parts all equal to 1, under the convention that $c_0 = 1 \in \mathcal{C}$, represented by the empty diagram. This has been known in principle for some time, but the proof in [1] was fairly circuitous and had to refer at one point to quantum group work. Kawagoe [3] used these Schur functions and proved that they were eigenvectors of φ, but his proof too was relatively complicated. Lukac [7] gives a much more satisfactory skein proof, using a further skein-based algebra, coupled with the knowledge that the eigenvalues of $\varphi|\mathcal{C}_n$ are distinct and the elements Q_λ can essentially be characterised as eigenvectors of φ.

The fact that Q_λ is also an eigenvector of every φ_Y, which is not at all clear at first sight for the Schur function $s_\lambda(\sum c_i t^i)$, is the feature that allows us to complete our analysis here, using the consequence that t_λ is a homomorphism.

For the sake of brevity of notation we write λ in place of Q_λ as an element of $\mathcal{C}$. As Schur functions of a power series these elements multiply according to the Littlewood-Richardson rules for Young diagrams. We make use of this later when we calculate the product in $\mathcal{C}$ of the elements c_i and d_j corresponding to a single row and column respectively.

From the series $E_\lambda(t)$, defined in (1) above, we can find $\langle \lambda, \mu \rangle$ by calculating its Schur function $s_\mu(E_\lambda(t))$ and using the following lemma.

Lemma 2.1 *We have*
$$s_\mu(E_\lambda(t)) = \frac{1}{\langle\lambda\rangle}\langle\lambda,\mu\rangle$$
for any partitions λ *and* μ.

Proof: Applying the homomorphism t_λ to the power series $\sum c_i t^i$ gives $E_\lambda(t) = t_\lambda(\sum c_i t^i)$. Then
$$s_\mu(E_\lambda(t)) = t_\lambda(s_\mu(\sum c_i t^i)) = t_\lambda(\mu) = \frac{1}{\langle\lambda\rangle}\langle\lambda,\mu\rangle.$$
□

We also define
$$H_\lambda(t) = \frac{1}{\langle\lambda\rangle}\sum_{j\geq 0}\langle\lambda, d_j\rangle t^j = t_\lambda(\sum d_j t^j)$$
for any partition λ, taking $d_0 = 1$.

Lemma 2.2 *We have*
$$E_\lambda(t)H_\lambda(-t) = 1$$
for any partition λ.

Proof: Apply the homomorphism t_λ to the two power series with coefficients in $\mathcal{C}$ in the equation
$$\left(\sum_{i\geq 0} c_i t^i\right)\left(\sum_{j\geq 0} d_j(-t)^j\right) = 1$$
proved by direct skein theory in [1]. The equation follows alternatively from the fact that the elements d_j in $\mathcal{C}$ are the Schur functions corresponding to single row diagrams. □

3 The Hopf link decorated with columns and rows

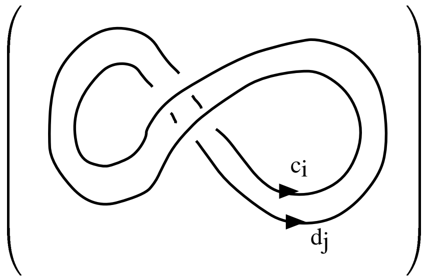
We now compute the series $E_{c_k}(t)$ for any integer $k \geq 0$. To do this, we start with a surprisingly simple formula for $\langle c_i, d_j\rangle$.

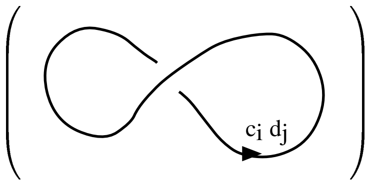
Lemma 3.1 *We have*

$$\langle c_i, d_j\rangle = \langle c_i\rangle\,\langle d_j\rangle\,\frac{v^{-1}(s^{2j}-s^{2(j-i)}+s^{-2i})-v}{v^{-1}-v}$$

for any integers $i \geq 0$ and $j \geq 0$.

Proof: We shall prove the lemma by expressing the Homfly polynomial of a certain decorated link in two different ways and comparing the results. The link in question is the 2-parallel of the unknot with framing 1 as shown below, decorated with c_i on one component and d_j on the other component. We denote its framed Homfly polynomial by R.

$R \;=\; \chi$ 

$= \chi$ 

The positive curl on n strings belongs to the centre of the Hecke algebra H_n, and so its product with any quasi-idempotent e_λ, $|\lambda| = n$, is a scalar multiple $f(\lambda)e_\lambda$. The scalar was calculated using skein theory in theorem 17 of [11] as

$$f(\lambda) = v^{-|\lambda|}s^{n_\lambda}$$

where $n_\lambda = 2\sum_{x\in\lambda} c(x)$ is twice the sum of the contents of all cells in the diagram of λ. By removing the curls on the two components of the 2-parallel we get

$$R = f(c_i)f(d_j)\langle c_i, d_j\rangle. \tag{3}$$

The other way to calculate R is to regard it as the Homfly polynomial of the unknot with framing 1 decorated by the product of c_i and d_j, as elements of $\mathcal{C}$. Since the elements Q_λ are Schur functions, by Lukac [7], they multiply in $\mathcal{C}$ according to the Littlewood-Richardson rules. We can then write $c_i d_j = \mu_{i,j+1} + \mu_{i+1,j}$ where $\mu_{a,b}$ is the simple hook Young diagram with a cells in the first column and b cells in the first row. Hence

$$R = f(\mu_{i,j+1})\,\langle\mu_{i,j+1}\rangle + f(\mu_{i+1,j})\,\langle\mu_{i+1,j}\rangle\,. \tag{4}$$

From the formula for $f(\lambda)$ above we get

$$f(c_i) = v^{-i}s^{-i(i-1)},\, f(d_j) = v^{-j}s^{j(j-1)},\, f(\mu_{a,b}) = v^{-(a+b-1)}s^{b(b-1)-a(a-1)},$$

and hence

$$f(c_{i+1}) = v^{-1}s^{-2i}f(c_i),\ \ f(d_{j+1}) = v^{-1}s^{2j}f(d_j),\ \ f(\mu_{i,j}) = vf(c_i)f(d_j).$$

The following relations were shown skein theoretically in [1].

$$\begin{aligned}
\langle c_{i+1}\rangle &= \frac{v^{-1}s^{-i} - vs^{i}}{s^{i+1} - s^{-i-1}}\langle c_i\rangle,\\
\langle d_{j+1}\rangle &= \frac{v^{-1}s^{j} - vs^{-j}}{s^{j+1} - s^{-j-1}}\langle d_j\rangle,\\
\langle \mu_{i,j}\rangle &= \frac{(s^{j} - s^{-j})(s^{i} - s^{-i})}{(v^{-1} - v)(s^{i+j-1} - s^{-i-j+1})}\langle c_i\rangle\langle d_j\rangle.
\end{aligned}$$

We can in fact deduce the product formula (10) for $\sum\langle c_i\rangle t^i$ using only the first of these, and hence the closed formula (12) for $\langle\lambda\rangle$.

Using these relations equation (4) gives

$$\begin{aligned}
R &= f(\mu_{i,j+1})\langle\mu_{i,j+1}\rangle + f(\mu_{i+1,j})\langle\mu_{i+1,j}\rangle\\
&= vf(c_i)v^{-1}s^{2j}f(d_j)\frac{(s^{i} - s^{-i})(v^{-1}s^{j} - vs^{-j})}{(v^{-1} - v)(s^{i+j} - s^{-i-j})}\langle c_i\rangle\langle d_j\rangle\\
&\quad + s^{-2i}f(c_i)f(d_j)\frac{(s^{j} - s^{-j})(v^{-1}s^{-i} - vs^{i})}{(v^{-1} - v)(s^{i+j} - s^{-i-j})}\langle c_i\rangle\langle d_j\rangle\\
&= \frac{v^{-1}(s^{2j} - s^{2(j-i)} + s^{-2i}) - v}{v^{-1} - v}f(c_i)f(d_j)\langle c_i\rangle\langle d_j\rangle. \qquad (5)
\end{aligned}$$

Equations (3) and (5) then give

$$\langle c_i, d_j\rangle = \langle c_i\rangle\langle d_j\rangle\frac{v^{-1}(s^{2j} - s^{2(j-i)} + s^{-2i}) - v}{v^{-1} - v}$$

since $f(c_i)$ and $f(d_j)$ are non-zero. □

Using our notation above, we have

Corollary 3.2

$$H_{c_k}(t) = \frac{1 - v^{-1}s^{-2k+1}t}{1 - v^{-1}st}H_\emptyset(t)$$

for any integer $k \geq 0$.

Proof: Since $H_\emptyset(t) = t_\emptyset(H(t)) = \sum_{j\geq 0}\langle d_j\rangle t^j$ we must show that

$$(1 - v^{-1}st)\frac{1}{\langle c_k\rangle}\sum_{j\geq 0}\langle c_k, d_j\rangle t^j = (1 - v^{-1}s^{-2k+1}t)\sum_{j\geq 0}\langle d_j\rangle t^j.$$

The constant terms of the power series in the above equation are equal to 1. To show that the coefficients of t^j on each side agree it is enough to show that

$$\frac{1}{\langle c_k\rangle}\langle c_k, d_j\rangle - v^{-1}s\frac{1}{\langle c_k\rangle}\langle c_k, d_{j-1}\rangle = \langle d_j\rangle - v^{-1}s^{-2k+1}\langle d_{j-1}\rangle . \tag{6}$$

By lemma 3.1 the left hand side of equation (6) is

$$\begin{aligned}&\langle d_j\rangle \frac{v^{-1}\big(s^{2j} - s^{2(j-k)} + s^{-2k}\big) - v}{v^{-1} - v}\\ &-v^{-1}s\,\langle d_{j-1}\rangle \frac{v^{-1}\big(s^{2(j-1)} - s^{2(j-1-k)} + s^{-2k}\big) - v}{v^{-1} - v}.\end{aligned}$$

Because

$$\langle d_j\rangle = \langle d_{j-1}\rangle \frac{v^{-1}s^{j-1} - vs^{-j+1}}{s^j - s^{-j}}$$

this can be rewritten as

$$\begin{aligned}\Bigg(&\frac{(v^{-1}s^{j-1} - vs^{-j+1})\big(v^{-1}\big(s^{2j} - s^{2(j-k)} + s^{-2k}\big) - v\big)}{(s^j - s^{-j})(v^{-1} - v)}\\ &-v^{-1}s\frac{v^{-1}\big(s^{2(j-1)} - s^{2(j-1-k)} + s^{-2k}\big) - v}{v^{-1} - v}\Bigg)\langle d_{j-1}\rangle .\end{aligned} \tag{7}$$

The right hand side of equation (6) is

$$\left(\frac{v^{-1}s^{j-1} - vs^{-j+1}}{s^j - s^{-j}} - v^{-1}s^{-2k+1}\right)\langle d_{j-1}\rangle . \tag{8}$$

It is straightforward to confirm that the expressions (7) and (8) are equal, and thus equation (6) follows. $\square$

An immediate consequence of corollary 3.2 and lemma 2.2 is

Corollary 3.3

$$E_{c_k}(t) = \frac{1 + v^{-1}st}{1 + v^{-1}s^{-2k+1}t}E_{\emptyset}(t)$$

for any integer $k \geq 0$.

4 The Hopf link decorated with any Young diagrams

We now in principle have a means of finding $\langle c_k, \lambda\rangle$ by calculating the Schur function $s_\lambda(E_{c_k}(t))$. This in turn gives the coefficients for the series $E_\lambda(t)$, whose Schur function s_μ finally gives $\langle \lambda, \mu\rangle$. The aim of this section is to give a simple formula for the series $E_\lambda(t) = \sum \langle \lambda, c_i\rangle t^i / \langle \lambda\rangle$ as a product of the known series $E_\emptyset(t) = \sum \langle c_i\rangle\, t^i$ and a rational function of t, in theorem 4.5. To do this we specialise the coefficients so that $E_{c_k}(t)$ becomes a polynomial, and we can apply the classical determinantal formulae for Schur functions. By comparing the coefficients in $E_\lambda(t)$ with those of our proposed product formula, and showing that they agree under sufficiently many of the specialisations we are able to deduce that they are identical.

The determinantal formulae which appear give us a very attractive expression for the specialised versions of the invariants $\langle \lambda, \mu\rangle$, which we describe in theorem 4.2.

4.1 The $sl(N)$ substitution

We use the substitution $v = s^{-N}$ to define a ring homomorphism from our coefficient ring Λ to the ring of Laurent polynomials in s with denominators $s^r - s^{-r}$. Denote the image of $\langle X, Y\rangle$ by $\langle X, Y\rangle_N$, and the image of $\langle X\rangle$ by $\langle X\rangle_N$.

The element $t_\lambda(Y)$ for $Y \in \mathcal{C}$ is an element of Λ. Applying the homomorphism to the equation $\langle \lambda\rangle\, t_\lambda(Y) = \langle \lambda, Y\rangle$ from lemma 1.1 shows that if $\langle \lambda\rangle_N = 0$ then $\langle \lambda, Y\rangle_N = 0$ also, for any Y.

We make use of some results of Macdonald to help in our evaluations. In exercise I.3.3 of [8] Macdonald observes that when a power series can be expressed as

$$E(t) = \prod_{i=0}^{\infty} \frac{1 + aq^i t}{1 + bq^i t}$$

then its Schur function s_λ is given in terms of the cells x in the diagram of λ by the formula

$$s_\lambda = q^{n(\lambda)} \prod_{x \in \lambda} \frac{a - bq^{c(x)}}{1 - q^{h(x)}}, \tag{9}$$

where $c(x)$ is the content of x, $h(x)$ is its hook length, and $n(\lambda)$ is given by

$$2n(\lambda) = \sum_{x \in \lambda} (h(x) - c(x) - 1).$$

The coefficients of the series $\langle \sum c_r t^r \rangle = \sum \langle c_r \rangle t^r$ were shown by Aiston in chapter 4 of [1] to satisfy the simple recursive relation

$$\langle c_{r+1} \rangle = \frac{v^{-1}s^{-r} - vs^r}{s^{r+1} - s^{-r-1}} \langle c_r \rangle ,$$

leading to the product formula

$$\sum \langle c_r \rangle t^r = \prod_{i=0}^{\infty} \frac{1 + vs^{2i+1}t}{1 + v^{-1}s^{2i+1}t}. \tag{10}$$

Applying Macdonald's formula (9) gives

$$\langle \lambda \rangle = s_\lambda \Big\langle \sum c_r t^r \Big\rangle = q^{n(\lambda)} \prod_{x \in \lambda} \frac{vs - v^{-1}sq^{c(x)}}{1 - q^{h(x)}} \tag{11}$$

taking $q = s^2$, $a = vs$ and $b = v^{-1}s$. This can be rewritten in the form

$$\langle \lambda \rangle = \prod_{x \in \lambda} \frac{v^{-1}s^{c(x)} - vs^{-c(x)}}{s^{h(x)} - s^{-h(x)}}, \tag{12}$$

which can equally be derived by skein theory for $\langle Q_\lambda \rangle$ from [11], without assuming that Q_λ can be expressed as a Schur function.

It is clear that the specialisation $v = s^{-N}$ gives

$$\begin{aligned} \langle \lambda \rangle_N = 0 \quad &\Longleftrightarrow \quad \text{some cell } x \in \lambda \text{ has content } -N \\ &\Longleftrightarrow \quad \lambda \text{ has } > N \text{ parts.} \end{aligned} \tag{13}$$

Consequently when $N \geq l(\lambda)$ we can always write $t_\lambda(Y)_N = \dfrac{\langle \lambda, Y \rangle_N}{\langle \lambda \rangle_N}$.

Lemma 4.1 *Let λ be a partition and let $N \geq l(\lambda)$. Then the specialisation $E^N_\lambda(t)$ is a polynomial in t of degree $\leq N$.*

Proof: We must show that $t_\lambda(c_r)_N = 0$ for $r > N$. Since $\langle \lambda \rangle_N \neq 0$ it is enough to show that $\langle \lambda, c_r \rangle_N = 0$ for $r > N$.

By (13) we know that $\langle c_r \rangle_N = 0$ for $r > N$, and so

$$\langle c_r, \lambda \rangle_N = t_{c_r}(\lambda)_N \langle c_r \rangle_N = 0.$$

The result follows since $\langle \lambda, c_r \rangle = \langle c_r, \lambda \rangle$. □

We now find a factorisation of this polynomial $E^N_\lambda(t)$ into linear factors in lemma 4.3, and use the Schur function $s_\mu(E^N_\lambda(t))$ to derive a determinantal formula for $\langle \lambda, \mu \rangle_N$ in theorem 4.2.

Recall first the determinantal description of the Schur function

$$s_\lambda(x_1, \ldots, x_N) = s_\lambda\left(\prod_{i=1}^{N}(1 + x_i t)\right).$$

For fixed $N \geq l(\lambda)$ define the index set

$$I_\lambda = \{\lambda_1 + N - 1, \lambda_2 + N - 2, \ldots, \lambda_N\}$$

consisting of N distinct integers ≥ 0.

Number the columns of the $N \times \infty$ matrix

$$\begin{pmatrix} 1 & x_1 & x_1^2 & \cdots \\ 1 & x_2 & x_2^2 & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_N & x_N^2 & \cdots \end{pmatrix}$$

starting with 0, so that its (i,j) entry is x_i^j.

Write $P_\lambda^N(\mathbf{x})$ for the $N \times N$ minor determined by the columns I_λ. The leading minor is $P_\emptyset^N(\mathbf{x})$, and the Schur function is given by

$$s_\lambda(x_1, \ldots, x_N) = P_\lambda^N(\mathbf{x})/P_\emptyset^N(\mathbf{x}). \tag{14}$$

The formula for $\langle\lambda, \mu\rangle_N$ can be found in terms of a function of $q = s^2$ defined similarly by $N \times N$ minors of the infinite Vandermonde matrix

$$V = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & q & q^2 & q^3 & q^4 & \cdots \\ 1 & q^2 & q^4 & q^6 & q^8 & \cdots \\ 1 & q^3 & q^6 & q^9 & q^{12} & \cdots \\ 1 & q^4 & q^8 & q^{12} & q^{16} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with (i,j) entry q^{ij}, numbering both rows and columns from 0.

Definition. For partitions λ, μ with $l(\lambda), l(\mu) \geq N$ take index sets I_λ and I_μ as above and define $P_{(\lambda,\mu)}^N$ to be the $N \times N$ minor of V with rows indexed by I_μ and columns by I_λ.

Theorem 4.2

$$\langle\lambda, \mu\rangle_N = s^{(1-N)(|\lambda|+|\mu|)} P_{(\lambda,\mu)}^N / P_{(\emptyset,\emptyset)}^N.$$

Proof: Using formula (14) for the Schur function, applied to the rows I_μ of V, we can write

$$s_\lambda\left(\prod_{i\in I_\mu}(1+q^i t)\right) = P^N_{(\lambda,\mu)}/P^N_{(\emptyset,\mu)}. \tag{15}$$

In particular

$$P^N_{(\lambda,\emptyset)}/P^N_{(\emptyset,\emptyset)} = s_\lambda\left(\prod_{i=0}^{N-1}(1+q^i t)\right).$$

We first derive a formula for the polynomial $E^N_\lambda(t)$ in the next lemma.

Lemma 4.3

$$\prod_{i\in I_\lambda}(1+q^i t) = E^N_\lambda(s^{N-1}t),$$

giving

$$E^N_\lambda(t) = \prod_{j=1}^{N}(1+s^{N+2\lambda_j-2j+1}t).$$

To finish the proof of theorem 4.2 we use equation (15), lemma 4.3 and lemma 2.1 to get

$$\begin{aligned} P^N_{(\lambda,\mu)}/P^N_{(\emptyset,\mu)} = s_\lambda\left(\prod_{i\in I_\mu}(1+q^i t)\right) &= s_\lambda(E^N_\mu(s^{N-1}t)) \\ &= s^{(N-1)|\lambda|}s_\lambda(E^N_\mu(t)) \\ &= s^{(N-1)|\lambda|}\langle\lambda,\mu\rangle_N/\left\langle\mu\right\rangle_N \end{aligned}$$

and

$$P^N_{(\mu,\emptyset)}/P^N_{(\emptyset,\emptyset)} = s_\mu(E^N_\emptyset(s^{N-1}t) = s^{(N-1)|\mu|}\left\langle\mu\right\rangle_N.$$

Multiply the two expressions to get

$$P^N_{(\lambda,\mu)}/P^N_{(\emptyset,\emptyset)} = s^{(N-1)(|\lambda|+|\mu|)}\langle\lambda,\mu\rangle_N.$$

The formula claimed in theorem 4.2 for $\langle\lambda,\mu\rangle_N$ in terms of the $N\times N$ minors of the Vandermonde matrix follows at once. $\square$

Proof of lemma 4.3:

We begin by establishing theorem 4.2 directly in the case $\mu = c_k$. The index set I_{c_k} is

$$I_{c_k} = \{N, N-1, \ldots, N-k+1, N-k-1, \ldots, 2, 1, 0\} = I_\emptyset \cup \{N\} - \{N-k\}.$$

Recall that equation (10) gives

$$E_\emptyset(t) = \sum_{r\geq 0} \langle c_r\rangle\, t^r = \prod_{i=0}^{\infty} \frac{1+vs^{2i+1}t}{1+v^{-1}s^{2i+1}t}.$$

The substitution $v = s^{-N}$ reduces this to the finite product

$$E_\emptyset^N(t) = \prod_{i=0}^{N-1}(1+s^{-N+2i+1}t).$$

Then

$$E_\emptyset^N(s^{N-1}t) = \prod_{i=0}^{N-1}(1+s^{2i}t) = \prod_{i\in I_\emptyset}(1+q^it). \tag{16}$$

Let k be an integer, $k \leq N$. By corollary 3.3 we have

$$E_{c_k}(t) = \frac{1+v^{-1}st}{1+v^{-1}s^{-2k+1}t} E_\emptyset(t).$$

Substituting $v = s^{-N}$ in the above equation gives

$$\begin{aligned} E_{c_k}^N(s^{N-1}t) &= \frac{1+s^{2N}t}{1+s^{2N-2k}t} E_\emptyset^N(s^{N-1}t) \\ &= \prod_{i\in I_{c_k}}(1+q^it). \end{aligned} \tag{17}$$

Now $s_\lambda\left(\prod_{i\in I_{c_k}}(1+q^it)\right) = P^N_{(\lambda,c_k)}/P^N_{(\emptyset,c_k)}$, by (15). Consequently

$$P^N_{(\lambda,c_k)}/P^N_{(\emptyset,c_k)} = s_\lambda(E^N_{c_k}(s^{N-1}t))$$

and

$$P^N_{(c_k,\emptyset)}/P^N_{(\emptyset,\emptyset)} = s_{c_k}(E^N_\emptyset(s^{N-1}t)).$$

Since $s_\lambda(E^N_{c_k}(t)) = \langle\lambda, c_k\rangle_N / \langle c_k\rangle_N$ by lemma 2.1, we can multiply the two expressions to get

$$\begin{aligned} P^N_{(\lambda,c_k)}/P^N_{(\emptyset,\emptyset)} &= s_\lambda(E^N_{c_k}(s^{N-1}t))s_{c_k}(E^N_\emptyset(s^{N-1}t)) \\ &= s^{(N-1)|\lambda|}\frac{\langle\lambda,c_k\rangle_N}{\langle c_k\rangle_N} s^{(N-1)|c_k|}\langle c_k\rangle_N \\ &= s^{(N-1)(|\lambda|+|c_k|)}\langle\lambda,c_k\rangle_N. \end{aligned} \tag{18}$$

To complete the proof of lemma 4.3 it is enough to show that

$$s_{c_k}\left(\prod_{i\in I_\lambda}(1+q^it)\right) = s_{c_k}(E^N_\lambda(s^{N-1}t))$$

for all $k \leq N$, so as to compare the coefficients of the two polynomials.

By (15) we have $s_{c_k}\left(\prod_{i\in I_\lambda}(1+q^i t)\right) = P^N_{(c_k,\lambda)}/P^N_{(\emptyset,\lambda)}$, while

$$\begin{aligned} s_{c_k}(E^N_\lambda(t)) &= \langle\lambda, c_k\rangle_N / \langle\lambda\rangle_N \\ &= s^{(1-N)|c_k|}P^N_{(\lambda,c_k)}/P^N_{(\lambda,\emptyset)} \end{aligned}$$

by (18).

Since $P^N_{(c_k,\lambda)} = P^N_{(\lambda,c_k)}$ and $s_{c_k}(E^N_\lambda(s^{N-1}t)) = s^{(N-1)|c_k|}s_{c_k}(E^N_\lambda(t))$ the result follows. □

We now deduce a formula for the power series $E_\lambda(t)$ with coefficients in the two variables s and v from the formula for $E^N_\lambda(t)$ in lemma 4.3.

Theorem 4.4 *We have*

$$E_\lambda(t) = \prod_{j=1}^{l(\lambda)} \frac{1+v^{-1}s^{2\lambda_j-2j+1}t}{1+v^{-1}s^{-2j+1}t}E_\emptyset(t)$$

for any partition λ.

Proof: For any integer $N \geq l(\lambda)$ we have $E^N_\lambda(s^{N-1}t) = \prod_{i\in I_\lambda}(1+q^i t)$ by lemma 4.3. Now

$$I_\lambda = I_\emptyset \cup_{j=1}^{l(\lambda)} \{\lambda_j + N - j\} - \cup_{j=1}^{l(\lambda)}\{N-j\}.$$

Hence

$$E^N_\lambda(s^{N-1}t) = \prod_{j=1}^{l(\lambda)} \frac{1+q^{\lambda_j+N-j}t}{1+q^{N-j}t}E^N_\emptyset(s^{N-1}t)$$

and so

$$\begin{aligned} E^N_\lambda(t) &= \prod_{j=1}^{l(\lambda)} \frac{1+s^{2(\lambda_j+N-j)}s^{1-N}t}{1+s^{2(N-j)}s^{1-N}t}E^N_\emptyset(t) \\ &= \prod_{j=1}^{l(\lambda)} \frac{1+s^{N+2\lambda_j-2j+1}t}{1+s^{N-2j+1}t}E^N_\emptyset(t). \end{aligned}$$

This means that the two power series $E_\lambda(t)$ and

$$\prod_{j=1}^{l(\lambda)} \frac{1+v^{-1}s^{2\lambda_j-2j+1}t}{1+v^{-1}s^{-2j+1}t}E_\emptyset(t)$$

agree for every substitution $v = s^{-N}$ with $N \geq l(\lambda)$. Since the coefficients are Laurent polynomials in v it follows that the two power series are equal when v is treated as an indeterminate. □

When we apply theorem 4.4 to the case $\lambda = c_k$ and compare the result with corollary 3.3 we note a number of cancellations in

$$\prod_{j=1}^{l(\lambda)} \frac{1+v^{-1}s^{2\lambda_j-2j+1}t}{1+v^{-1}s^{-2j+1}t}.$$

In our final theorem we give a simpler expression for the rational function $E_\lambda(t)/E_\emptyset(t)$, in terms of the Frobenius notation $\lambda = (a_1, \ldots, a_{d(\lambda)}|b_1, \ldots, b_{d(\lambda)})$ for the partition λ. As in [8] the Frobenius notation describes λ in terms of the lengths $\{a_i\}$ of the 'arms' and $\{b_i\}$ of the 'legs' of its Young diagram, counting right or down from the $d(\lambda)$ cells on the main diagonal. Thus $a_i = \lambda_i - i$ and $b_i = \lambda'_i - i$ for $i = 1, \ldots, d(\lambda)$.

Theorem 4.5 *We have*

$$E_\lambda(t) = \prod_{i=1}^{d(\lambda)} \frac{1+v^{-1}s^{2a_i+1}t}{1+v^{-1}s^{-2b_i-1}t} E_\emptyset(t)$$

for any partition $\lambda = (a_1, \ldots, a_{d(\lambda)}|b_1, \ldots, b_{d(\lambda)})$ in Frobenius notation.

Proof: The result follows from exactly the same argument as in theorem 4.4 once we observe that we can write the index set I_λ as

$$I_\lambda = I_\emptyset \cup_{i=1}^{d(\lambda)} \{N + a_i\} - \cup_{j=1}^{d(\lambda)} \{N - b_j - 1\}.$$

To confirm this, write $a_i = \lambda_i - i, b_j = \lambda'_j - j$ for all $i, j \le N$. Then

$$\begin{aligned} a_i \ge 0 &\iff i \le d_\lambda, \\ b_j \ge 0 &\iff j \le d_\lambda. \end{aligned}$$

The integers $a_i, 1 \le i \le N$ are all distinct, and $I_\lambda = \{N + a_i, i \le N\}$. Since $N + a_i \ge N \iff i \le d_\lambda$ there are exactly d_λ numbers in I_λ which are not in $I_\emptyset$, and hence d_λ numbers in $I_\emptyset$ which are not in I_λ. We can identify these as $\{N - b_j - 1 : j \le d_\lambda\}$ by showing that

$$N - b_j - 1 \ne N + a_i, \text{ for any } i, j \le N.$$

Consider the cell in row i and column j, $1 \le i, j \le N$, in relation to the Young diagram of λ. If the cell lies in λ then $\lambda_i \ge j$ and $\lambda'_j \ge i$. Add these to get $a_i + b_j \ge 0$ and so $a_i + b_j \ne -1$. Otherwise $\lambda_i \le j - 1$ and $\lambda'_j \le i - 1$, giving $a_i + b_j \le -2$ and again $a_i + b_j \ne -1$.

This completes the proof; indeed the rational function

$$\prod_{i=1}^{d(\lambda)} \frac{1+v^{-1}s^{2a_i+1}t}{1+v^{-1}s^{-2b_i-1}t} = E_\lambda(t)/E_\emptyset(t)$$

admits no further cancellation, since the exponents of s in the numerator are all positive, while those in the denominator are negative. $\square$

Having used this result to get the power series for $E_\lambda(t)$ with explicit coefficients in Λ, the general invariant $\langle \lambda, \mu \rangle$ is then found by calculating Schur functions of the series, using the formula

$$\langle \lambda, \mu \rangle = s_\mu(E_\lambda(t)) \langle \lambda \rangle .$$

Remark. The rational function above can be expressed in terms of the *content polynomial* $C_\lambda(t)$ defined by

$$C_\lambda(t) = \prod_{x\in\lambda} (1+q^{c(x)}t).$$

It is easy to check, by decomposing λ as a disjoint union of simple hooks based on the diagonal cells, that

$$\prod_{i=1}^{d(\lambda)} \frac{1+v^{-1}s^{2a_i+1}t}{1+v^{-1}s^{-2b_i-1}t} = C_\lambda(v^{-1}st)/C_\lambda(v^{-1}s^{-1}t).$$

This leads to an alternative proof of theorem 4.5 using theorem 3.10 of [10], once it is established that the product of the Gyoja-Aiston idempotent e_λ, with $|\lambda| = n$, and the polynomial $EM(t)$, whose coefficients are the elementary symmetric functions of the Murphy operators in H_n, satisfies the equation

$$EM(t)e_\lambda = C_\lambda(t)e_\lambda.$$

4.2 An explicit example

We illustrate our results by calculating the framed Homfly polynomial $\langle \ydiagram{3,2}, \ydiagram{2,2} \rangle$.

Let μ be a partition whose conjugate μ' has r parts. The Jacobi-Trudy formula gives the Schur function s_μ of the series $\sum e_i t^i$ as the determinant of the $r \times r$ matrix whose (i,j) entry is e_k, with $k = \mu'_i + j - i$, taking $e_k = 0$ for $k < 0$. The diagonal entries in the matrix are thus the coefficients corresponding to the columns of the diagram of μ, and each row of the matrix is completed by taking consecutive coefficients from the series. Then

$$s_{\ydiagram{2,2}} = \det \begin{pmatrix} e_2 & e_3 \\ e_1 & e_2 \end{pmatrix} = e_2^2 - e_1e_3$$

and it is enough to expand the series $E_{\yng(3,1)}(t)$ as far as the term in t^3 in order to calculate $s_{\yng(2,2)}(E_{\yng(3,1)}(t))$.

For the diagram $\lambda = \yng(3,1)$ we have $d(\lambda) = 1$ with $a_1 = 2, b_1 = 1$ so that

$$E_{\yng(3,1)}(vs^{-1}t) = \frac{1+q^2t}{1+q^{-2}t}E_{\emptyset}(vs^{-1}t).$$

Using (12) with $\lambda = c_i$ we can write

$$\begin{aligned} E_{\emptyset}(vs^{-1}t) &= 1 + \frac{1-v^2}{q-1}t + \frac{(1-v^2)(1-qv^2)}{(q-1)(q^2-1)}t^2 \\ &+ \frac{(1-v^2)(1-qv^2)(1-q^2v^2)}{(q-1)(q^2-1)(q^3-1)}t^3 + O(t^4). \end{aligned}$$

We have

$$\frac{1+q^2t}{1+q^{-2}t} = 1 + (q^2-q^{-2})t - (1-q^{-4})t^2 + (q^{-2}-q^{-6})t^3 + O(t^4)$$

and so

$$\begin{aligned} E_{\yng(3,1)}(vs^{-1}t) &= 1 + \left(\frac{1-v^2}{q-1} + q^2 - q^{-2}\right)t \\ &+ \left(\frac{(1-v^2)(1-qv^2)}{(q-1)(q^2-1)} + (q^2-q^{-2})\frac{1-v^2}{q-1} - (1-q^{-4})\right)t^2 \\ &+ \left(\frac{(1-v^2)(1-qv^2)(1-q^2v^2)}{(q-1)(q^2-1)(q^3-1)} + (q^2-q^{-2})\frac{(1-v^2)(1-qv^2)}{(q-1)(q^2-1)}\right. \\ &- \left.(1-q^{-4})\frac{1-v^2}{q-1} + (q^{-2}-q^{-6})\right)t^3 + O(t^4) \\ &= 1 + e_1t + e_2t^2 + e_3t^3 + O(t^4). \end{aligned}$$

Now

$$\begin{aligned} s_{\yng(2,2)}(E_{\yng(3,1)}(t)) &= (vs^{-1})^{-4}s_{\yng(2,2)}(E_{\yng(3,1)}(vs^{-1}t)) \\ &= (vs^{-1})^{-4}(e_2^2 - e_1e_3), \end{aligned}$$

where e_1, e_2 and e_3 are the coefficients in the series above.

Combined with the expression

$$\langle \yng(3,1) \rangle = \frac{(v^{-1}-v)(v^{-1}s-vs^{-1})(v^{-1}s^2-vs^{-2})(v^{-1}s^{-1}-vs)}{(s-s^{-1})^2(s^2-s^{-2})(s^4-s^{-4})}$$

from (12) we get

$$\begin{aligned}
\langle \yng(3,1), \yng(2,2) \rangle &= (vs^{-1})^{-4}(e_2^2 - e_1 e_3)\,\langle \yng(3,1) \rangle \\
&= v^{-8}\frac{(v^2-1)^2(v^2-q)(v^2-q^2)(v^2q-1)}{(q-1)^3(q^2-1)^3(q^3-1)(q^4-1)}C \qquad (19)
\end{aligned}$$

with

$$\begin{aligned}
C &= -q^{14}+q^{13}+q^{12}-q^{11}-q^{10}-q^{9}+2q^{8}+q^{7}-q^{6}-2q^{4}+2q^{2}-1 \\
&\quad +(q^{13}-q^{10}-q^{9}+2q^{8}+q^{7}+q^{6}-2q^{4}+q^{2}+q)v^2 \\
&\quad +(-q^{10}-q^{9}+q^{8}-q^{4}-q^{3})v^4+q^6v^6.
\end{aligned}$$

Remark. Since $\langle\lambda,\mu\rangle = s_\mu(E_\lambda(t))\,\langle\lambda\rangle = s_\lambda(E_\mu(t))\,\langle\mu\rangle$ the calculation in our example could have been done with the roles of λ and μ interchanged. In this case we would have had a more complicated expression for $E_\mu(t)$ since $d(\mu) = 2$, and in addition the expression for s_λ would involve terms up to degree 4 from the Jacobi-Trudy formula

$$s_{\yng(3,1)} = \det\begin{pmatrix} e_2 & e_3 & e_4 \\ 1 & e_1 & e_2 \\ 0 & 1 & e_1 \end{pmatrix}.$$

In general $\langle\lambda,\mu\rangle$ contains some form of least common multiple of $\langle\lambda\rangle$ and $\langle\mu\rangle$, and a 'core' part, such as the expression C in the example above.

To find $\langle\lambda,\mu\rangle_N$ we can make the substitution $v = s^{-N}$. When either $\langle\lambda\rangle_N = 0$ or $\langle\mu\rangle_N = 0$ this will vanish. This happens when one of the partitions has more than N parts. In our example this is the case when $N = 1$, as we can see from the factor $v^2q - 1$ in (19).

We can equally calculate $\langle\lambda,\mu\rangle_N$ from Vandermonde minors, using theorem 4.2, and we compare the two results for our example in the case $N = 3$.

The substitution with $N = 3$ in equation (19) gives

$$\langle \yng(3,1), \yng(2,2) \rangle_3 = (q^2+q+1)(q^8+q^4+q^3-q^2+1)(q^2+1)(q^4+q^3+q^2+q+1)/q^3.$$

On the other hand taking $I_{\yng(3,1)} = \{0,2,5\}$ and $I_{\yng(2,2)} = \{0,3,4\}$, gives the minor

$$\begin{aligned}
P^3_{(\yng(3,1),\yng(2,2))} &= \det\begin{pmatrix} 1 & 1 & 1 \\ 1 & q^6 & q^{15} \\ 1 & q^8 & q^{20} \end{pmatrix} = q^{26}-q^{23}-q^{20}+q^{15}+q^8-q^6 \\
&= q^6(q+1)(q^2+1)(q^2+q+1) \\
&\qquad (q^4+q^3+q^2+q+1)(q^8+q^4+q^3-q^2+1)(q-1)^3,
\end{aligned}$$

and so

$$
\begin{aligned}
\langle \ydiagram{3,1}, \ydiagram{2,2} \rangle_3 &= q^{-8} P^3_{(\ydiagram{3,1}, \ydiagram{2,2})} / P^3_{(\emptyset,\emptyset)}, \text{ by theorem 4.2,} \\
&= q^{-8} \frac{q^{26} - q^{23} - q^{20} + q^{15} + q^8 - q^6}{(q-1)(q^2-1)(q^2-q)} \\
&= (q^2+1)(q^2+q+1) \\
&\quad (q^4+q^3+q^2+q+1)(q^8+q^4+q^3-q^2+1)/q^3,
\end{aligned}
$$

as before.

4.3 The $sl(N)_q$ invariants of the Hopf link

Irreducible $sl(N)_q$ modules$\{V_\lambda\}$ are indexed by partitions with at most N parts. The framed $sl(N)_q$ invariant $(J_H; V_\lambda, V_\mu)$ for the Hopf link H coloured by the irreducible modules V_λ and V_μ is given, up to a fractional power of s, from our Homfly invariant $\langle \lambda, \mu \rangle$ by applying the $sl(N)$ specialisation $v = s^{-N}$. The exact formula, noted in [6], is

$$(J_H; V_\lambda, V_\mu) = x^{2|\lambda||\mu|} \langle \lambda, \mu \rangle_N, \text{ with } x = s^{-\frac{1}{N}}.$$

The correction factor arises from the isomorphism between the Hecke algebras used here and the version based on the R-matrix in $sl(N)_q$.

The $sl(N)_q$ invariants of the Hopf link are then essentially the Laurent polynomials in q determined by $N \times N$ minors of the Vandermonde matrix (q^{ij}).

In the simplest case $N = 2$ the most general minor is

$$\det \begin{pmatrix} q^{ij} & q^{i(j+b)} \\ q^{(i+a)j} & q^{(i+a)(j+b)} \end{pmatrix} = q^{2ij+ib+aj}(q^{ab} - 1),$$

arising from partitions λ with parts $(b+j-1, j)$ and μ with parts $(a+i-1, i)$.

Then $\langle \lambda, \mu \rangle_2 = \dfrac{q^{ab} - 1}{q - 1}$, up to a power of q. Calculation of this power and the correction factor gives the formula

$$(J_H; V_\lambda, V_\mu) = [ab] \tag{20}$$

for the $sl(2)_q$ invariants of the Hopf link, where $[k]$ is the quantum integer $[k] = \dfrac{s^k - s^{-k}}{s - s^{-1}}$.

The $sl(2)_q$ modules V_λ and V_μ in this case do not depend on i or j, and have dimension a and b respectively. They correspond to the single row diagrams d_{a-1} and d_{b-1}. The formula (20) for the $sl(2)_q$ Hopf link invariants was given in [12] and [4]. It allows a simple development of the properties for the related 3-manifold invariants where q is replaced by a root of unity, [13, 4].

For $N \geq 3$ the $sl(N)_q$ invariants of the Hopf link do not have such an immediately memorable form. Their expression by Vandermonde minors was given by Kohno and Takata, [5], only in the case where q is a root of unity. Our determination of them here for generic q in theorem 4.2 and the eventual 2-variable formulation for $\langle \lambda, \mu \rangle$ given in theorem 4.5 was initially inspired by our reading of [5].

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