Idempotents of the Hecke algebra become Schur functions in the skein of the annulus

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Abstract

The Hecke algebra H_n can be described as the skein R_n^n of (n, n)-tangle diagrams with respect to the framed Homfly relations. This algebra R_n^n contains well-known idempotents E_λ which are indexed by Young diagrams λ with n cells. The geometric closures Q_λ of the E_λ in the skein of the annulus are known to satisfy a combinatorial multiplication rule which is identical to the Littlewood–Richardson rule for Schur functions in the ring of symmetric functions. This fact is derived from two skein theoretic lemmas by using elementary determinantal arguments. Previously known proofs depended on results for quantum groups.

1. *Introduction*

Sections 2 and 3 provide an overview on skeins, Young diagrams and symmetric functions. In Section 4 the skein R_n^n of (n, n)-tangles is presented. R_n^n contains idempotents E_λ indexed by Young diagrams λ with n cells. These idempotents were described algebraically by Gyoja [**5**] and were presented in a skein theoretic formulation by Yokota [**14**] and by Aiston and Morton [**1**, **2**].

The main ideas start in Section 5 where the skein $\mathcal{C}$ of the annulus which has a commutative multiplicative structure is introduced. The canonical closure operation induces a map from R_n^n to $\mathcal{C}$ and the closure of E_λ is denoted by Q_λ.

The skein $\mathcal{C}$ of the annulus is well known from the calculation of invariants of satellite knots because the framed Homfly polynomial of a satellite knot depends (beside the companion knot) only on the skein class of the decoration. The skein elements Q_λ are a basis of the important subring $\mathcal{C}_+$ of skein elements whose components have a strictly counter-clockwise orientation. The understanding of the multiplication rule for the Q_λ is therefore of great interest. For example, it allows in [**11**] the computation of the framed Homfly polynomial of the decorated Hopf link. The substitution of the variables of these polynomials by appropriate roots of unity leads to the realm of 3-manifold invariants (see e.g. [**1**, **3**, **7**, **9**, **14**] for more details and references).

In Section 6, I consider the homomorphism from the ring of symmetric functions to $\mathcal{C}$ which maps the ith complete symmetric function h_i to $Q_{(i)}$ where (i) denotes a row diagram with i cells. The homomorphic image of the Schur function s_λ is denoted by S_λ and is expressed by an explicit determinantal formula. Only in Theorem 10·1

will I be able to prove that S_λ is equal to Q_λ for any Young diagram λ which immediately implies the main Theorem 10·3 that the multiplication of the Q_λ follows the Littlewood–Richardson rule for Schur functions. This was proved first by Aiston [1, theorem 4.8.8] using quantum groups.

The idea for proving the equality of S_λ and Q_λ is to construct a linear map $\Gamma: \mathcal{C}_+ \to \mathcal{C}_+$ in such a way that the skein elements S_λ and Q_λ are eigenvectors of Γ and their eigenvalues are equal for any Young diagram λ. If the eigenvalues for all the Q_λ are pairwise different then S_λ is a scalar multiple of Q_λ. The linear map Γ used in this paper is the encirclement with a single loop (following Kawagoe [6]). Q_λ is an eigenvector of Γ because the identity braid with an encircling loop is a central element of R_n^n and is therefore swallowed by any E_λ at the expense of a scalar [2, lemma 5.2]. These scalars are the eigenvalues and they are pairwise different.

The variant skein $\mathcal{C}'$ of the annulus with two boundary points is defined in Section 7. This skein was introduced by Morton [10] and was shown by him to be a commutative ring. It is the appropriate setting for proving the skein theoretic Lemmas 8·3 and 8·4 in Section 8. In Section 9, I deduce from these two lemmas by purely determinantal calculations that S_λ is an eigenvector of the map Γ and that the eigenvalues for S_λ and Q_λ are equal for any Young diagram λ. As mentioned before, it follows that S_λ is a scalar multiple of Q_λ.

In the proof of [1, theorem 4·8·8] Aiston stated that, after the inclusion of the annulus into the plane, the framed Homfly polynomials of S_λ (in her notation $\theta(\lambda)$) and Q_λ are equal and non-zero. Since she used results for quantum groups I cannot follow this direct route to prove the equality of S_λ and Q_λ. Instead, an explicit semi-simple decomposition of R_n^n described by Blanchet [3] gives enough information to conclude in Section 10 that S_λ is equal to Q_λ for any Young diagram λ.

2. *The skein of a planar surface*

Let F be an oriented planar surface with designated n incoming and n outgoing boundary points for some integer $n \geqslant 0$. A *tangle* T in F is a regular diagram that consists of oriented closed curves and n oriented arcs whose boundary points agree with the designated boundary points of F. The arcs of T are oriented towards the incoming boundary points of F. The orientation of the closed components of T is not restricted. We remark that a *tangle* as we use it is usually called a *tangle diagram* in the literature.

The *skein* $\mathcal{S}(F)$ is defined as the module of linear combinations of tangles in F quotiented by regular isotopy (i.e. Reidemeister moves II and III), the two local relations in Figure 1 where v and s are variables, and the relation that a null-homotopic disjoint simple closed curve can be removed from a tangle at the expense of multiplication with the scalar $\delta = (v^{-1} - v)/(s - s^{-1})$. This last relation for removing a circle is a consequence of the other two relations except when the remaining tangle is the empty tangle. We remark that the relation for removing a positive curl in Figure 1 has to be required for both of the two possible orientations.

The set of scalars for a skein can be chosen as $\mathbb{Z}[v^{\pm 1}, s^{\pm 1}]$ with powers of $s - s^{-1}$ in the denominators. In Section 4 we have to divide a quasi-idempotent by a scalar to turn it into an idempotent. Therefore, we shall consider throughout this paper the field of rational functions in v and s as the set of scalars.
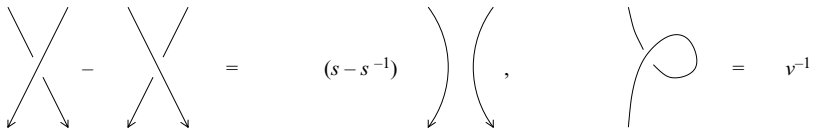
Fig. 1. Skein relations.

For the skein of any oriented planar surface F we define a map $\rho\colon S(F) \to S(F)$. It is induced by switching all crossings of a tangle and replacing s by s^{-1} and v by v^{-1} which preserves the skein relations. We remark that $\rho^2 = \mathrm{id}$. The map ρ is additive but not linear because it changes the scalars.

It is possible to define an apparently more general version of the skein relations by disturbing the skein relations with an additional variable x (see e.g. [7, section 2·1] for details). In the case $x = 1$ this agrees with our version. In fact, the two skeins are always isomorphic by mapping $D \mapsto x^{-\mathrm{wr}(D)}D$ where $\mathrm{wr}(D)$ is the writhe of a tangle D, i.e. the weighted sum of positive and negative crossings of D.

3. *Young diagrams and symmetric functions*

3·1. *Young diagrams*

A *partition* of a non-negative integer n is a non-strictly decreasing sequence of non-negative integers that add up to n. We consider partitions to be equal if they differ only by some zeros at the end.

A *Young diagram* is a graphical depiction of a partition by square cells. A partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is displayed as a flush-left arrangement of square cells with λ_i cells in the ith row for each $1 \leqslant i \leqslant k$. The rows are numbered top-down.

We define a partial ordering for partitions $\lambda = (\lambda_1, \ldots, \lambda_k)$ and $\mu = (\mu_1, \ldots, \mu_m)$ by saying that $\lambda \subset \mu$ if $\lambda_i \leqslant \mu_i$ for each $1 \leqslant i \leqslant k$ where $\mu_j = 0$ for $m + 1 \leqslant j \leqslant k$. This ordering coincides in the graphical depiction with λ being a subset of μ.

We define $|\lambda|$ to be the number of cells of λ, i.e. for $\lambda = (\lambda_1, \ldots, \lambda_k)$ we have $|\lambda| = \lambda_1 + \cdots + \lambda_k$.

The *length* $l(\lambda)$ of a Young diagram λ is the number of non-zero rows of λ.

3·2. *Symmetric functions*

Following the exposition of Macdonald [8], we consider the polynomial ring in n variables $x_1, \ldots, x_n$. For a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ with $k \leqslant n$ we define

$$s_\lambda(x_1, \ldots, x_n) = \frac{\det{(x_i^{\lambda_j + n - j})_{1 \leqslant i, j \leqslant n}}}{\det{(x_i^{n-j})_{1 \leqslant i, j \leqslant n}}}.$$

The two determinants in this fraction are skew-symmetric polynomials and therefore the quotient is a symmetric polynomial in $x_1, \ldots, x_n$. It is a polynomial, indeed. In the inverse limit of the polynomial rings $\mathbb{Z}[x_1, \ldots, x_n]$ for $n \to \infty$ one can define an element s_λ, called the λ-*Schur function*, for any Young diagram λ. This element specialises to $s_\lambda(x_1, \ldots, x_n)$ when we set $x_i = 0$ for all $i \geqslant n + 1$ provided that n is greater than or equal to the length of λ. The set of Schur functions s_λ for all Young diagrams λ is a $\mathbb{Z}$-basis of the ring of symmetric functions.

The most basic Young diagrams are on the one hand a row diagram with, say, n cells and on the other hand a column diagram with, say, k cells. One can check that

$s_{(n)}$ is the nth complete symmetric function h_n and that $s_{(1^k)}$ is the kth elementary symmetric function e_k.

The Littlewood–Richardson rule from [**8**] states that

$$s_\lambda s_\mu = \sum_{|\nu|=|\lambda|+|\mu|} a_{\lambda\mu}^\nu s_\nu \tag{3.1}$$

with integers $a_{\lambda\mu}^\nu \geqslant 0$ for any Young diagrams λ and μ.

4. *The skein R_n^n of a disk with $2n$ boundary points*

We consider a rectangle F with n outgoing points on the top edge and n incoming points on the bottom edge, $n \geqslant 0$. We denote the skein of this surface by R_n^n.

R_n^n gets an algebra structure by defining the product $D \cdot E$ as putting the tangle D over the tangle E so that the incoming points of D match with the outgoing points of E. Clearly, R_n^n is generated as an algebra by the elementary braids $\sigma_1, \ldots, \sigma_{n-1}$ where σ_i agrees with the identity braid up to a single positive crossing connecting the boundary points i and $i+1$ (numbered from left to right). R_n^n is known to be isomorphic to the Hecke algebra H_n.

As a module, R_n^n is spanned linearly by the positive permutation braids ω_π on n strings where π runs through the permutations on n letters (see [**9**] or [**7**, section 2·3] for details).

R_n^n has a well known quasi-idempotent a_n which is given in terms of the positive permutation braids as

$$a_n = \sum_{\pi \in S_n} s^{l(\pi)} \omega_\pi$$

where $l(\pi)$ is the minimal number of transpositions to represent π. This integer $l(\pi)$ is equal to the writhe of the braid ω_π as defined at the end of Section 2. We understand a_0 to be the empty tangle in R_0^0. This special case needs some additional, but trivial, arguments in this section.

It turns out that the element a_n 'swallows' an elementary braid σ_i at the expense of the scalar s as stated in the following lemma which is proved in [**9**].

LEMMA 4·1. *$a_n \sigma_i = \sigma_i a_n = s a_n$ for any integers n and i with $n \geqslant i + 1 \geqslant 2$.*

This Lemma implies that a_n is central in R_n^n because the elementary braids $\sigma_1, \ldots, \sigma_{n-1}$ generate R_n^n. The lemma also implies that $a_n a_n = \alpha_n a_n$ for a scalar α_n for any integer $n \geqslant 1$. This scalar is non-zero and can be computed inductively as

$$\alpha_n = s^{\frac{n(n-1)}{2}} [n][n-1] \cdots [1]$$

where the *quantum integer i* is defined by $[i] = (s^i - s^{-i})/(s - s^{-1})$ for any integer $i \geqslant 1$ (see e.g. [**7**, lemma 2·4·2]). Hence, the element $(1/\alpha_n)a_n$ is an idempotent of R_n^n for any integer $n \geqslant 1$. It is invariant under the map ρ from Section 2. The same holds for $n = 0$ when we define $\alpha_0 = 0$.

LEMMA 4·2. *$\rho((1/\alpha_n)a_n) = (1/\alpha_n)a_n$ for any integer $n \geqslant 0$.*

Proof. The map $\rho \colon R_n^n \to R_n^n$ is a ring homomorphism but not an algebra homomorphism because it changes the scalars. Nevertheless, the multiplicativity
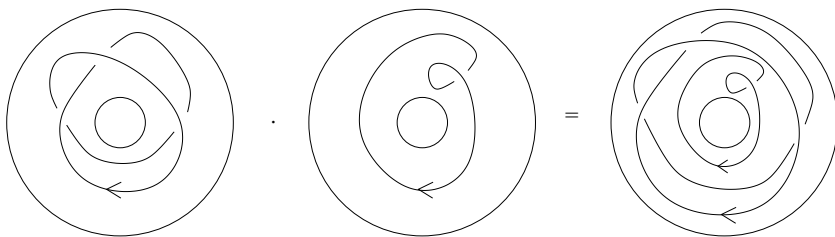
Fig. 2. The multiplication in the skein $\mathcal{C}$ of the annulus.

$\rho(fg) = \rho(f)\rho(g)$ for any elements $f, g \in R_n^n$ together with Lemma 4·1 implies that

$$\sigma_i \rho(a_n) = \rho(\sigma_i^{-1})\rho(a_n) = \rho(\sigma_i^{-1} a_n) = \rho(s^{-1} a_n) = s\rho(a_n)$$

in R_n^n. The equation $\sigma_i \rho(a_n) = s\rho(a_n)$ implies that $a_n \rho(a_n) = \alpha_n \rho(a_n)$ for the same scalar α_n as above. When we apply the map ρ to this equation we get $\rho(a_n)a_n = \rho(\alpha_n)a_n$. The left-hand sides of the previous two equations are equal because a_n is central. Hence, $\alpha_n \rho(a_n) = \rho(\alpha_n)a_n$ and thus $\rho((1/\alpha_n)a_n) = (1/\alpha_n)a_n$.

Aiston and Morton [**2**] constructed idempotents E_λ in R_n^n for any Young diagram λ with n cells. To do this, they considered a three-dimensional version of R_n^n where the distinguished boundary points are lined up along the cells of the Young diagram λ. In this setting, Aiston and Morton arranged idempotents a_{λ_i} along the rows of λ and variant idempotents $b_{\lambda_j^\vee}$ along the columns of λ. Here, $\lambda^\vee$ derives from λ by transposing, i.e. interchanging columns and rows. The resulting skein element is denoted by e_λ in the equivalent standard two-dimensional R_n^n-setting where it satisfies $e_\lambda e_\lambda = \alpha_\lambda e_\lambda$ for a scalar α_λ. In order to prove that $e_\lambda \neq 0$ and $\alpha_\lambda \neq 0$ it is sufficient to prove it for the well known specialisation $v = s = 1$ which turns R_n^n into the symmetric-group algebra (see [**1**]).

The element $E_\lambda = (1/\alpha_\lambda)e_\lambda$ is an idempotent of R_n^n. In particular, we have $E_{(n)} = (1/\alpha_n)a_n$ for any integer $n \geqslant 0$. In this paper we consider the whole ring of rational functions in v and s as the set of scalars. One has to be careful about the scalars e.g. in the context of 3-manifold invariants. Then, the variables v and s are substituted by some roots of unity and the denominators might become zero (see e.g. [**1**] and [**7**]).

We indicate a second proof that $e_\lambda \neq 0$ in R_n^n. We consider the inclusion of the closure of e_λ into the skein of the plane. We make the specialisation $v = s = 1$ and take care of δ by keeping it as a variable. Over- and undercrossings become interchangeable and the closure of e_λ becomes the empty tangle multiplied with a polynomial in δ. The coefficient of δ^n can be shown to be equal to 1 which proves that the closure of e_λ is non-zero [**7**, lemma 2·4·6].

5. *The skein $\mathcal{C}$ of the annulus*

The skein of the annulus is denoted by $\mathcal{C}$. Let D_1 and D_2 be tangles in the annulus $S^1 \times [0, 1]$. We can bring D_1 into $S^1 \times \left(0, \frac{1}{2}\right)$ and D_2 into $S^1 \times \left(\frac{1}{2}, 1\right)$ by regular isotopies. Then the product of D_1 and D_2 is defined as the tangle $D_1 \cup D_2$. An example is shown in Figure 2. The multiplication is commutative because $D_1 \cdot D_2$ and $D_2 \cdot D_1$ differ by a regular isotopy. The skein $\mathcal{C}$ is therefore a commutative ring with the empty tangle
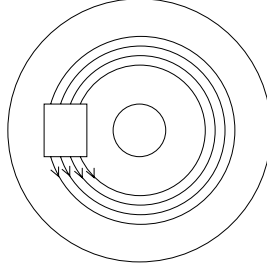
Fig. 3. The additional n arcs for the closure map $\Delta_n \colon R_n^n \to \mathcal{C}$.
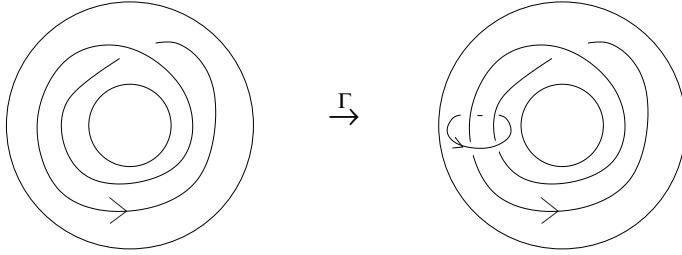

Fig. 4. Encirclement of a tangle in the annulus.

as the identity element. Furthermore, the skein $\mathcal{C}$ can be regarded as a commutative algebra over our chosen ring of scalars, namely the rational functions in v and s.

Figure 3 depicts an annulus with a set of n oriented arcs. A rectangle is removed from the annulus so that we can insert a tangle from R_n^n. This factors to a map $\Delta_n \colon R_n^n \to \mathcal{C}$ for any $n \geqslant 0$. This is a special case of a wiring. We abbreviate Δ_n by Δ if the index n is obvious from the context.

We denote by $\mathcal{C}_n$ the image of R_n^n under the closure map Δ_n. It is a submodule of $\mathcal{C}$. In particular, $\mathcal{C}_0$ is the submodule of scalar multiples of the empty tangle. We have $\mathcal{C}_n \cdot \mathcal{C}_m \subset \mathcal{C}_{n+m}$ for any integers $n, m \geqslant 0$.

By $\mathcal{C}_+$ we denote the submodule of $\mathcal{C}$ spanned by the set of $\mathcal{C}_i$ for all $i \geqslant 0$.

We define Q_λ to be the closure of the idempotent E_λ of R_n^n where n is the number of cells of λ,

$$Q_\lambda = \Delta_n(E_\lambda) \in \mathcal{C}_n.$$

At the end of Section 4 we remarked that the closure of e_λ as an element of the skein of the plane is non-zero. Hence, the inclusion of Q_λ into the skein of the plane gives a non-zero element which implies that Q_λ is non-zero in $\mathcal{C}$.

We define a linear map $\Gamma \colon \mathcal{C} \to \mathcal{C}$ that encircles a tangle in the annulus with a single loop which has a specified orientation as shown in Figure 4. It is clear that $\Gamma(\mathcal{C}_i) \subset \mathcal{C}_i$ for any integer $i \geqslant 0$ and thus $\Gamma(\mathcal{C}_+) \subset \mathcal{C}_+$.

In R_n^n, the identity braid on n strings with an encircling loop can be written as a linear combination of Jucys–Murphy operators as explained in [**10**]. Each of these summands is swallowed by e_λ at the expense of a scalar which is computed skein theoretically in [**2**]. Summing up these scalars immediately implies the following lemma. For details see [**7**, lemma 2·4·7].

LEMMA 5·1. *For any Young diagram λ we have $\Gamma(Q_\lambda) = c_\lambda Q_\lambda$ in $\mathcal{C}_+$ with the scalar*

$$c_\lambda = \frac{v^{-1} - v}{s - s^{-1}} + vs^{-1} \sum_{k=1}^{l(\lambda)} (s^{2(k-\lambda_k)} - s^{2k}).$$

Remark. It is easy to confirm that the eigenvalues c_λ are pairwise different.

LEMMA 5·2. *The set $\{Q_\lambda \mid \lambda \text{ has } n \text{ cells}\}$ is a basis of $\mathcal{C}_n$ for any integer $n \geqslant 0$.*

Proof. We define $X_i^+ \in \mathcal{C}_i$ as the closure of the braid $\sigma_{i-1}\sigma_{i-2}\cdots\sigma_1 \in R_i^i$. The *weighted degree* of a monomial $(X_{i_1}^+)^{j_1}\cdots(X_{i_k}^+)^{j_k}$ is defined as $i_1 j_1 + \cdots + i_k j_k$.

Any tangle in the annulus can be written inductively via the skein relations as a linear combination of tangles each of which is totally descending. This means that $\mathcal{C}_n$ is spanned linearly by the monomials in $X_1^+, \ldots, X_n^+$ of weighted degree n. Hence, the dimension of $\mathcal{C}_n$ is at most $\pi(n)$, by which we denote the number of partitions of n.

On the other hand, any Q_λ lies in $\mathcal{C}_n$ provided that the Young diagram λ has n cells. The Q_λ are non-zero and they are linearly independent because they have pairwise different eigenvalues with respect to the map Γ by Lemma 5·1. By definition, there are $\pi(n)$ Young diagrams with n cells. Hence, there are at least $\pi(n)$ linearly independent elements in $\mathcal{C}_n$. Since the dimension of $\mathcal{C}_n$ is at most $\pi(n)$ by the above argument, the dimension of $\mathcal{C}_n$ is exactly $\pi(n)$ and the set of Q_λ where λ has n cells is a basis.

COROLLARY 5·3. *The set $\{Q_\lambda \mid \text{all Young diagrams } \lambda\}$ is a basis of $\mathcal{C}_+$.*

Proof. $\mathcal{C}_+$ consists of linear combinations of elements of $\mathcal{C}_n$, $n \geqslant 0$. It follows from Lemma 5·2 that $\mathcal{C}_+$ is spanned by the set $\{Q_\lambda \mid \text{all Young diagrams } \lambda\}$. The Q_λ are linearly independent because they are non-zero and they have pairwise different eigenvalues with respect to the map Γ.

Remark. It follows from the proof of Lemma 5·2 and from Corollary 5·3 that the set of elements X_i^+ for all $i \geqslant 0$ generates $\mathcal{C}_+$ freely as a commutative algebra. This is a special case of Turaev's result [**12**] that the whole skein $\mathcal{C}$ is generated freely as a commutative algebra by the set of X_i^+ and X_i^- for all $i \geqslant 0$ where X_i^- derives from X_i^+ by reversing the orientation.

6. *The skein $\mathcal{C}$ of the annulus and symmetric functions*

We recall that (i) denotes the row diagram with i cells. We denote the skein element $Q_{(i)} \in \mathcal{C}_+$ by h_i for any integer $i \geqslant 0$. Since $E_{(i)} = (1/\alpha_i)a_i$, we have $h_i = (1/\alpha_i)\Delta(a_i)$ for any integer $i \geqslant 0$. We define $h_i = 0 \in \mathcal{C}_+$ for any $i \leqslant -1$.

The ring of symmetric functions (in countably many variables) is freely generated as a commutative algebra by the set of complete symmetric functions h_i for all $i \geqslant 0$. For any $i \leqslant -1$ we set $h_i = 0$. The multiple use of the letter h is an abuse of notation but by considering the context it is possible to distinguish between a complete symmetric function h_i and a skein element h_i.

We induce a well-defined ring homomorphism from the ring of symmetric functions to the skein of the annulus by mapping the complete symmetric function h_i to the skein element $h_i \in \mathcal{C}$. We denote the image of the Schur function s_λ under this map
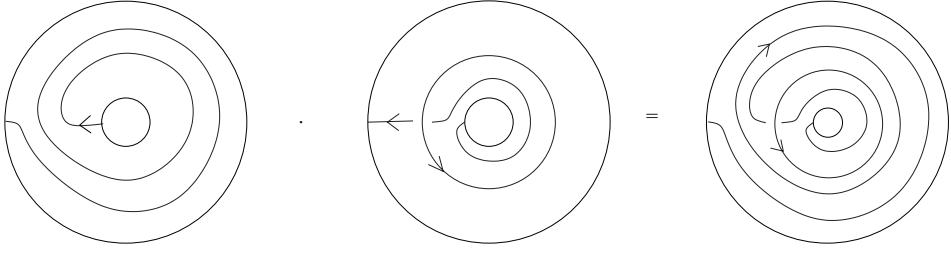
Fig. 5. The multiplication in the variant skein $\mathcal{C}'$ of the annulus.

by S_λ. The Schur function s_λ can be expressed in terms of the complete symmetric functions via the formula $s_\lambda = \det(h_{\lambda_i+j-i})_{1 \leqslant i,j \leqslant l(\lambda)}$ from [8]. Therefore, we have

$$S_\lambda = \det(h_{\lambda_i+j-i})_{1 \leqslant i,j \leqslant l(\lambda)} \in \mathcal{C}_n \qquad (6 \cdot 1)$$

where $n = |\lambda|$. We deduce that $S_{(i)} = h_i$ for any integer $i \geqslant 0$. Hence, $S_\lambda = Q_\lambda$ for any row diagram λ.

Only in Theorem 10·1 will we be able to prove that $S_\lambda = Q_\lambda$ for any Young diagram λ. We remark that Aiston [1, theorem 4·8·4] gave a skein theoretic proof that $S_\lambda = Q_\lambda$ for any hook-shaped Young diagram λ, i.e. $\lambda = (a, 1^b)$ for positive integers a and b.

It follows from the above determinantal formula for S_λ and from the formula for the scalars α_i from Section 4 that S_λ can be written as a linear combination of tangles in such a way that the denominators of the scalars are products of quantum integers $[i]$ where $1 \leqslant i \leqslant \lambda_1 + l(\lambda) - 1$. The upper bound $\lambda_1 + l(\lambda) - 1$ is also known as the *hook length* of λ.

It is possible to prove at this stage that S_λ is non-zero in $\mathcal{C}$ for any Young diagram λ. To do this, one considers the inclusion of the annulus into the plane. The inclusion maps S_λ and h_i to skein elements $\tilde{S}_\lambda$ and $\tilde{h}_i$, respectively. The framed Homfly polynomial of $\tilde{h}_i$ is well known and the framed Homfly polynomial of $\tilde{S}_\lambda$ is the λ-Schur function of these values which is quickly shown to be non-zero (see [7, lemma 3·6·1]). In what follows we do not need this observation.

7. *The variant skein $\mathcal{C}'$ of the annulus*

The skein of the annulus $\mathcal{C}'$ with two boundary points appeared e.g. in [4] and [6]. We are using the version introduced by Morton in [10] apart from a turn over of the annulus. His version has the benefit of a commutative multiplication as we explain now.

We equip the annulus with two distinguished boundary points. An output point on the inner circle and an input point on the outer circle. The resulting skein $\mathcal{C}'$ becomes an algebra by defining the product $A \cdot B$ of tangles A and B as putting the inward circle of A next to the outward circle of the shrunk tangle B so that the two involved distinguished boundary points become a single point in the interior of the new annulus. An example for the multiplication is depicted in Figure 5. The single straight arc e that connects the two marked points as shown in Figure 6 is the identity element.

Any arc that connects the two marked points without crossing itself is regularly isotopic to an integer power of the tangle a which is depicted in Figure 7.
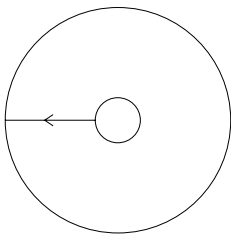
Fig. 6. The identity e in $\mathcal{C}'$.
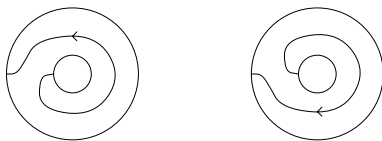
Fig. 7. The arc a (on the left) and its inverse a^{-1} (on the right).

The tangles $A \cdot B$ and $B \cdot A$ are not regularly isotopic in general but they are equal modulo the skein relations as will be shown in Lemma 7·1.

We note that $\mathcal{C}'$ can be turned into an algebra over $\mathcal{C}$ in two ways. Let A be a tangle in $\mathcal{C}'$ and γ be a tangle in $\mathcal{C}$. Then γA is defined as putting γ over A, and $A\gamma$ is defined as putting γ under A. For example, Figure 5 reads $a^{-2} \cdot (h_1 a) = h_1 a^{-1}$. Morton uses in [**10**] the notation $l(\gamma, A)$ and $r(A, \gamma)$ for the operation of $\mathcal{C}$ on $\mathcal{C}'$ on the left and on the right, respectively.

LEMMA 7·1 ([**10**]). *$\mathcal{C}'$ is commutative.*

Proof. Let D be a tangle in $\mathcal{C}'$. We remark that D contains a single arc and possibly several closed components. By induction on the number of crossings of D, we can write D via the skein relations as a scalar linear combination of tangles in $\mathcal{C}'$, say, $D = \alpha_1 D_1 + \cdots + \alpha_k D_k$, in such a way that the single arc of any tangle D_i is totally descending and that this arc lies completely below the closed components of D_i. Hence, $D = \gamma_1 a^{j_1} + \cdots + \gamma_k a^{j_k}$ where $\gamma_1, \ldots, \gamma_k$ are scalar multiples of tangles in $\mathcal{C}$ and $j_1, \ldots, j_k$ are integers.

The commutativity of $\mathcal{C}'$ now follows from the observation that terms of the kind γa^j commute with each other where $\gamma \in \mathcal{C}$ and $j \in \mathbb{Z}$.

For any integer $n \geqslant 1$ we have a linear map $\Delta'_n \colon R^n_n \to \mathcal{C}'$ as shown in Figure 8. This wiring Δ'_n is defined as follows. For a tangle D in R^n_n, the top right boundary point of D is joined with the inner boundary point of $\mathcal{C}'$ and the bottom left boundary point of D is joined with the outer boundary point of $\mathcal{C}'$. The remaining points on the top of D are joined with the remaining points on the bottom of D without crossings.

With this notation, the tangle a from Figure 7 is the image of the identity braid 1_2 of R^2_2 under the map Δ'_2. We abbreviate Δ'_n by Δ' if the index n is obvious from the context.

We denote the image of R^n_n under the map Δ'_n by $\mathcal{C}'_n$ which is a submodule of $\mathcal{C}'$. In particular, $\mathcal{C}'_1$ is the submodule of scalar multiples of the identity e. We have $\mathcal{C}'_n \cdot \mathcal{C}'_m \subset \mathcal{C}'_{n+m-1}$ for any integers $n, m \geqslant 1$.
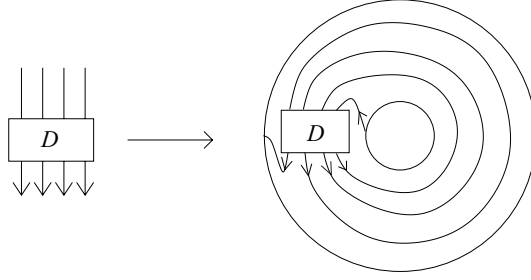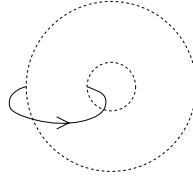
Fig. 8. Map Δ'_n from R^n_n to $\mathcal{C}'$.

Fig. 9. The additional arc for the closure.

By $\mathcal{C}'_+$ we denote the submodule of $\mathcal{C}'$ spanned by the set of $\mathcal{C}'_i$ for all $i \geqslant 1$. We remark that Δ'_0 and $\mathcal{C}'_0$ are not defined.

Remark. The proof of Lemma 7·1 suggests that $\mathcal{C}'_+$ is the polynomial ring in a with coefficients in $\mathcal{C}_+$ on the left (or on the right). This turns out to be true. One has to check that the powers of a are linearly independent over the scalars $\mathcal{C}_+$ (see [**4**] and [**7**, lemma 3·3·3] for two different methods).

We define a closure operation $r \mapsto \hat{r}$ from $\mathcal{C}'$ to $\mathcal{C}$ by putting the arc in Figure 9 over a tangle r. To make this possible, the annulus for $\mathcal{C}$ has to be slightly larger than the annulus for $\mathcal{C}'$. We remark that this closure operation is not a ring homomorphism.

The encirclement map $\Gamma : \mathcal{C} \to \mathcal{C}$ from Section 5 can be expressed in our new terminology as $\Gamma(f) = (f \cdot e)^\wedge$ for any $f \in \mathcal{C}$ where e is the identity element of $\mathcal{C}'$. This simply means that adding an arc under f and then adding an arc over f is the same as putting a loop around f. This will be of importance in Section 10.

8. *Basic skein relations in* $\mathcal{C}'$

We recall from Section 4 that the quasi-idempotent $a_i \in R^i_i$ satisfies $a_i a_i = \alpha_i a_i$ for a non-zero scalar α_i for any integer $i \geqslant 0$. In Section 5 we defined $h_i = (1/\alpha_i)\Delta(a_i) \in \mathcal{C}_i$ for any integer $i \geqslant 0$. We defined $h_i = 0 \in \mathcal{C}_+$ for any integer $i \leqslant -1$.

We now define analogue elements in the variant skein $\mathcal{C}'$. We define $h'_i = (1/\alpha_i)\Delta'(a_i) \in \mathcal{C}'_i$ for any integer $i \geqslant 1$. We define $h'_i = 0 \in \mathcal{C}'_+$ for any integer $i \leqslant 0$. Recall that $\mathcal{C}'_0$ is not defined. The following lemma is depicted in Figure 10.

LEMMA 8·1 ([**10**]). *We have*

$$[i+1]h'_{i+1} = eh_i + s^{-1}[i]h'_i a$$
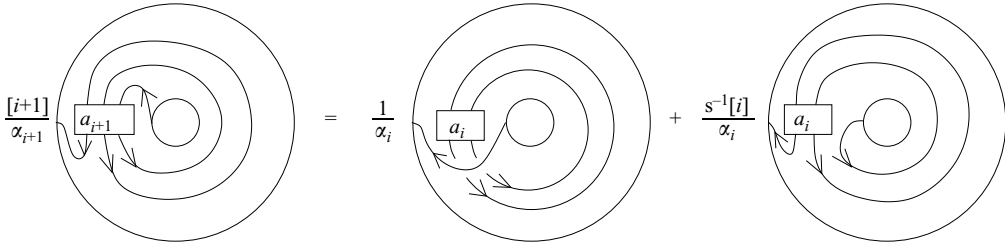
in $\mathcal{C}'_+$ *for any integer* i.

Fig. 10. Depiction of Lemma 8·1.

Proof. In the case $i \leqslant -1$ the lemma simply reads $0 = 0 + 0$ by the definitions of h_i and h'_i. In the case $i = 0$ it reads $h'_1 = eh_0 + 0$ which is true because $h'_1 = e$ and h_0 is equal to the empty tangle in $\mathcal{C}$.

For the remaining case $i \geqslant 1$ we use the well-known decomposition

$$a_{i+1} = (a_i \otimes 1_1)(1_{i+1} + s\sigma_i + s^2\sigma_i\sigma_{i-1} + \cdots + s^i\sigma_i\sigma_{i-1}\cdots\sigma_1)$$

in R_{i+1}^{i+1} where the tensor product $\otimes$ denotes the juxtaposition of braids. Each braid product $\sigma_i\sigma_{i-1}\cdots\sigma_k$ where $2 \leqslant k \leqslant i$ can be moved anti-clockwise in the annulus to get on top of $a_i \otimes 1$ where it is swallowed at the expense of a scalar. It follows that $\Delta'(a_{i+1})$ is a linear combination of the two skein elements $\Delta'((a_i \otimes 1_1)(\sigma_i\sigma_{i-1}\cdots\sigma_1))$ and $\Delta'(a_i \otimes 1_1)$ which are equal to $e\Delta(a_i)$ and $\Delta'(a_i)a$, respectively. Hence, h'_{i+1} is a linear combination of eh_i and $h'_i a$. The scalars in this linear combination are computed in [**10**] and [**7**, lemma 3·4·1].

We define

$$t_i = h_i e - e h_i \in \mathcal{C}'_+$$

for any integer i. We remark that $t_i = 0$ for any $i \leqslant 0$.

LEMMA 8·2. *We have*

$$t_i = (s^{-i} - s^i)h'_i a$$

in $\mathcal{C}'_+$ for any integer i.

Proof. We have

$$[i+1]h'_{i+1} = eh_i + s^{-1}[i]h'_i a$$

by Lemma 8·1 for any integer i. By applying the map ρ from Section 2 we get

$$[i+1]h'_{i+1} = h_i e + s[i]h'_i a$$

because h_i, h'_i and h'_{i+1} are invariant under ρ by Lemma 4·2. These two equations imply that $t_i = h_i e - e h_i = (s^{-1}[i] - s[i])h'_i a$ which is equivalent to our claim.

The closure of t_i is a scalar multiple of h_i.

LEMMA 8·3. *We have*

$$\hat{t}_i = (s^{1-2i} - s)v h_i$$

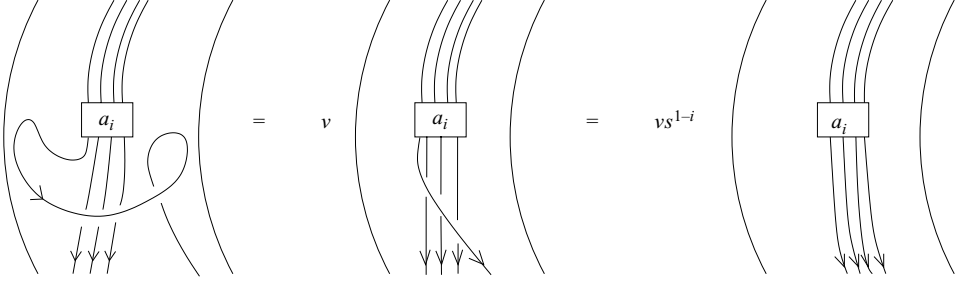*in $\mathcal{C}_+$ for any integer i.*

Fig. 11. The closure of the element $\Delta'(a_i)a \in \mathcal{C}'_{i+1}$ is an element of $\mathcal{C}_i$.

Proof. The skein relation $(\Delta'(a_i)a)^\wedge = vs^{1-i}a_i$ depicted in Figure 11 is a consequence of Lemma 4·1. By this skein relation we deduce from Lemma 8·2 that

$$\hat{t}_i = (s^{-i} - s^i)(h'_i a)^\wedge$$
$$= (s^{-i} - s^i)vs^{1-i}h_i$$
$$= (s^{1-2i} - s)vh_i$$

for any integer $i \geqslant 0$. This equation holds for negative integers i as well because h_i and t_i are equal to zero in this case.

In what follows we consider determinants. We understand a determinant in a commutative ring to be defined via the Leibniz rule.

Lemma 8·4. *We have*

$$\begin{vmatrix} h_i e & t_{i+1} \\ h_j e & t_{j+1} \end{vmatrix} = s^2 \begin{vmatrix} eh_i & t_{i+1} \\ eh_j & t_{j+1} \end{vmatrix}$$

in $\mathcal{C}'_+$ for any integers i and j.

Proof. We substitute in Lemma 8·1 the term $h'_i a$ by $(s^{-i} - s^i)^{-1}(h_i e - eh_i)$ using Lemma 8·2 and get

$$(s^{-i-1} - s^{i+1})h'_{i+1} = s^{-1}(h_i e) - s(eh_i).$$

In the above equation we multiply the right-hand side by t_{j+1} and the left-hand side by $(s^{-j-1} - s^{j+1})h'_{j+1}a$ (which is equal to t_{j+1} by Lemma 8·2) and get

$$(s^{-i-1} - s^{i+1})(s^{-j-1} - s^{j+1})h'_{i+1}h'_{j+1}a = (s^{-1}(h_i e) - s(eh_i))t_{j+1}.$$

The left-hand side of the above equation is invariant under the interchange of i and j because $\mathcal{C}'$ is commutative, and thus the right-hand side is invariant under this interchange. Hence,

$$(s^{-1}(h_i e) - s(eh_i))t_{j+1} = (s^{-1}(h_j e) - s(eh_j))t_{i+1}.$$

This is equivalent to our claim

$$(h_i e)t_{j+1} - (h_j e)t_{i+1} = s^2(eh_i)t_{j+1} - s^2(eh_j)t_{i+1}.$$

Lemma 8·4 is the stepping stone from skein calculations to the following algebraic calculations.

9. *Determinantal calculations in* $\mathcal{C}'$

Lemma 9·1. *For any integer $r \geqslant 1$ and any integers $i_1, i_2, \ldots, i_r$ we have an equality of $(r \times r)$-determinants in $\mathcal{C}'_+$*

$$\begin{vmatrix} h_{i_1}e & \cdots & h_{i_1+r-2}e & t_{i_1+r-1} \\ \vdots & \vdots & \vdots & \vdots \\ h_{i_r}e & \cdots & h_{i_r+r-2}e & t_{i_r+r-1} \end{vmatrix} = s^{2(r-1)} \begin{vmatrix} eh_{i_1} & \cdots & eh_{i_1+r-2} & t_{i_1+r-1} \\ \vdots & \vdots & \vdots \\ eh_{i_r} & \cdots & eh_{i_r+r-2} & t_{i_r+r-1} \end{vmatrix}.$$

Proof. For $r = 1$ we understand the lemma to be read as $\det(t_{i_1}) = s^{2(1-1)} \det(t_{i_1})$ which is trivially true. The case $r = 2$ was proved in Lemma 8·4. From now on let $r \geqslant 3$.

Since $t_i = h_i e - e h_i$, we deduce from Lemma 8·4 by the multilinearity of a determinant that

$$\begin{vmatrix} t_i & t_{i+1} \\ t_j & t_{j+1} \end{vmatrix} = (s^2 - 1) \begin{vmatrix} eh_i & t_{i+1} \\ eh_j & t_{j+1} \end{vmatrix} \tag{9·1}$$

and

$$\begin{vmatrix} t_i & t_{i+1} \\ t_j & t_{j+1} \end{vmatrix} = (1 - s^{-2}) \begin{vmatrix} h_i e & t_{i+1} \\ h_j e & t_{j+1} \end{vmatrix}. \tag{9·2}$$

We see that

$$\begin{vmatrix} t_{i_1} & t_{i_1+1} & t_{i_1+2} & \cdots & t_{i_1+r-1} \\ \vdots & \vdots & \vdots & & \vdots \\ t_{i_r} & t_{i_r+1} & t_{i_r+2} & \cdots & t_{i_r+r-1} \end{vmatrix} = (s^2 - 1) \begin{vmatrix} eh_{i_1} & t_{i_1+1} & t_{i_1+2} & \cdots & t_{i_1+r-1} \\ \vdots & \vdots & \vdots & & \vdots \\ eh_{i_r} & t_{i_r+1} & t_{i_r+2} & \cdots & t_{i_r+r-1} \end{vmatrix}$$

by developing the determinant on the left-hand side by the first two columns, applying equation (9·1) to each of the $\binom{r}{2}$ resulting summands, and redeveloping the determinant. By doing this successively for the columns 1 and 2, then 2 and 3, $\ldots$, and finally for the columns $(r - 1)$ and r, we deduce that

$$\begin{vmatrix} t_{i_1} & \cdots & t_{i_1+r-2} & t_{i_1+r-1} \\ \vdots & & \vdots & \vdots \\ t_{i_r} & \cdots & t_{i_r+r-2} & t_{i_r+r-1} \end{vmatrix} = (s^2 - 1)^{r-1} \begin{vmatrix} eh_{i_1} & \cdots & eh_{i_1+r-2} & t_{i_1+r-1} \\ \vdots & & \vdots & \vdots \\ eh_{i_r} & \cdots & eh_{i_r+r-2} & t_{i_r+r-1} \end{vmatrix}.$$

On the other hand, if we use equation (9·2) instead of equation (9·1) in the above argument, we get

$$\begin{vmatrix} t_{i_1} & \cdots & t_{i_1+r-2} & t_{i_1+r-1} \\ \vdots & & \vdots & \vdots \\ t_{i_r} & \cdots & t_{i_r+r-2} & t_{i_r+r-1} \end{vmatrix} = (1 - s^{-2})^{r-1} \begin{vmatrix} h_{i_1}e & \cdots & h_{i_1+r-2}e & t_{i_1+r-1} \\ \vdots & & \vdots & \vdots \\ h_{i_r}e & \cdots & h_{i_r+r-2}e & t_{i_r+r-1} \end{vmatrix}.$$

The above two equations imply that

$$\begin{vmatrix} h_{i_1}e & \cdots & h_{i_1+r-2}e & t_{i_1+r-1} \\ \vdots & & \vdots & \vdots \\ h_{i_r}e & \cdots & h_{i_r+r-2}e & t_{i_r+r-1} \end{vmatrix} = s^{2(r-1)} \begin{vmatrix} eh_{i_1} & \cdots & eh_{i_1+r-2} & t_{i_1+r-1} \\ \vdots & & \vdots & \vdots \\ eh_{i_r} & \cdots & eh_{i_r+r-2} & t_{i_r+r-1} \end{vmatrix}.$$

We shall prove now that the skein element $S_\lambda \in \mathcal{C}$ as defined in Section 6 is an eigenvector of the encirclement map Γ from Section 5. Furthermore, the eigenvalues for S_λ and Q_λ are equal.

LEMMA 9·2. *For any Young diagram λ we have $\Gamma(S_\lambda) = q_\lambda S_\lambda$ in $\mathcal{C}_+$ with the scalar*

$$q_\lambda = \frac{v^{-1} - v}{s - s^{-1}} + vs^{-1} \sum_{k=1}^{l(\lambda)} \left(s^{2(k-\lambda_k)} - s^{2k} \right).$$

Proof. We first consider the empty Young diagram. The corresponding skein element $S_{(0)}$ is the empty tangle in the annulus. Hence, $\Gamma(S_{(0)})$ is a null-homotopic, simple closed curve in the annulus which can be removed via the skein relations at the expense of multiplication with the scalar δ. Hence, $\Gamma(S_{(0)}) = ((v^{-1}-v)/(s-s^{-1}))S_{(0)}$ as claimed.

In what follows we fix a non-empty Young diagram λ and abbreviate its length $l(\lambda)$ by r. For any elements α and β of the skein of the annulus $\mathcal{C}$ we have $(\alpha e) \cdot (\beta e) = (\alpha\beta)e$ in $\mathcal{C}'$ where e is the identity of $\mathcal{C}'$. By this equality we deduce from equation (6·1) that in $\mathcal{C}'_+$

$$S_\lambda e = \det(h_{\lambda_i + j - i}e)_{1 \leqslant i,j \leqslant r}.$$

Similarly,

$$eS_\lambda = \det(eh_{\lambda_i + j - i})_{1 \leqslant i,j \leqslant r}.$$

We denote the element $S_\lambda e - eS_\lambda \in \mathcal{C}'_+$ by $\mathbf{\Omega}_\lambda$. We remark that $\mathbf{\Omega}_{(i)} = t_i$ for any integer $i \geqslant 0$.

By the multilinearity of a determinant we can write the difference of any two $(r \times r)$-determinants as a telescope sum of r determinants of size $(r \times r)$,

$$\begin{vmatrix} y_{11} & \cdots & y_{1r} \\ \vdots & & \vdots \\ y_{r1} & \cdots & y_{rr} \end{vmatrix} - \begin{vmatrix} z_{11} & \cdots & z_{1r} \\ \vdots & & \vdots \\ z_{r1} & \cdots & z_{rr} \end{vmatrix} = \sum_{j=1}^{r} \begin{vmatrix} y_{1\,1} & \cdots & y_{1\,j-1} & \iota_{1\,j} & z_{1\,j+1} & \cdots & z_{1\,r} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ y_{r\,1} & \cdots & y_{r\,j-1} & \iota_{r\,j} & z_{r\,j+1} & \cdots & z_{r\,r} \end{vmatrix}$$

where $\iota_{i\,k} = y_{i\,k} - z_{i\,k}$. We apply this formula to the determinants for $S_\lambda e$ and eS_λ and get

$$\begin{aligned} \mathbf{\Omega}_\lambda &= S_\lambda e - eS_\lambda \\ &= \det(h_{\lambda_i + j - i}e)_{1 \leqslant i,j \leqslant r} - \det(eh_{\lambda_i + j - i})_{1 \leqslant i,j \leqslant r} \\ &= \sum_{j=1}^{r} \begin{vmatrix} h_{\lambda_1}e & \cdots & h_{\lambda_1 + j - 2}e & t_{\lambda_1 + j - 1} & eh_{\lambda_1 + j} & \cdots & eh_{\lambda_1 + r - 1} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ h_{\lambda_r + 1 - r}e & \cdots & h_{\lambda_r + j - 1 - r}e & t_{\lambda_r + j - r} & eh_{\lambda_r + j + 1 - r} & \cdots & eh_{\lambda_r} \end{vmatrix} \end{aligned}$$

since $t_i = h_i e - eh_i$ by definition. In the above sum, we develop the jth summand by the first j columns, apply Lemma 9·1 to each of the $\binom{r}{j}$ resulting summands, and redevelop the determinant. We thus get

$$\mathbf{\Omega}_\lambda = \sum_{j=1}^{r} s^{2j-2} \begin{vmatrix} eh_{\lambda_1} & \cdots & eh_{\lambda_1 + j - 2} & t_{\lambda_1 + j - 1} & eh_{\lambda_1 + j} & \cdots & eh_{\lambda_1 + r - 1} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ eh_{\lambda_r + 1 - r} & \cdots & eh_{\lambda_r + j - 1 - r} & t_{\lambda_r + j - r} & eh_{\lambda_r + j + 1 - r} & \cdots & eh_{\lambda_r} \end{vmatrix}.$$

The r determinants appearing in this sum are very special because each of them can be written in $\mathcal{C}'_+$ by the Leibniz rule as a sum of terms of the form of a t_i above a product in the h_k. Therefore, the closure of each of these summands is a $\hat{t}_i$ multiplied

with a product in the h_k. Explicitly, we can write the closure of Ω_λ in $\mathcal{C}_+$ as

$$\widehat{\Omega}_\lambda = \sum_{j=1}^{r} s^{2j-2} \begin{vmatrix} h_{\lambda_1} & \cdots & h_{\lambda_1+j-2} & \hat{t}_{\lambda_1+j-1} & h_{\lambda_1+j} & \cdots & h_{\lambda_1+r-1} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ h_{\lambda_r+1-r} & \cdots & h_{\lambda_r+j-1-r} & \hat{t}_{\lambda_r+j-r} & h_{\lambda_r+j+1-r} & \cdots & h_{\lambda_r} \end{vmatrix}.$$

We know by Lemma 8·3 that $\hat{t}_i$ is a scalar multiple of h_i. Hence,

$$\widehat{\Omega}_\lambda = \sum_{j=1}^{r} \begin{vmatrix} h_{\lambda_1} & \cdots & h_{\lambda_1+j-2} & \beta_{1\,j}h_{\lambda_1+j-1} & h_{\lambda_1+j} & \cdots & h_{\lambda_1+r-1} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ h_{\lambda_r+1-r} & \cdots & h_{\lambda_r+j-1-r} & \beta_{r\,j}h_{\lambda_r+j-r} & h_{\lambda_r+j+1-r} & \cdots & h_{\lambda_r} \end{vmatrix}$$

where $\beta_{i\,j} = s^{2j-2}(s^{1-2(\lambda_i+j-i)} - s)v$.

If β_{ij} depended only on j then $\widehat{\Omega}_\lambda$ would be a scalar multiple of S_λ because of the multilinearity of a determinant. But even in our case we can extract β_{ij} step by step. This is due to the fact that β_{ij} can be written as the sum of two terms that depend only on i and j, respectively. Explicitly, we have $\beta_{ij} = \pi_i + \gamma_j$ where $\pi_i = s^{2i-2\lambda_i-1}v$ and $\gamma_j = -s^{2j-1}v$. The terms γ_j can be handled by the multilinearity of a determinant in its columns. For the remaining summands we need the formula

$$\sum_{j=1}^{r} \begin{vmatrix} w_{1\,1} & \cdots & w_{1\,j-1} & \pi_1 w_{1\,j} & w_{1\,j+1} & \cdots & w_{1\,r} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ w_{r\,1} & \cdots & w_{r\,j-1} & \pi_r w_{r\,j} & w_{r\,j+1} & \cdots & w_{r\,r} \end{vmatrix} = (\pi_1 + \cdots + \pi_r) \begin{vmatrix} w_{1\,1} & \cdots & w_{1\,r} \\ \vdots & & \vdots \\ w_{r\,1} & \cdots & w_{r\,r} \end{vmatrix}$$

which is valid for variables π_k and w_{ij} in any commutative ring. To prove this formula, consider the determinant $\det((1 + \pi_i t)w_{ij})_{1\leqslant i,j\leqslant r}$ where t is a variable. This determinant can be written by the multilinearity in the rows as $(1+\pi_1 t)\cdots(1+\pi_r t)\det(w_{ij})_{1\leqslant i,j\leqslant r}$. On the other hand, it can be written by the multilinearity in the columns as a power series in t whose coefficients are sums of determinants in the π_k and w_{ij}. The claimed formula follows from comparing the coefficients of t^1 in both expressions. Similar formulas with other elementary symmetric functions in the π_k as coefficients follow from the higher degree terms in t.

We simplify the above formula for $\widehat{\Omega}_\lambda$ by this method and get

$$\widehat{\Omega}_\lambda = (\pi_1 + \cdots + \pi_r)S_\lambda + (\gamma_1 + \cdots + \gamma_r)S_\lambda$$
$$= (\beta_{1\,1} + \cdots + \beta_{r\,r})S_\lambda.$$

The closure $(S_\lambda e)^\wedge$ is equal to $\Gamma(S_\lambda)$ as we remarked at the end of Section 7. Furthermore, the closure $(eS_\lambda)^\wedge$ is equal to S_λ with a disjoint circle which can be removed at the expense of the scalar δ. Hence,

$$\Gamma(S_\lambda) = (S_\lambda e)^\wedge$$
$$= (eS_\lambda)^\wedge + (S_\lambda e)^\wedge - (eS_\lambda)^\wedge$$
$$= (eS_\lambda)^\wedge + \widehat{\Omega}_\lambda$$
$$= \delta S_\lambda + (\beta_{1\,1} + \cdots + \beta_{r\,r})S_\lambda$$
$$= \left(\frac{v^{-1} - v}{s - s^{-1}} + vs^{-1} \sum_{k=1}^{r} \left(s^{2(k-\lambda_k)} - s^{2k}\right) \right) S_\lambda.$$

LEMMA 9·3. *S_λ is a scalar multiple of Q_λ for any Young diagram λ.*

Proof. Let λ be a Young diagram. The set of Q_μ for all Young diagrams μ is a basis of $\mathcal{C}_+$ by Corollary 5·3. Their eigenvalues with respect to the map Γ are pairwise different by Lemma 5·1. Since the eigenvalues for Q_λ and S_λ are equal by Lemma 9·2, we deduce that S_λ is a scalar multiple of Q_λ. This scalar is a rational function in v and s, and it is possibly equal to zero.

10. *Proof that $S_\lambda = Q_\lambda$*

We have to introduce some notation. Let λ be a Young diagram. A *standard tableau* of λ is a labelling of the cells of λ with the integers $1, 2, \ldots, |\lambda|$ which is increasing in each row from left to right and in each column from top to bottom. The Young diagram underlying a standard tableau t is denoted by $\lambda(t)$. The number of different standard tableaux for a Young diagram λ is denoted by d_λ.

THEOREM 10·1. *S_λ is equal to Q_λ for any Young diagram λ.*

Proof. When λ is the empty Young diagram then Q_λ and S_λ are equal because they are both equal to the empty tangle in the annulus.

From now on, we fix an integer $n \geqslant 1$ and prove that S_λ is equal to Q_λ for any Young diagram λ with n cells. We denote the Young diagram consisting of a single cell by.

Macdonald states in [**8**, example I·3·11] the multiplication rule for the product of $s_\square$ and s_μ,

$$s_\square s_\mu = \sum_{\substack{\mu \subset \eta \\ |\eta|=|\mu|+1}} s_\eta$$

for any Young diagram μ. This is a special case of the Littlewood–Richardson rule for the multiplication of Schur functions. By applying this formula repeatedly we deduce that the nth power of $s_\square$ is an integer linear combination of all the s_λ where λ has n cells. Explicitly, we have

$$s_\square^n = \sum_{|\lambda|=n} d_\lambda s_\lambda$$

where d_λ is the number of standard tableaux of λ. Therefore, we have

$$S_\square^n = \sum_{|\lambda|=n} d_\lambda S_\lambda \tag{10·1}$$

in $\mathcal{C}_n$. We remark that $S_\square$ and $Q_\square$ are both equal to the single core circle of the annulus with anticlockwise orientation. $S_\square^n$ consists of n parallel copies of the core circle of the annulus.

We cite an explicit semi-simple decomposition of R_n^n which was observed by Blanchet [**3**]. The lemmas he uses for this construction are also contained in [**2**] as explained in [**7**, section 2·5]. Generalising results of Wenzl [**13**], Blanchet constructs elements $\alpha_{t\tau} \in R_n^n$ (which are denoted $\alpha_t \beta_\tau$ in [**3**]) for standard tableaux t and τ with $\lambda(t) = \lambda(\tau)$ and $|\lambda(t)| = n$. These elements satisfy $\alpha_{t\tau}\alpha_{s\sigma} = \delta_{\tau s}\alpha_{t\sigma}$ where $\delta_{\tau s}$ is the Kronecker delta, i.e. the elements $\alpha_{t\tau}$ multiply like matrix units. An important observation is that the closure $\Delta(\alpha_{t\tau}) \in \mathcal{C}_+$ is equal to zero if $t \neq \tau$, and $\Delta(\alpha_{t\tau}) = Q_{\lambda(t)}$

if $t = \tau$. Furthermore, the identity braid $\mathrm{id}_n \in R_n^n$ is equal to the sum of all the α_{tt},

$$\mathrm{id}_n = \sum_{|\lambda(t)|=n} \alpha_{tt}.$$

We apply the closure map $\Delta \colon R_n^n \to \mathcal{C}_n$ to the above equation and we get

$$Q_\Box^n = \sum_{|\lambda|=n} d_\lambda Q_\lambda. \tag{10·2}$$

Since $S_\Box = Q_\Box$, equations (10·1) and (10·2) imply that

$$\sum_{|\lambda|=n} d_\lambda (S_\lambda - Q_\lambda) = 0.$$

This equation implies that $S_\lambda = Q_\lambda$ for any Young diagram λ with n cells because of the following three arguments. First, any S_λ differs from Q_λ by a scalar as shown in Lemma 9·3. Second, the set $\{Q_\lambda \mid \lambda \text{ has } n \text{ cells}\}$ is a basis of $\mathcal{C}_n$ by Lemma 5·2. And third, the integers d_λ are non-zero because they are greater than or equal to 1.

Since n was any positive integer, we have $S_\lambda = Q_\lambda$ for any non-empty Young diagram λ. We considered the empty Young diagram at the beginning and we have thus proved that $S_\lambda = Q_\lambda$ for any Young diagram λ.

THEOREM 10·2. *We consider the ring of symmetric functions as an algebra over the rational functions in v and s. Then, the linear map induced by $s_\lambda \mapsto Q_\lambda$ from the algebra of symmetric functions to the subalgebra $\mathcal{C}_+$ of the skein $\mathcal{C}$ of the annulus is an algebra isomorphism.*

Proof. The linear map induced by $s_\lambda \mapsto S_\lambda$ is by definition the ring homomorphism ϕ induced by $h_i \mapsto Q_{(i)}$. By Theorem 10·1 we have $S_\lambda = Q_\lambda$. Hence the linear map induced by $s_\lambda \mapsto Q_\lambda$ is equal to the ring homomorphism ϕ.

This map is injective because the Schur functions s_λ are a basis of the symmetric functions and the Q_λ are linearly independent in $\mathcal{C}$. The image of this map is equal to $\mathcal{C}_+$ because the Q_λ are a basis of $\mathcal{C}_+$ by Corollary 5·3.

From equation (3·1) and Theorem 10·1 we deduce

THEOREM 10·3.

$$Q_\lambda Q_\mu = \sum_{|\nu|=|\lambda|+|\mu|} a_{\lambda\mu}^\nu Q_\nu$$

in $\mathcal{C}$ for any Young diagrams λ and μ. The non-negative integers $a_{\lambda\mu}^\nu$ are the Littlewood–Richardson coefficients.

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