M.Sc. Pure Mathematics
Main Dissertation

—

Knots, Tangles and Algebras

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Abstract

We aim to use the work in [MT90] which gives a geometric realisation of the Hecke algebras. This then allows us to consider more general tangles which will also be considered in an algebraic context. This will be achieved using the Homfly polynomial to construct families of algebras based on linear combinations of these tangles. The work described is an expansion of the work of [MW].

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Chapter 1

Introduction

In this work we aim to use the geometric realisation of the Hecke algebras found in [MT90] through the positive permutation braids to enable us to use a more general type of tangle in a similar algebraic construction.

In this chapter we introduce some of the concepts used throughout the following chapters. Then in Chapter 2 we give a survey of the work immediately preceding the main work described in Chapter 3. We then end with a brief discussion of some possible future work, and some work carried out by other parties with a slightly different emphasis.

Acknowledgments. My thanks first go to Dr H.R. Morton, for his supervision throughout the year, and particularly during the writing of this dissertation. I also thank him for giving me this very interesting topic to study, I have thoroughly enjoyed it. Also, thank you to anyone who asked me about my topic (and perhaps regrets having done so), I'm sorry if I bored you, but it certainly helped **me** understand things better. Finally, thank you to Dr N.P. Kirk for supplying me with some very useful LaTeX macros.

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1.1 The Homfly polynomial

We shall make use of the Homfly polynomial to construct families of algebras based on linear combinations of certain classes of oriented tangles.

The Homfly polynomial is often just referred to as the 2-variable polynomial P for an oriented link, or even as the Homflypt polynomial. It was found by several groups of mathematicians during 1984 and described in [FYH+85] and in [PT87]. The discovery followed the construction of a simpler polynomial invariant V, the so-called Jones polynomial, [Jon85] found using von Neumann algebras and braid groups.

The 2-variable polynomial was found through two different approaches. The first was through considering the Hecke algebras H_n (to be defined in Chapter 2) and the second was a more combinatorial approach making use of knot diagrams and linear skein theory.

1.2 Tangles

1.2.1 Linear skein theory

We begin by introducing the ideas of linear skein theory. Let F be a planar surface. If F has a boundary then we may specify a finite (but possibly empty) set of points on it. Now, a diagram in F consists of arcs joining the boundary points and closed curves. The choice of F is usually one of the following three: (i) $F = \mathbb{R}^2$, (ii) $F = S^1 \times I$, the annulus, and (iii) $F = R \cong I \times I$, a rectangle. It is the last of these we will concentrate on in this work.

The curves and arcs in the diagrams may cross in a finite number of crossing points where two strands cross. We distinguish between the under- and over-crossings in the usual way. We will then be interested in Λ-linear combinations of diagrams in F for a specified ring Λ, the set of which we will denote $\mathcal{D}(F)$. The linear skein $\mathcal{S}(F)$ is then the quotient of $\mathcal{D}(F)$ by linear relations required for the two variable polynomial P. We will return to a more specific consideration of this in the main part of the text.

1.2.2 Tangles

The notion of tangles were originally introduced by J.H. Conway [Con70], however his definition resulted from study of tangles in a different context. For example some of the work had particular regard to the study of mutants of knots, and thus the definition of a tangle was principally limited to identifying four points on a circular boundary, as in Figure 1.1, although more general

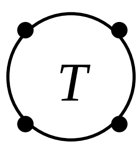


Figure 1.1: Conway's notion of a tangle.

types were discussed. We use a different definition in this work as it allows

a more natural transition to an algebraic approach:

Definition. An (m, n)-*tangle* is a piece of a knot diagram in a rectangle $R \cong I \times I$, in the plane. It consists of m points specified at the top and n points specified at the bottom. The interior of the rectangle then consists of closed curves and arcs such that each of the $m + n$ points specified coincide with the end of the arcs.

We give an example of a $(4, 2)$-tangle in Figure 1.2.

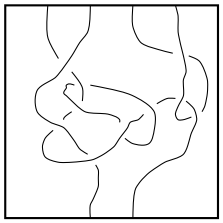


Figure 1.2: A $(4, 2)$-tangle.

We now redefine the idea of isotopy used in knot theory which can be found in almost any introductory book such as [BZ85] or [GP94]. We use this new definition to state what it means for two tangles to be *the same*.

Definition. Two tangles are *ambient isotopic* if one can be changed to the other through a finite sequence of Reidemeister moves R_I, R_{II}, R_{III} (see Figure 1.3) together with isotopies of R fixing its boundary. Two tangles are *regularly isotopic* if the use of move R_I is not required.

Tangles can now be split into numerous classes depending upon the following factors:

- whether we consider the tangles up to regular or ambient isotopy;
- whether or not we consider the arcs and closed curves to have an orientation;
- whether we have the arcs entering at the top and exiting at the bottom or a "mixture" of entry or exit points at the top and bottom.

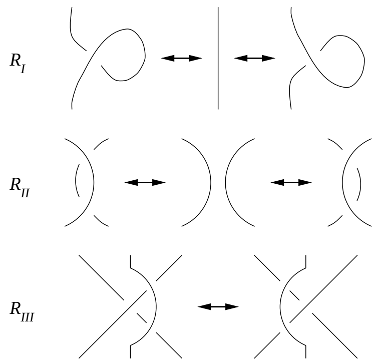


Figure 1.3: The Reidemeister moves R_I, R_{II}, R_{III}.

In the following work we will limit our attention to two specific classes of tangles. Firstly we will be required to consider oriented (n, n)-tangles such the each point at the top is an entry point which is joined to an exit point at the bottom, so the orientation is running from top to bottom. We consider such tangles up to ambient isotopy. We will abbreviate the notation here and refer to such tangles simply as n-tangles.

Notation. Write $\bar{\mathcal{T}}_n$ for the set of such tangles up to ambient isotopy.

Next we consider, up to regular isotopy, oriented (m, n)-tangles, combined with an orientation $\mathcal{O}$ to specify the orientation of the boundary points (or in other words, specifying which are entry and exit points).

Notation. Write $\mathcal{T}_n^m(\mathcal{O})$ for the set of such tangles with orientation $\mathcal{O}$ up to regular isotopy.

Remark. The most well known tangles are the geometric braids. The set of braids B_n can clearly be seen to be a subset of $\bar{\mathcal{T}}_n$ since a braid can be thought of as a tangle which necessarily has no closed components and with a monotonic decrease in the height co-ordinate of each arc. Also $B_n \subset \mathcal{T}_n^m(\mathcal{O})$ from setting $m = n$ and fixing $\mathcal{O}$ such that all points at the top are entry points and all points at the bottom are exit points. So we have:

$$\begin{aligned} B_n &\subset \bar{\mathcal{T}}_n \\ B_n &\subset \bar{\mathcal{T}}_n^m(\mathcal{O}). \end{aligned}$$

Remark. It should also be noted that these choices of classes of tangles are due to our specific interest in the invariant P. Consideration of other invariants may require a different class of tangles. See for example [MT90] in which both P and the Kauffman polynomial is discussed from this tangle viewpoint.

We will return to a more detailed discussion of tangles in the relevant sections of the main text.

1.2.3 Other notes on tangles

We will now only give a brief informal introduction to some topics as they will be treated in a rigorous way in the main part of the text.

We compose tangles T and T', say, by placing T' at the bottom of T. It is intuitively obvious that in order to do this, the tangles must have the same number of strings and at the points where arcs join, the orientations must coincide. This will be immediate in the case of tangles from $\bar{\mathcal{T}}_n$, however, tangles from $\bar{\mathcal{T}}_n^m(\mathcal{O})$ require more care. The orientation $\mathcal{O}$ will need to be such that composition is possible. We will refer to orientations which allow this as being *matched*.

The basic idea behind wiring diagrams is to place one planar surface F inside another F'. So in terms of tangles, we are putting tangles into other tangles such that their arcs join at prescribed points. An example of such a wiring for a 2-tangle is shown in Figure 1.4.

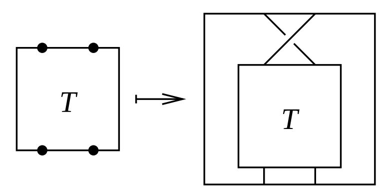


Figure 1.4: An example of a simple wiring.

Finally, we define an operation on tangles, analogous to that used in braid theory to obtain the *closure of the tangle*. We denote this map by $\wedge$, and when it is applied to a tangle we obtain a knot or a link. Clearly there are restrictions on the choice of $\mathcal{O}$ to enable closure to take place, but this is exactly as one would expect. So we have:

$$\wedge \ : \ \bar{\mathcal{T}}_n \mapsto \bar{\mathcal{T}}_0,$$

$$\text{and} \quad \wedge \ : \ \mathcal{T}_n^n(\mathcal{O}) \mapsto \mathcal{T}_0^0(\mathcal{O}'),$$

where $\mathcal{O}'$ is the trivial orientation. We finally note that for some tangle T, the result $\wedge(T)$ will mostly be seen written as $\hat{T}$.

Chapter 2

A Survey Of Previous Material

2.1 The work of Morton and Traczyk

We now give a brief survey of work on tangles in $\bar{\mathcal{T}}_n$. The work we shall discuss was carried out be H.R. Morton and P. Traczyk and comes from [MT90]. This paper covers the 2-variable polynomial P. It also adopts a similar approach for the 2-variable Kauffman polynomial. Our interest will mainly lie with the former topic.

Given an oriented link K, the polynomial $P(K)$ will then lie in $\mathbb{Z}[v^{\pm 1}, z^{\pm 1}]$. For this section we shall take the defining skein relation for the polynomial to be

$$v^{-1}P(K^+) - vP(K^-) = zP(K^0).$$

In fact, the construction of P from either of the methods available indicates that P will lie in a subring of $\mathbb{Z}[v^{\pm 1}, z^{\pm 1}]$ which is isomorphic to the ring $\Lambda = \mathbb{Z}[v^{\pm 1}, z, \delta]/ < v^{-1} - v = z\delta >$. This quotient ring is mapped injectively to $\mathbb{Z}[v^{\pm 1}, z^{\pm 1}]$ by the assignment

$$\delta = \frac{v^{-1} - v}{z}.$$

Remark. In [MT90], the convention for tangles in $\bar{\mathcal{T}}_n$ is to have 'inputs' at the bottom. Here we change this to harmonise this chapter with the others and adopt the convention of having 'inputs' at the top. For the same reason, we refer to the algebra constructed as M_n instead of L_n.

2.1.1 The algebra M_n

From P and the set of tangles $\bar{\mathcal{T}}_n$, an algebra M_n is constructed. This M_n is then shown to be isomorphic to the Hecke algebra H_n, which we will now define.

Definition. The *Hecke algebra,* $H_n(z)$ for $z \in \mathbb{C}$ is defined to be an algebra over the ring $\mathbb{Z}[z]$ with generators c_i, $1 \leq i \leq n-1$ and relations:

- $c_i^2 = c_i + 1$,
- $c_i c_j = c_j c_i$, $|i-j| \geq 2$,
- $c_i c_{i+1} c_i = c_{i+1} c_i c_{i+1}$, $1 \leq i \leq n-2$.

So, in the notation of the Introduction, we have that $F = \bar{\mathcal{T}}_n$. Then the free Λ-module $\mathcal{D}(F) = \Lambda[\bar{\mathcal{T}}_n]$ consists of all the Λ-linear combinations of tangles in $\bar{\mathcal{T}}_n$. We then define a Λ-module $\mathcal{S}(F) = M_n$ by factoring out the skein relations

$$\begin{aligned} v^{-1}T^+ - vT^- &= zT^0 \\ T \sqcup \bigcirc &= \delta T \end{aligned}$$

from the free Λ-module $\Lambda[\bar{\mathcal{T}}_n]$.

Remark. The tangles T^+, T^- and T^0 differ only as shown in Figure 2.1. Also $T \sqcup \bigcirc$ consists of a tangle T and a disjointly embedded unknotted component.

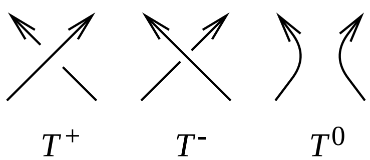


Figure 2.1: T^+, T^- and T^0 differ only as shown.

We then have the following almost immediate facts (not to be proved here): (1) since composition of tangles induce a Λ-linear multiplication on M_n, then M_n is an algebra over Λ; (2) the map $P \circ \wedge : \bar{\mathcal{T}}_n \mapsto \Lambda$ induces a Λ-linear map $M_n \mapsto \Lambda$.

A free basis for M_n can now be found. This basis has $n!$ elements, each corresponding to a permutation in S_n. We begin with defining a totally descending tangle, but it should be noted that due to the more limited scope of this exposition, the definition given here for our purposes differs from that used in [MT90].

Definition. Given a tangle $T \in \bar{\mathcal{T}}_n$, choose a sequence of base-points, consisting firstly of the n 'inputs' on the top taken in order and then one point on each of the closed components, taken in any order. Then T is *totally descending (with this choice of base-points)* if, on traversing all the strands of T starting from the base point of each arc and component in order, each crossing is first met as an overcrossing.

We then have:

Theorem 2.1 *The algebra M_n is linearly generated by totally descending tangles. Moreover, M_n is linearly generated by totally descending tangles without closed components.*

Proof (using the techniques used by Lickorish and Millett in [LM87]): Let T be a tangle representing an element of M_n. If T is not a totally descending tangle then when one reaches the first non-descending crossing, apply the first skein relation to it. This will result with T being written as a linear combination of two tangles, one with fewer crossings, the other with fewer non-descending crossings. Then the first part of the theorem follows by induction, firstly on the number of crossings, then on the number of non-descending crossings.

Moreover, it T is totally descending, with r closed components, then applying the second relation yields $T = \delta^r T'$ in M_n, where T' is simply T without the r closed components. $\square$

2.1.2 Permutations

We can clearly associate to each tangle $T \in \bar{\mathcal{T}}_n$ a permutation $\pi_T \in S_n$ in the obvious way by comparing the top and bottom points of the n arcs.

Given some permutation $\pi \in S_n$, a tangle T_π which satisfies certain conditions has been studied, first in [Elr88] (and subsequently elsewhere including [EM94] and [Had99]). These tangles are presented as braids in B_n and are called *positive permutation braids.* As their name suggests, they must consist only of positive crosings and induce the permutation π on their strings. In addition, each pair of strings may only cross at most once.

Such braids have the following nice property.

Lemma 2.2 *If the braids β_1 and β_2 are both positive permutation braids which both induce the same permutation on their strings, then $\beta_1 = \beta_2$. Moreover, for each permutation $\pi \in S_n$, there is a positive permutation braid which induces that permutation.*

Proofs of this can be found in all three of the references given above.

We now give the result:

Theorem 2.3 *The algebra M_n is linearly generated by $\{T_\pi\}$, for $\pi \in S_n$.*

Now, as an algebra, M_n can be generated by elements g_i which correspond to the elementary braids $\sigma_i \in \bar{\mathcal{T}}_n$ since each T_π for $\pi \in S_n$ is a composite of these. Clearly these generators g_i satisfy the usual braid relations

- $g_i g_j = g_j g_i$, for $|i - j| \geq 2$,
- $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$, for $1 \leq i \leq n-2$.

We also have the relation:

- $v^{-1} g_i - v g_i^{-1} = z$,

which comes directly from the first skein relation being applied in M_n to a tangle $T = \sigma_i$ corresponding to an element $t \in M_n$. Then we have $T^+ = \sigma_i$, $T^- = \sigma_i^{-1}$ and $T^0 =$ identity tangle.

We know that the Hecke algebra H_n can be defined by these generators and relations with the identification $c_i = v^{-1} g_i$. It is then possible to establish that the $n!$ linear generators of H_n map to the positive permutation braids $\{T_\pi\}$ with $\pi \in S_n$. Therefore M_n is a quotient of H_n.

This quotient map is seen to be an isomorphism of the algebras since we have already stated that $\{T_\pi\}$ for $\pi \in S_n$ are independent. Therefore, we may consider M_n as a concrete realisation of H_n.

2.2 Morton-Short computer implementation

We shall not consider the practicalities of the program discussed in [MS90], only a simplification of the reasons why this Hecke algebra work is so suitable to allow such an algorithm to be written.

It is possible to define the Homfly polynomial in terms of a trace function on the Hecke algebra. We shall refrain from giving details here, the interested reader is encouraged to read [MS90].

Therefore, the program is in effect taking the braid inputted and rewriting it in terms of a linear combination of positive permutation braids, from which it will have information stored on how to calculate the Homfly polynomial for the closure of the original braid. The reason this is able to work quite efficiently is now discussed through considering how a general braid can be thought of as a positive permutation braid which is multiplied by additional elementary braids.

2.2.1 Multiplication in M_n

Right multiplication of a basis element of M_n by a generator g_i corresponds to placing an elementary braid σ_i at the end of a positive permutation braid. This then splits into two cases. The first is that the word for the original positive permutation braid does not contain the element σ_i. In this case the new braid formed by the composition is simply another positive permutation braid.

Otherwise, if the word for the positive permutation braid does contain the element σ_i, the we rewrite this word, using the braid relations, such that this element in at the end of the word. We then apply the skein relations to the composition and obtain a linear combination of two positive permutation braids. The first of this linear combination will be the original positive permutation braid and the second will be the original positive permutation braid without the element σ_i.

We can also achieve this from considering the permutations induced by the braids involved. Let the original positive permutation braid induce a permutation π. The effect of multiplication by σ_i on the right is then determined in the following way: the elementary braid σ_i induces a transposition $\tau_i = (i \quad i+1)$, so if the braid does not involve this crossing, then the result is the positive permutation braid representing the permutation $\tau_i \pi$, otherwise we have a linear combination of positive permutation braids inducing the permutations π and $\tau_i \pi$. The reason this simplification is possible is a direct consequence of Lemma 2.2.

Left multiplication is analogous, the only change is that for the second case, we rewrite the word with the appropriate σ_i at the beginning of the word.

Chapter 3

Tangle Algebras

Returning again to the notation used in the introduction, for this section we set $F = \mathcal{T}_n^m(\mathcal{O})$, where, as previously stated, $\mathcal{O}$ specifies the orientation of the boundary points. We then have a free Λ-module $\mathcal{D}(F) = \Lambda[\mathcal{T}_n^m(\mathcal{O})]$ consisting of all Λ-linear combinations of such tangles.

We then define a Λ-module $\mathcal{S}(F) = M_n^m(\mathcal{O})$ by factoring out the skein relations

$$\begin{aligned} T^+ - T^- &= zT^0 \\ \lambda T^{\text{right}} &= T \\ T \sqcup \bigcirc &= \delta T \end{aligned}$$

from the free Λ-module $\Lambda[\mathcal{T}_n^m(\mathcal{O})]$. Hence set

$$\Lambda = \mathbb{Z}[z, \lambda^{\pm 1}, \delta] / < \lambda^{-1} - \lambda = z\delta > .$$

Remark. We have that T^+, T^-, T^0 and $T \sqcup \bigcirc$ are as defined in Chapter 2. Also, T^{right} differs only from a tangle T as shown in Figure 3.1.

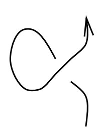


Figure 3.1: T^{right} differs only as shown.

The work that follows comes from work begun by H.R. Morton and A.J. Wassermann [MW].

3.1 Wiring diagrams

We now offer a more formal and specific treatment of wiring diagrams. We have that $\mathcal{T}_n^m(\mathcal{O})$ is the set of oriented (m, n)-tangles with orientation $\mathcal{O}$. We may then place the rectangle which contains such a tangle inside an oriented (m', n')-tangle, with a maybe different orientation $\mathcal{O}'$, by means of a *wiring diagram*. Another example of a wiring diagram is shown in Figure 3.2.

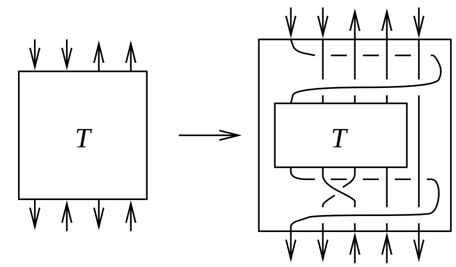


Figure 3.2: An example of a wiring diagram.

Therefore, if we apply a given wiring diagram, W say, to a tangle T, then the result is a new tangle $W(T)$. Hence we have a map

$$W : \mathcal{T}_n^m(\mathcal{O}) \longrightarrow \mathcal{T}_{n'}^{m'}(\mathcal{O}').$$

The map W then induces a Λ-module homomorphism

$$W : M_n^m(\mathcal{O}) \longrightarrow M_{n'}^{m'}(\mathcal{O}'),$$

since the skein relations are clearly satisfied.

In some cases, it is possible, and may be useful, to define an inverse wiring. This is only possible if $m + n = m' + n'$, then the inverse W^{-1} is then the wiring which when surrounding W yields a tangle isotopic to the original. If such a construction is possible, the induced Λ-module homomorphism is then an isomorphism. Such a construction will be required in later work.

Theorem 3.1 *Let $m + n = 2r$. The module $M_n^m(\mathcal{O})$ is a free Λ-module of dimension $r!$ for each choice of boundary orientation $\mathcal{O}$.*

Proof: Our aim is to show, through the use of a wiring diagram, that there is an isomorphism from the module $M_n^m(\mathcal{O})$ to the module $M_r^r(\bar{\mathcal{O}})$ where the orientation $\bar{\mathcal{O}}$ is such that all the entry points are at the top and the exit points are at the bottom. It is then well known from the work in [MT90] that such a Λ-module is free and of dimension $r!$.

We first observe that we can always find a wiring diagram for a general tangle in $M_n^m(\mathcal{O})$ such that the resulting tangle has an orientation (for which we retain the label $\mathcal{O}$) where all entry and exit points on the top are together on the left and right respectively. Similarly, all entry and exit points together on the bottom are together on the left and right respectively. This can clearly be achieved using a braid type wiring which contains no closed loops. Above the rectangle the wiring joins the first points on the left to the inputs and the last on the right to the outputs and below the rectangle it joins the inputs to the first points on the left and the outputs to the last points on the right. Hence there is an inverse wiring available and the wiring is an isomorphism of the modules concerned. The result of such a wiring diagram is shown in Figure 3.3.

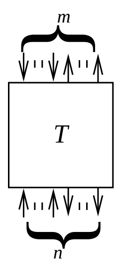


Figure 3.3: The entry and exit points together.

We now consider two wiring diagrams,

$$\begin{array}{rrcl} & L : M_n^m(\mathcal{O}) & \longrightarrow & M_{n-1}^{m+1}(\mathcal{O}'), \\ \text{and} & R : M_n^m(\mathcal{O}) & \longrightarrow & M_{n+1}^{m-1}(\mathcal{O}''), \end{array}$$

with inverses L^{-1} and R^{-1} respectively, giving an isomorphism of the modules. These wirings are shown in Figure 3.4.

We now simply make repeated applications of the isomorphisms L and R, the composition of these will then yield the required isomorphism from $M_n^m(\mathcal{O})$ to $M_r^r(\bar{\mathcal{O}})$. □

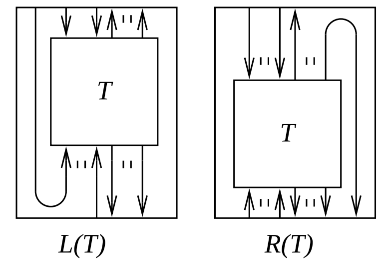


Figure 3.4: The wiring diagrams L and R.

3.2 Algebras

It is clear that we can compose a tangle from $\mathcal{T}_n^m(\mathcal{O})$ with one from $\mathcal{T}_p^n(\mathcal{O}')$ by placing one on top of the other, modulo the matching of the orientations where the tangles join. We then obtain a tangle from $\mathcal{T}_p^m(\mathcal{O}'')$ where $\mathcal{O}''$ is the expected orientation. We then have a bilinear composition

$$M_n^m(\mathcal{O}) \times M_p^n(\mathcal{O}') \longrightarrow M_p^m(\mathcal{O}''). \tag{3.1}$$

Definition. An orientation $\mathcal{O}$ on (r,r)-tangles is said to be *matched* if the orientation of the top matches that of the bottom in such a way that they could be place on top of each other.

In any matched orientation as described above, the top will consist of n entry points and $p = r - n$ exit points in some order and the bottom will consist of n exit points and p entry point in the same order.

Suppose that $M_r^r(\mathcal{O})$ has a matched orientation. We then have a bilinear composition of the form of (3.1) with $r = m = n = p$ and $\mathcal{O} = \mathcal{O}' = \mathcal{O}''$, giving $M_r^r(\mathcal{O})$ the structure of an algebra over Λ. Composition in this algebra is induced by composition of tangles.

We will adopt the convention of writing $M_{n,p}$ for the algebra $M_r^r(\mathcal{O})$ where the orientation is matched and the top points consists of n entry points together on the left and p exit points together on the right. This configuration is shown in Figure 3.5.

Theorem 3.2 *For each matched orientation $\mathcal{O}$ on (r,r)-tangles with n entry and p exit points at the top, the algebra $M_r^r(\mathcal{O})$ is isomorphic as an algebra to $M_{n,p}$.*

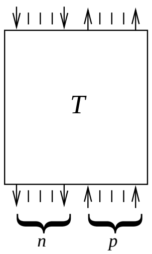


Figure 3.5: A tangle in $M_{n,p}$.

Proof: Given any element $T \in \mathcal{T}_r^r(\mathcal{O})$. We choose a braided (r,r)-tangle, α, with no loops, only the $r = n + p$ arcs (compare this with the proof of Theorem 3.1). We then fix α such that the first n points at the top join the top entry points of T and the last p points join the top exit points of T. Then the tangle α^{-1} is the inverse of the braid α. It is then clear that the tangle $\alpha T \alpha^{-1}$ defines an element in $M_{n,p}$. The map resulting from this construction

$$\alpha_* : M_r^r(\mathcal{O}) \longrightarrow M_{n,p}$$

is an isomorphism of algebras. If we were to choose a different α which had the same effect then the new isomorphism would only differ by an inner automorphism. $\square$

We are interested in studying the algebras $M_{n,p}$ for different n and p. We already know from Theorem 3.1 that $M_{n,p}$ has dimension $(n+p)!$. It should be noted here that if we consider the special cases with $p = 0$ or $n = 0$, then the algebras $M_{n,0}$ or $M_{0,p}$ respectively are known to be isomorphic to the Hecke algebra $H_n(z)$ with coefficient ring extended to Λ.

Remark. The algebra M_n described in Chapter 2 was constructed using tangles up to ambient isotopy, whereas the algebra $M_{n,0}$ for example considers tangles up to regular isotopy. It should be noted that they are both in fact isomorphic and therefore both isomorphic to the Hecke algebra since in this setting, we have no need for considering Reidemeister move R_I as every tangle can be thought of as a linear combination of braids.

Definition. We define the algebra $M_{2n}^{2n}(\mathcal{O})$ in which the orientation $\mathcal{O}$ specifies that the entry and exit point alternate, to be the *string algebra on* $2n$ *strings* which we will denote $S_{n,n}$.

A direct consequence of Theorem 3.2 is that the algebra $M_{n,n}$ is isomorphic to the string algebra on $2n$ strings, $S_{n,n}$. Further discussion of the string algebra and its possible uses will take place in Section 4.1.

3.3 The algebra $M_{n,p}$

It is clear from what has already been said that the algebras $M_{n,p}$ contain subalgebras isomorphic to $M_{n,0} \times M_{0,p}$, which is a product of two Hecke algebras. This leads to the inclusions $M_{n,0} \subset M_{n,p}$ and $M_{0,p} \subset M_{n,p}$. These inclusions come from adding p straight strings to the right of an (n,n)-tangle and n straight strings to the left of a (p,p)-tangle. The wiring diagrams required are shown in Figure 3.6.

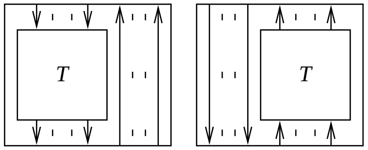


Figure 3.6: The inclusions $M_{n,0} \subset M_{n,p}$ and $M_{0,p} \subset M_{n,p}$.

We now want to describe small parts of a tangle in $M_{n,p}$ using a method similar to the elementary braids used in braid theory. We therefore label the points as follows: the first n points by $-n, \ldots, -1$ and the last p points by $1, \ldots, p$. We then write σ_i where $-n < i < 0$ for the braid positively interchanging strings $i-1$ and i, and σ_i where $0 < i < p$ for the braid positively interchanging strings i and $i+1$. We adopt the notation $g_i \in M_{m,p}$ $(-n < i < p,\ i \neq 0)$, for the element of the algebra represented by σ_i.

It is well known that the elements g_i $(-n < i < 0)$ generate the subalgebra isomorphic to $M_{n,0}$ and similarly, the g_i $(0 < i < p)$ generate the subalgebra isomorphic to $M_{0,p}$. These algebras clearly commute.

We must now introduce another elementary tangle F_k, which we shall say represents the element $f_k \in M_{n,p}$. The tangle F_k is shown in Figure 3.7. As one can see, there are strings between the points $-i$ and i at the top and $-i$ and i at the bottom for $1 \leq i \leq k$. The tangle F_0 (or the element of the algebra, f_0) is simply the identity in their respective contexts.

Proposition 3.3 *The following intertwining relations hold for* $1 \leq i < k$*:*

$$\begin{aligned} g_i f_k &= g_{-i} f_k, \\ f_k g_i &= f_k g_{-i}. \end{aligned}$$

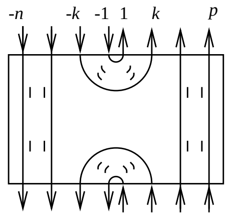


Figure 3.7: The tangle F_k.

Proof: It is enough to consider the tangles these elements of $M_{n,p}$ represent, some simple diagrams will then suffice to prove the relations given. Such a diagram is shown in Figure 3.8 for the first relation. A similar picture is required for the second. □

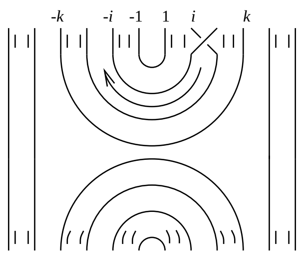


Figure 3.8: The first intertwining relation.

Remark. It is easy to observe that we may write F_2 in terms of F_1 and the braid generator σ_1. We have

$$\begin{aligned} F_2 &= F_1\sigma_1\sigma_{-1}^{-1}F_1 \\ &= F_1\sigma_1^{-1}\sigma_{-1}F_1. \end{aligned}$$

Simple diagrams will confirm these relations. We can extend this further and write

$$\begin{aligned} F_k &= F_{k-1}\sigma_{k-1}\sigma_{k-2}\cdots\sigma_1\sigma_{-(k-1)}^{-1}\sigma_{-(k-2)}^{-1}\cdots\sigma_{-1}^{-1}F_1 \\ &= F_{k-1}\sigma_{-(k-1)}^{-1}\sigma_{-(k-2)}^{-1}\cdots\sigma_{-1}^{-1}\sigma_{k-1}\sigma_{k-2}\cdots\sigma_1 F_1 \\ &= F_1\sigma_1\sigma_2\cdots\sigma_{k-1}\sigma_{-1}^{-1}\sigma_{-2}^{-1}\cdots\sigma_{-(k-1)}^{-1}F_{k-1}. \end{aligned}$$

Therefore, in the algebra $M_{n,p}$, we can write f_k inductively in terms of f_1 and the elements g_i,

$$\begin{aligned} f_k = (f_1g_1g_2\cdots g_{k-1}g_{-1}^{-1}g_{-2}^{-1}\cdots g_{-(k-1)}^{-1})(f_1g_1g_2\cdots g_{k-2}g_{-1}^{-1}g_{-2}^{-1}\cdots g_{-(k-2)}^{-1})\cdots \\ \cdots(f_1g_1g_{-1}^{-1})(f_1). \end{aligned}$$

3.4 Wrap-up maps

Definition. A *wrap-up map* is one of the following two maps

$$\begin{aligned} W : M_n &\longrightarrow M_{n-1}, \\ \text{and} \qquad W' : M_n &\longrightarrow M_{n-1}, \end{aligned}$$

given by the wiring diagrams shown in Figure 3.9.

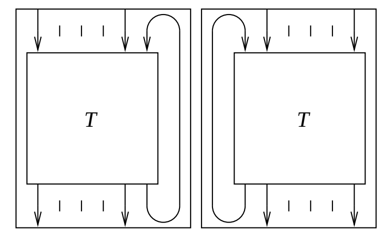


Figure 3.9: The wrap-up maps W and W'.

We could generalise this notion further for some (r, s)-tangle, provided the orientation is suitable. In fact, it is such a map that is used in the final stages of the algorithm mentioned in Chapter 2 and discussed in more detail in [MS90].

Returning to the maps defined on tangles in M_n, we have that W as the projection of the Hecke algebra, up to a scalar, on the subalgebra $M_{n-1} \subset M_n$ (where inclusion is achieved through the addition of an extra string). In other words, if we consider a tangle on $n-1$ strings and an element $T \in M_{n-1}$ in the algebra. Then thinking of it as a element of M_n, applying the map W gives $W(T) = \delta T$.

Now take $T \in M_n$ and retain the same label T for the tangle with p straight strings added to the right, now with $T \in M_{n,p}$. In a similar way, we write $W(T)$ for the tangle in $M_{n,p}$ with $p+1$ straight strings added to the right of the original tangle $W(T) \in M_{n-1}$.

Proposition 3.4 *For any $T \in M_n$ we have*

$$\begin{aligned} f_1 T f_1 &= W(T) f_1 \\ &= f_1 W(T), \end{aligned}$$

in $M_{n,p}$

Proof: Result immediately seen from comparing tangles. □

In a similar way, a tangle $S \in M_p$ with the wrap-up map W' can be seen to satisfy $f_1 S f_1 = W'(S) f_1 = f_1 W'(S)$.

Corollary 3.5 *Let $U \in M_n \times M_p \subset M_{n,p}$ be represented by a tangle in which either string 1 or -1 are straight. Then for each $k \geq 1$,*

$$f_1 U f_k = U' f_k,$$

for some $U' \in M_n \times M_p$.

Proof: We combine the previous result with the fact that $f_k = f_1 h$ for some h, and since U can be written as the product of two tangles in M_n and M_p, then by the hypothesis on the strings, one will commute in $M_{n,p}$ with f_1. □

Theorem 3.6 *The algebra $M_{n,p}$ is generated by f_1 and the braid elements g_i, $-(n-1) \leq i \leq p-1$, $i \neq 0$.*

Proof: We introduce the wiring diagram V which represents and isomorphism of modules $V : M_r \mapsto M_{n,p}$, where $r = n + p$. The wiring diagram V is shown in Figure 3.10.

Now it is well-known that the algebra M_r is spanned by the positive permutation braids on r strings, [MT90]. Each of these braids can be written in the form $B \Sigma_k B'$ for some k, where $B = B_1 B_2$ and $B' = B_1' B_2'$ with B_1, B_1' as positive permutation braids on the first n strings and B_2, B_2' are ppbs on the last p strings. Also, Σ_k only involves the strings $-1, \ldots, -k$ crossing over the strings $1, \ldots, k$. The positive permutation braid drawn in this form is shown schematically in Figure 3.11.

We then apply V to this braid. We can clearly redraw this tangle such that it looks like the tangle shown in Figure 3.12. Then B_2 and B_2' can be turned upside down and made to incorporate the relevant half-twists to become C_2' and C_2. Our tangle is then of the form $C f_k C'$ where $C, C' \in M_{n,0} \times M_{0,p}$. It is known that both C and C' can be written as a monomial in $\{g_i\}$ and f_k can be written as a monomial in f_1 and g_i. Also, since V is an isomorphism the elements $V(B \Sigma_k B')$ span the algebra $M_{n,p}$, then the result follows. □

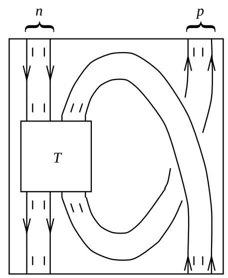


Figure 3.10: The wiring diagram V.

In the course of this proof we have actually proved the following:

Corollary 3.7 *The algebra $M_{n,p}$ is spanned by the subsets:*

$$\{B f_i B'\}; i = 0, \ldots, min(n,p); B, B' \in M_{n,0} \times M_{0,p};$$

where f_0 is the identity element of the algebra.

3.5 An explicit spanning set for $M_{n,p}$

We have already seen before that for the cases $p = 0$ and $n = 0$, the spanning sets for M_n and M_p consist of the positive permutation braids, $\{\beta_\pi\}$, $\pi \in S_n$ and $\{\beta_\rho'\}$, $\rho \in S_p$ respectively. As noted, the convenient nature of the spanning set has enabled an algorithm to be written to find P_k for a knot presented as the closure of a braid, [MS90].

We recall that a positive permutation braid is such that each pair of strings cross at most once, and each crossing is a positive crossing. It was shown in [Elr88] and subsequently in [EM94] that any such braid depends only on the permutation it induces on its strings. Therefore, if two positive permutation braids induce the same permutation, then they are equal braids.

We now have need for defining a special class of positive permutation braids.

Definition. A *Lorenz (l,r) braid* is a positive permutation braid on $l + r$ strings, where reading from the bottom, neither the left hand group of l strings, nor the right hand group of r strings cross among themselves. We can then define a *reverse Lorenz (l,r) braid* in a similar way, except we now read from the top.

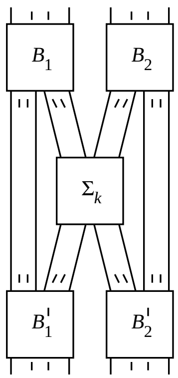


Figure 3.11: A positive permutation braid in the form $B\Sigma_k B'$.

An example of a Lorenz $(4,2)$ braid is shown in Figure 3.13, together with its corresponding reverse Lorenz $(4,2)$ braid.

Remark. Any positive permutation braid on $n = l+r$ strings can be written as a Lornez (l,r) braid composed with a positive permutation braid on the first l strings and a positive permutation braid on the last r strings.

We now come to the main theorem of this section.

Theorem 3.8 *The algebra $M_{n,p}$ can be spanned by the following $r! = (n+p)!$ elements:*

$$\{B_1\beta_2 f_i\gamma_1 C_2\},\ \textit{for } i = 0,\ldots,\min(n,p),$$

where B_1 and C_2 are positive permutation braids on the first n and the last p strings respectively, while β_2 is a Lorenz $(i, p-i)$ braid on the last p strings and γ_1 is a reverse Lorenz $(n-i, i)$ braid on the first n strings. These elements form a free Λ-basis for $M_{n,p}$.

Proof: We can write such a tangle more symmetrically if we use the decomposition of positive permutation braids described in the above remark. We apply this decomposition to the braids B_1 and C_2. The result is therefore of the form $\beta_1\alpha_1\delta_1\beta_2 f_i\gamma_1\alpha_2\delta_2\gamma_2$, which is best explained by means of Figure 3.14.

Now, we know from Corollary 3.7 that the algebra $M_{n,p}$ is spanned by elements of the form $B_1B_2f_iB_1'B_2'$ where B_1, B_2, B_1', B_2' are all positive

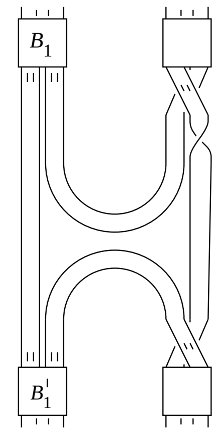


Figure 3.12: The tangle $V(B\Sigma_k B')$.

permutation braids on the appropriate numbers of strings. Decomposing each of these braids in the now familiar way yields a tangle of the form shown in Figure 3.15. It is then easy to see that we can move α_2 and α_1' around to join α_1 and α_2' respectively. Similarly, we move δ_2 and δ_1' to join δ_1 and δ_2' respectively. Applications of appropriate relations result in a linear combination of tangles of the form of Figure 3.14.

A count then shows us that there are $(n+p)!$ such tangles, and we know this to be the dimension of $M_{n,p}$, so the result follows. □

3.6 Multiplication in $M_{n,p}$

We are interested in trying to determine how to multiply each basis element of $M_{n,p}$ by the generators g_i (for $-n < i < p$, $i \neq 0$) and f_1. We aim to find the result as a linear combination of other basis elements. The consequence of such information would be that we would know, subject to doing a little work, how to calculate all products.

It is important to be aware of the case for the algebra M_n, which is summarised in Chapter 2. This case will be made use of in the subsequent sections.

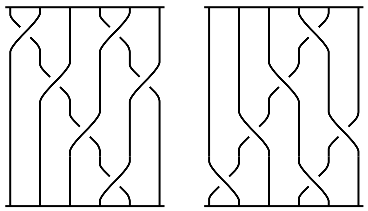


Figure 3.13: A Lorenz and corresponding reverse Lorenz $(4, 2)$ braid.

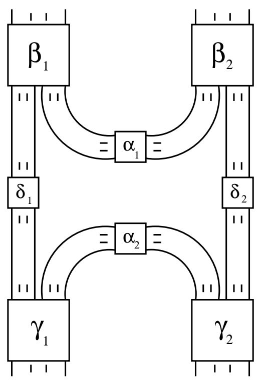


Figure 3.14: The decomposition of $B_1\beta_2 f_i \gamma_1 C_2$.

3.6.1 Multiplication by the generators g_i

Recall that we have shown that a basis of $M_{n,p}$ is the set of tangles of the form $\{B_1\beta_2 f_i \gamma_1 C_2\}$ where B_1 and C_2 are positive permutation braids on n and p strings respectively, β_2 is a Lorenz $(i, p-i)$ braid and γ_1 is a reverse Lorenz $(n-i, i)$ braid.

With knowledge of the work described in Chapter 2, left multiplication of such a tangle by g_i is straightforward. First, if $i \leq -1$, then the crossing will interact with the positive permutation braid B_1 and a subsequent application of the skein relation will result in a linear combination of tangles from the basis. If $i \geq 1$ then the crossing will either interact with the Lorenz braid β_2 or otherwise with one of the positive permutation braids C_2 or B_1, the

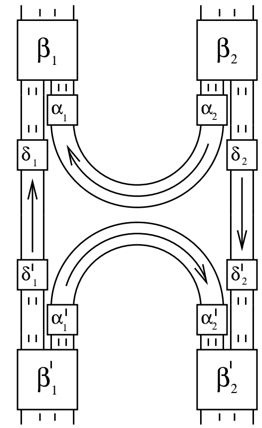


Figure 3.15: The decomposition of $B_1 B_2 f_i B_1' B_2'$.

latter interaction being made via f_i through an application of the appropriate intertwining relation described in Proposition 3.3.

Right multiplication requires a similar approach to the above.

Remark. To simplify the rewriting involved here, it is likely to be beneficial to consider B_1 and C_2 decomposed as $\beta_1\alpha_1\delta_1$ and $\alpha_2\delta_2\gamma_2$ where α_i and δ_i are positive permutation braids, β_1 is a Lorenz braid and γ_2 is a reverse Lorenz braid. This will result in the process being more efficient since the positive permutation braids involved are on smaller numbers of strings.

3.6.2 Multiplication by the generator f_i

We again consider the basis elements in the decomposed form with B_1 written as $\beta_1\alpha_1\delta_1$ and C_2 written as $\alpha_2\delta_2\gamma_2$ as before. This case is more complicated, and we break it down into four smaller cases after the following observation.

Remark. The Lorenz $(n-i, i)$ braid β_1 joins string -1 at the top to either -1 or $-(i+1)$. Similarly, the Lorenz $(i, p-i)$ braid joins string 1 at the top to either 1 or $i+1$.

We now need to call on the wrap-up maps described above. The first case to be considered is if β_1 joins -1 to -1 and β_2 joins 1 to 1. Then a simple picture will immediately show that

$$f_1\beta_1\alpha_1\delta_1\beta_2 f_i = \beta_1 W(\alpha_1)\delta_1\beta_2 f_i.$$

The result can be rewritten as a linear combination of basis elements made up of components of a similar form to the original.

The following two cases are similar. Here we deal with one completely and leave the second for the reader to verify using exactly an analogous method. Consider the case where β_1 joins -1 to -1 and β_2 joins -1 to $-(i+1)$. Consideration of the diagrams shown in Figure 3.16 will convince the reader that

$$f_1\beta_1\alpha_1\delta_1\beta_2 f_i = \beta_1\delta_1 W'(\beta_2\bar{\alpha_1})f_i,$$

where $\bar{\alpha_1}$ is the positive permutation braid α_1 turned upside down. It is then

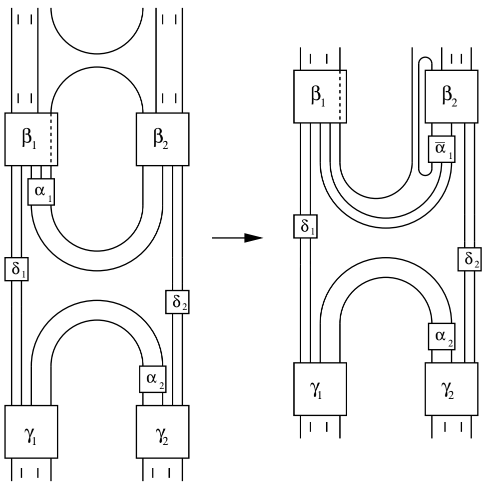


Figure 3.16: $f_1\beta_1\alpha_1\delta_1\beta_2 f_i = \beta_1\delta_1 W'(\beta_2\bar{\alpha_1})f_i$.

possible to rewrite the result as a linear combination of the basis elements since application of the skein relations on $W'(\beta_2\bar{\alpha_1})$ will result in a linear combination on positive permutation braids, then we decompose these positive permutation braids in the usual way so we are left with a Lorenz braid in the top right corner. The associated positive permutation braids on i and $p-i$ strings are then moved to interact with B_1 (via f_i) and C_2, further

applications of the skein relations will almost immediately result in a linear combination of tangles of the form of the basis elements.

The previous three cases can be summarised from a more practical viewpoint in the following way: we begin by finding if β_2 joins 1 to 1, otherwise we find if β_1 joins -1 to -1. We then have:

$$f_1\beta_1\alpha_1\delta_1\beta_2 f_i = \begin{cases} W(\beta_1\alpha_1)\delta_1\beta_2 f_i & \text{if } \beta_2 \text{ joins 1 to 1,} \\ \beta_1\delta_1 W'(\beta_2\bar{\alpha_1})f_i & \text{if } \beta_1 \text{ joins } -1 \text{ to } -1. \end{cases}$$

The braid $\bar{\alpha_1}$ can be thought of as the result of an anti-automorphism we shall call η, defined by:

$$\begin{aligned} \eta(g_i^{\pm 1}) &= g_i^{\pm 1}, \\ \eta(f_1) &= f_1, \end{aligned}$$

where given a tangle with word length m we have,

$$\eta\left(\prod_{j=1}^{m} x_j\right) = \prod_{j=1}^{m} x_{m-j+1},$$

with $x_j \in \{g_i^{\pm 1}, f_1\}$.

Remark. In fact, it is not strictly necessary to have explicit knowledge of W', therefore making the computation slightly more straightforward. Instead we must introduce a simple automorphism we shall call ρ which turns the tangle over side-to-side. We define ρ as follows:

$$\begin{aligned} \rho(g_i^{\pm 1}) &= g_{-i}^{\pm 1}, \\ \rho(f_1) &= f_1. \end{aligned}$$

For example, $\rho(g_1 f_1 g_{-2} g_3) = g_{-1} f_1 g_2 g_{-3}$. If β_1 joins -1 to -1, then we could use this map to turn the tangle over, use W as in the other case, and after appropriate applications of the skein relations we apply the map again to turn the tangle over again.

Finally, the most troublesome case is when β_1 joins -1 to $-(i+1)$ and β_2 joins 1 to $i+1$. Figure 3.17 shows schematically that here we are mainly interested in resolving, with the skein relations, a diagram which contains a section we shall refer to as a *clasp*. More descriptively, we shall refer to a clasp involving i strings from the top and j strings from the bottom as an (i,j)-*clasp*, as shown in Figure 3.18.

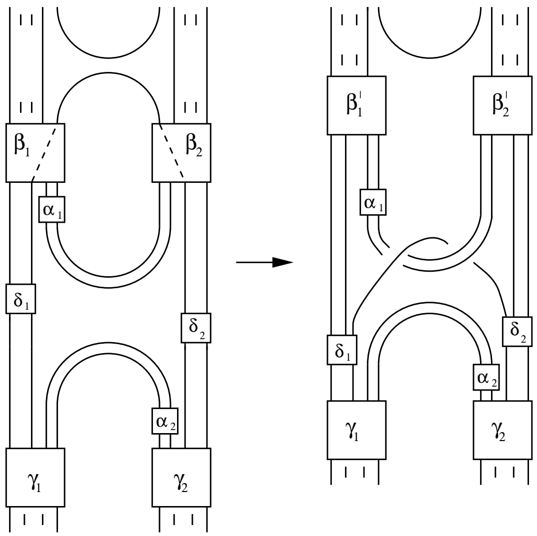
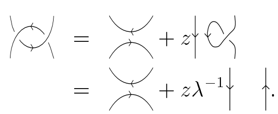
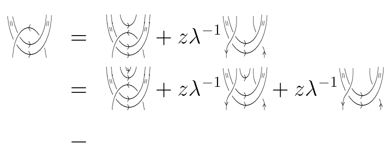


Figure 3.17: We obtain a clasp.

Therefore the clasp we are particularly interested in is a $(i, 1)$-clasp. Before moving to this more general case we consider a $(1, 1)$-clasp. The skein relations tell us that:



So in the case of the $(i, 1)$-clasp, repeated applications of the skein relations will allow the clasp to pull free at the expense of combining it with a linear combination of other tangles. In general, the linear combination will look like:



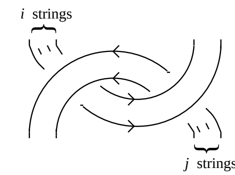


Figure 3.18: A (i, j)-clasp.

We now consider such a linear combination in the context of the original tangle. We have one contribution of the form $\beta_1''\alpha_1'\beta_2''f_{i+1}\delta_1\gamma_1C_2'$, shown in Figure 3.19.

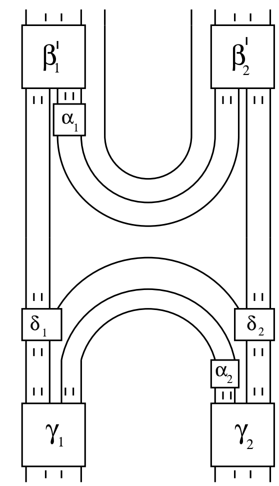


Figure 3.19: One contribution of this type.

The other i contributions are all multiplied by $z\lambda^{-1}$ with the main areas of interest shown in Figure 3.20.

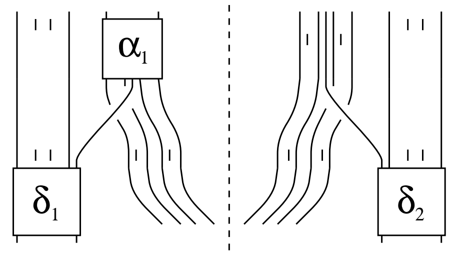


Figure 3.20: The other i contributions involve areas of this type.

It is clear that due to the nature of positive permutation braids, after sufficient rewriting with the skein relations, the result will be a linear combination of elements of the basis. The question therefore becomes how do we resolve this as a linear combination in an efficient and uniform way? The answer is not immediate. We therefore only offer the following comments on a far from ideal method.

Remark. With the first contribution shown in Figure 3.19, we know that δ_1 is a positive permutation braid so we can "pull out" the end string since positive permutation braids are in a sense layered and this end string lies in the top layer. We then push the crossings obtained through γ_1. The result of this is that we still must deal with the bottom right corner with the skein relations to obtain a linear combination of the basis elements, all composed with these extra crossings. We then multiply by the g_i as described above.

Remark. In a similar approach to that described above, we can limit the area to be dealt with by the skein relations "inside" the tangle at the expense of having to multiply by some g_i. The area of interest after such a re-drawing is shown in Figure 3.21.

The above work almost immediately gives the following result.

Proposition 3.9 *The subspace spanned by the subsets*

$$\{Bf_iB'\}; i = k, \ldots, min(n,p); B, B' \in M_{n,0} \times M_{0,p};$$

is the ideal generated by f_k.

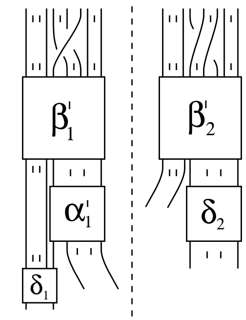


Figure 3.21: Redrawing of areas in Figure 3.20.

Proof: We have seen that the subspace spanned by the elements $\{Bf_iB'\}$ for $i \geq k$ is stable under left-multiplication by the generators of the algebra. Symmetry tells us that this is also true for right-multiplication. Hence the subspace is a 2-sided ideal which is contained in the ideal generated by f_k and also contains f_k. □

3.7 Algebraic properties of $M_{n,p}$

We end this chapter with a look at some of the algebraic properties of $M_{n,p}$.

3.7.1 A filtration of $M_{n,p}$

Let $M_{n,p}^{(j)}$ denote the algebra spanned by elements of the form $\{B_1\beta_2 f_i\gamma_1 C_2\}$ where $i \geq j$. In other words, the spanning set in question contains elements with "at least" i strings turning back. This yields the filtration:

$$M_{n,p} = M_{n,p}^{(0)} \triangleright M_{n,p}^{(1)} \triangleright \cdots \triangleright M_{n,p}^{(k)},$$

where $k = \min(n,p)$. This is similar to the Hanlon-Wales filtration discussed in [MW89], which relates to the work of P. Hanlon and D. Wales on Brauer centralizer algebras.

Proposition 3.10 *The quotient algebra* $M_{n,p}^{(i)}/M_{n,p}^{(i+1)}$*, where exactly* i *strings turn back, has dimension*

$$\frac{(n!)^2(p!)^2}{(i!)^2(n-i)!(p-i)!}.$$

Proof: We consider the tangles in the form $B_1\beta_2 f_i\gamma_1 C_2$, where i is fixed, B_1, C_2 are positive permutation braids and β_2, γ_1 are the usual types of Lorenz braids.

Then for the braid B_1, there is a choice of $n!$ and similarly for C_2 there is a choice of $p!$. Also, for β_2 and γ_1 there are $\binom{n}{i}$ and $\binom{p}{i}$ choices respectively.

Simplification of the product of these will then yield the required dimension. □

3.7.2 A presentation of the algebra $M_{n,p}$

We now state and prove the main theorem of this work. It offers a presentation of the algebra $M_{n,p}$.

Main Theorem. *The algebra $M_{n,p}$ has a presentation with generators g_i, for $-(n-1) \leq i \leq p-1$, $i \neq 0$ and f_1, and relations:*

1. *Far commutativity:*
   - $g_i g_j = g_j g_i$, *with* $|i-j| \geq 2$ *and* $-(n-1) \leq i, j \leq -1$,
   - $g_i g_j = g_j g_i$, *with* $|i-j| \geq 2$ *and* $1 \leq i, j \leq p-1$,
   - $g_i g_j = g_j g_i$, *with* $-(n-1) \leq i \leq -1$ *and* $1 \leq j \leq p-1$,
   - $g_i f_1 = f_1 g_i$, *with* $|i| \geq 2$;
2. *Braid relation:*
   - $g_i g_{i-1} g_i = g_{i-1} g_i g_{i-1}$, *with* $-(n-2) \leq i \leq -1$,
   - $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$, *with* $1 \leq i \leq p-2$;
3. *Twist relation:*
   - $f_1 g_1^{\varepsilon} f_1 = \lambda^{-\varepsilon} f_1$, *with* $\varepsilon = \pm 1$;
4. *Skein relation:*
   - $g_i - g_i^{-1} = z$;
5. *Equivalent definitions of an element f_2:*
   - $f_1 g_1 g_{-1}^{-1} f_1 = f_1 g_1^{-1} g_{-1} f_1$ $(= f_2,\ say)$;
6. *Intertwining relation for f_2:*
   - $g_1 f_2 = g_{-1} f_2$,

   - $f_2 g_1 = f_2 g_{-1}$.

Proof: Let $A_{n,p}$ be the abstract algebra represented by the above generators and relations. There is an obvious surjective homomorphism, ι say, from $A_{n,p}$ to $M_{n,p}$ which is clearly well defined, since the relations hold for both algebras. We need to show that ι is an isomorphism and then we are done.

We begin by considering the cases when $n = 0$ or $p = 0$. Then the algebras $A_{0,p}$ and $A_{n,0}$ are isomorphic to the Hecke algebras H_p and H_n respectively. It is well known, from the work in for example [MT90], that these algebras are isomorphic to $M_{0,p}$ and $M_{n,0}$ respectively, derived from tangles when the coefficient ring is extended.

Now define the elements $f_k \in A_{n,p}$ inductively by:

$$\begin{aligned} f_0 &= \text{identity}, \\ f_1 &= f_1, \\ f_k &= f_1 g_1 g_2 \cdots g_{k-1} g_{-1}^{-1} g_{-2}^{-1} \cdots g_{-(k-1)}^{-1} f_{k-1},\ k \geq 2. \end{aligned}$$

We now require the following two lemmas:

Lemma 3.11 *The following intertwining relations:*

$$\begin{aligned} g_i f_k &= g_{-i} f_k, \\ f_k g_i &= f_k g_{-i}, \end{aligned}$$

hold in $A_{n,p}$ for $1 \leq i < k$.

Proof: There are two cases to consider each splitting further into two subcases. Before embarking on proving this we must make the following observations on two ways of rewriting the element f_k.

Remark. Due to the inductive nature of the element f_k, we can rewrite it as follows:

$$\begin{aligned} f_k &= f_1 g_1 \cdots g_{k-1} g_{-1}^{-1} \cdots g_{-(k-1)}^{-1} f_{k-1} \\ &= f_1 g_1 \cdots g_{k-1} g_{-1}^{-1} \cdots g_{-(k-1)}^{-1} f_1 g_1 \cdots g_{k-2} g_{-1}^{-1} \cdots g_{-(k-2)}^{-1} f_{k-2} \\ &= f_1 g_1 g_{-1}^{-1} f_1 \beta f_{k-2} \\ &= f_2 \beta f_{k-2} \end{aligned}$$

where $\beta = g_2 g_1 g_3 g_2 \cdots g_{k-1} g_{k-2} g_{-2}^{-1} g_{-1}^{-1} g_{-3}^{-1} g_{-2}^{-1} \cdots g_{-(k-1)}^{-1} g_{-(k-2)}^{-1}$. A picture of β is shown in Figure 3.22.

Finally, it is not too difficult to see that an inductive argument on the above definition of f_k will allow us to rewrite it as follows:

$$f_k = f_{k-1} g_{k-1} \cdots g_1 g_{-(k-1)}^{-1} \cdots g_{-1}^{-1} f_1.$$

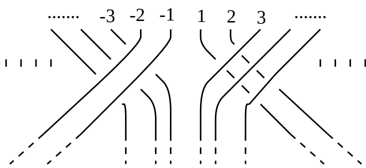


Figure 3.22: The braid β.

We now return to proving the Lemma.
CASE 1: We are required to show that $g_i f_k = g_{-i} f_k$ for $1 \leq i < k$.
(a) $i = 1$: It is now easy to see that:

$$\begin{aligned} g_1 f_k &= g_1 f_2 \beta f_{k-2} \\ &\quad \text{(by Remark)} \\ &= g_{-1} f_2 \beta f_{k-2} \\ &\quad \text{(by (6))} \\ &= g_{-1} f_k. \end{aligned}$$

(b) $2 \leq i < k$: We use induction on k. We know from (6) that for $k = 2$,

$$g_1 f_2 = g_{-1} f_2.$$

Now, assume that for $k = \eta - 1$,

$$g_i f_{\eta-1} = g_{-i} f_{\eta-1}, \quad (\text{for } 1 \leq i < \eta - 1).$$

Now suppose that $1 \leq i < \eta$. Then,

$$\begin{aligned} g_i \underline{f_\eta} &= \underline{g_i}(f_1 g_1 \cdots g_{\eta-1} g_{-1}^{-1} \cdots g_{-(\eta-1)}^{-1}) f_{\eta-1} \\ &\quad \text{(by definition)} \\ &= (f_1 g_1 \cdots \underline{g_i g_{i-1} g_i} \cdots g_{\eta-1} g_{-1}^{-1} \cdots g_{-(\eta-1)}^{-1}) f_{\eta-1} \\ &\quad \text{(by (1))} \\ &= (f_1 g_1 \cdots g_{i-1} g_i \underline{g_{i-1}} \cdots g_{\eta-1} g_{-1}^{-1} \cdots g_{-(\eta-1)}^{-1}) f_{\eta-1} \\ &\quad \text{(by (2))} \\ &= (f_1 g_1 \cdots g_{\eta-1} g_{-1}^{-1} \cdots g_{-(\eta-1)}^{-1}) \underline{g_{i-1} f_{\eta-1}} \\ &\quad \text{(by (1))} \\ &= (f_1 g_1 \cdots g_{\eta-1} g_{-1}^{-1} \cdots g_{-(\eta-1)}^{-1}) \underline{g_{-(i-1)}} f_{\eta-1} \end{aligned}$$

$$\begin{aligned} &\quad \text{(by inductive hypothesis, since } i - 1 < \eta - 1) \\ &= (f_1 g_1 \cdots g_{\eta-1} g_{-1}^{-1} \cdots \underline{g_{-(i-1)}^{-1} g_{-i}^{-1} g_{-(i-1)}} \cdots g_{-(\eta-1)}^{-1}) f_{\eta-1} \\ &\quad \text{(by (1))} \\ &= (f_1 g_1 \cdots g_{\eta-1} g_{-1}^{-1} \cdots \underline{g_{-i}} g_{-(i-1)}^{-1} g_{-i}^{-1} \cdots g_{-(\eta-1)}^{-1}) f_{\eta-1} \\ &\quad \text{(by (2))} \\ &= g_{-i} \underline{(f_1 g_1 \cdots g_{\eta-1} g_{-1}^{-1} \cdots g_{-(\eta-1)}^{-1}) f_{\eta-1}} \\ &\quad \text{(by (1))} \\ &= g_{-i} f_\eta. \end{aligned}$$

Hence the result holds by induction.
CASE 2: We are required to show that $f_k g_i = f_k g_{-i}$ for $1 \leq i < k$. We follow a similar approach to CASE 1.
(a) $1 \leq i < k - 1$: We use induction on k. We know from (6) that for $k = 2$,

$$f_2 g_1 = f_2 g_{-1}.$$

Now, assume that for $k = \eta - 1$,

$$f_{\eta-1} g_i = f_{\eta-1} g_{-i}, \quad (\text{for } 1 \leq i < \eta - 1).$$

Now suppose that $1 \leq i < \eta$. Then following a symmetrical argument to that used above for CASE 1 (b), we obtain the result:

$$f_\eta g_i = f_\eta g_{-i}, \quad (\text{for } 1 \leq i < \eta).$$

Hence the result holds by induction.
(b) $i = k - 1$: We again use the above Remark combined with our knowledge of manipulation of braids. Such an argument yields:

$$\begin{aligned} f_k g_{k-1} &= f_2 \beta f_{k-2} g_{k-1} \\ &\quad \text{(by Remark)} \\ &= f_2 \beta g_{k-1} f_{k-2} \\ &\quad \text{(by (1))} \\ &= f_2 g_1 \beta f_{k-2} \\ &\quad \text{(by Figure 3.22)} \\ &= f_2 g_{-1} \beta f_{k-2} \\ &\quad \text{(by (6))} \\ &= f_2 \beta g_{-(k-1)} f_{k-2} \end{aligned}$$

$$
\begin{aligned}
& \quad \text{(by Figure 3.22)} \\
&= f_2 \beta f_{k-2} g_{-(k-1)} \\
& \quad \text{(by (1))} \\
&= f_k g_{-(k-1)}.
\end{aligned}
$$

□

Lemma 3.12 *The sets*

$$W_j = (A_{n,0} \times A_{0,p}) f_j (A_{n,0} \times A_{0,p}),$$

for $j = 0, \ldots, \min(n, p)$ *span* $A_{n,p}$.

Proof: It is enough to show that left-multiplication of an element of W_j by each of the generators $\{g_i\}$ and f_1 gives a linear combination of elements in $T \in W_k$. Clearly, from knowledge of work such as [MT90], given some tangle $T \in W_j$, $g_i T \in W_j$ for each j. Again consider some tangle $T \in W_j$, then we must show that the product $f_1 T \in W_j$. We will actually see that this lies in the set spanned by W_k with $k \geq j$.

We can see that W_j is spanned by elements of the form $T = B_1 B_2 f_j B_1' B_2'$ where B_1 and B_1' run through a spanning set of $A_{n,0}$, and B_2 and B_2' run through a spanning set of $A_{0,p}$.

On the tangle front, we know that $A_{n,0}$ and $A_{0,p}$ are spanned by appropriately sized positive permutation braids. An equally valid spanning set for these algebras are the negative permutation braids, defined in the obvious way. We will consider $A_{n,0}$ to be spanned by the negative permutation braids and $A_{0,p}$ to be spanned by the positive permutation braids.

So we may assume that

$$B_1 = \beta_1 g_{-1}^{-1} \cdots g_{-l}^{-1} \quad \text{and} \quad B_2 = \beta_2 g_1 \cdots g_{l'}$$

with $\beta_1 = \beta_1(g_{-2}^{-1} \cdots g_{-l}^{-1})$ and $\beta_2 = \beta_2(g_2 \cdots g_{l'})$.

It is immediate that β_1 and β_2 commute with f_1 by relation (1), so we have

$$f_1 B_1 B_2 f_j B_1' B_2' = \beta_1 \beta_2 f_1 g_{-1}^{-1} \cdots g_{-l}^{-1} g_1 \cdots g_{l'} f_j B_1' B_2'.$$

We now split the problem into three cases:

CASE 1: If $l, l' \geq j$ then we have:

$$
\begin{aligned}
f_1 B_1 B_2 f_j B_1' B_2' &= \beta_1 \beta_2 f_1 g_1 \cdots g_j \underline{g_{j+1} \cdots g_{l'}} g_{-1}^{-1} \cdots g_{-j}^{-1} \underline{g_{-(j+1)}^{-1} \cdots g_{-l}^{-1}} f_j B_1' B_2' \\
&= \beta_1 \beta_2 \underline{f_1 g_1 \cdots g_j g_{-1}^{-1} \cdots g_{-j}^{-1} f_j} g_{-(j+1)}^{-1} \cdots g_{-l}^{-1} B_1' g_{j+1} \cdots g_{l'} B_2' \\
& \quad \text{(by (1))}
\end{aligned}
$$

$$
\begin{aligned}
&= \beta_1 \beta_2 f_{j+1} B_1'' B_2'' \\
& \quad \text{(by definition)} \\
&\in W_{j+1}.
\end{aligned}
$$

CASE 2: If $l = l'$ and $1 \leq l < j$, then we use induction on l:

The case for $l = 1$ is as follows:

$$
\begin{aligned}
f_1 B_1 B_2 f_j B_1' B_2' &= \beta_1 \beta_2 f_1 g_1 \underline{g_{-1}^{-1} f_j} B_1' B_2' \\
&= \beta_1 \beta_2 f_1 g_1 g_1^{-1} f_j B_1' B_2' \\
& \quad \text{(by Lemma 3.11)} \\
&= \beta_1 \beta_2 f_1 f_j B_1' B_2' \\
& \quad \text{(by cancellation)} \\
&= \beta_1 \beta_2 f_1 f_1 g_1 \cdots g_{j-1} g_{-1}^{-1} \cdots g_{-(j-1)}^{-1} f_{j-1} B_1' B_2' \\
& \quad \text{(by definition)} \\
&= \beta_1 \beta_2 \delta f_1 g_1 \cdots g_{j-1} g_{-1}^{-1} \cdots g_{-(j-1)}^{-1} f_{j-1} B_1' B_2' \\
& \quad \text{(by (3) and (4) where } \delta = \tfrac{\lambda^{-1} - \lambda}{z}\text{)} \\
&= \delta \beta_1 \beta_2 f_j B_1' B_2' \\
& \quad \text{(by definition)} \\
&\in W_j.
\end{aligned}
$$

Now assume that for $l = \mu - 1$

$$f_1 B_1 B_2 f_j B_1' B_2' = \delta \beta_1 \beta_2 f_j B_1' B_2' \subset W_j.$$

Now consider $l = \mu$:

$$
\begin{aligned}
f_1 B_1 B_2 f_j B_1' B_2' &= \beta_1 \beta_2 f_1 g_1 \cdots g_{\mu-1} g_\mu g_{-1}^{-1} \cdots g_{-(\mu-1)}^{-1} g_{-\mu}^{-1} f_j B_1' B_2' \\
&= \beta_1 \beta_2 f_1 g_1 \cdots g_{\mu-1} g_\mu g_{-1}^{-1} \cdots g_{-(\mu-1)}^{-1} g_\mu^{-1} f_j B_1' B_2' \\
& \quad \text{(by Lemma 3.11)} \\
&= \beta_1 \beta_2 f_1 g_1 \cdots g_{\mu-1} g_\mu g_\mu^{-1} g_{-1}^{-1} \cdots g_{-(\mu-1)}^{-1} f_j B_1' B_2' \\
& \quad \text{(by (1))} \\
&= \beta_1 \beta_2 f_1 g_1 \cdots g_{\mu-1} g_{-1}^{-1} \cdots g_{-(\mu-1)}^{-1} f_j B_1' B_2' \\
& \quad \text{(by cancellation)} \\
&= \delta \beta_1 \beta_2 f_j B_1' B_2' \\
& \quad \text{(by inductive hypothesis)} \\
&\in W_j.
\end{aligned}
$$

The result follows by induction.
CASE 3: If $1 \leq l < j$ and $l < l'$ then we use induction on l (an analagous argument is required for when the roles of l and l' are reversed):

We first make the following observation.

Remark. For this case, we can always manipulate the tangle thus:

$$\begin{aligned}
f_1B_1B_2f_jB_1'B_2' &= \beta_1\beta_2 f_1 g_1\cdots g_l g_{-1}^{-1}\cdots g_{-(l+1)}^{-1}g_{-(l+2)}^{-1}\cdots g_{-(j-1)}^{-1}\underline{g_{-j}^{-1}\cdots}\\
&\qquad \underline{\cdots g_{-l'}^{-1}}f_jB_1'B_2'\\
&= \beta_1\beta_2 f_1 g_1\cdots g_l g_{-1}^{-1}\cdots g_{-(l+1)}^{-1}\underline{g_{-(l+2)}^{-1}\cdots g_{-(j-1)}^{-1}}f_j\underline{g_{-j}^{-1}\cdots}\\
&\qquad \underline{\cdots g_{-l'}^{-1}B_1'}B_2'\\
&= \beta_1\underline{\beta_2 g_{j-1}^{-1}\cdots g_{l+2}^{-1}}f_1g_1\cdots g_l g_{-1}^{-1}\cdots g_{-(l+1)}^{-1}B_1''B_2'\\
&= \beta_1\beta_2' f_1g_1\cdots g_l g_{-1}^{-1}\cdots g_{-l}^{-1}g_{-(l+1)}^{-1}B_1''B_2'.
\end{aligned}$$

The base case for $l = 1$ is as follows:

$$\begin{aligned}
f_1B_1B_2f_jB_1'B_2' &= \beta_1\beta_2f_1g_1g_{-1}^{-1}\underline{g_{-2}^{-1}f_j}B_1'B_2'\\
&= \beta_1\beta_2f_1g_1\underline{g_{-1}^{-1}}g_2^{-1}f_jB_1'B_2'\\
&\qquad \text{(by Lemma 3.11)}\\
&= \beta_1\beta_2f_1g_1g_2^{-1}\underline{g_{-1}^{-1}f_j}B_1'B_2'\\
&\qquad \text{(by (1))}\\
&= \beta_1\beta_2f_1\underline{g_1g_2^{-1}g_1^{-1}}f_jB_1'B_2'\\
&\qquad \text{(by (2))}\\
&= \beta_1\beta_2f_1\underline{g_2^{-1}}g_1^{-1}g_2f_jB_1'B_2'\\
&\qquad \text{(by (1))}\\
&= \underline{\beta_1g_2^{-1}}\beta_2f_1g_1^{-1}\underline{g_2f_j}B_1'B_2'\\
&\qquad \text{(by Lemma 3.11)}\\
&= \beta_1'\beta_2f_1g_1^{-1}\underline{g_{-2}}f_jB_1'B_2'\\
&\qquad \text{(by (1))}\\
&= \beta_1'\underline{\beta_2g_{-2}}f_1g_1^{-1}\underline{f_j}B_1'B_2'\\
&\qquad \text{(by definition)}\\
&= \beta_1'\beta_2'\underline{f_1g_1^{-1}f_1}g_1\cdots g_{j-1}g_{-1}^{-1}\cdots g_{-(j-1)}^{-1}f_{j-1}B_1'B_2'\\
&\qquad \text{(by (3))}\\
&= \beta_1'\beta_2'\lambda\underline{f_1g_1\cdots g_{j-1}g_{-1}^{-1}\cdots g_{-(j-1)}^{-1}f_{j-1}}B_1'B_2'
\end{aligned}$$

$$\begin{aligned}
&\qquad \text{(by definition)}\\
&= \lambda\beta_1'\beta_2'f_jB_1'B_2'\\
&\in W_j.
\end{aligned}$$

Now assume that for $l = \mu - 1$:

$$f_1B_1B_2f_jB_1'B_2' = \lambda\beta_1'\beta_2''f_jB_1'B_2' \subset W_j.$$

Now consider $l = \mu$:

$$\begin{aligned}
f_1B_1B_2f_jB_1'B_2' &= \beta_1\beta_2'f_1g_1\cdots g_{\mu-1}g_\mu g_{-1}^{-1}\cdots g_{-(\mu-1)}^{-1}g_{-\mu}^{-1}\underline{g_{-(\mu+1)}^{-1}f_j}B_1'B_2'\\
&= \beta_1\beta_2'f_1g_1\cdots g_{\mu-1}g_\mu g_{-1}^{-1}\cdots g_{-(\mu-1)}^{-1}g_{-\mu}^{-1}\underline{g_{\mu+1}^{-1}}f_jB_1'B_2'\\
&\qquad \text{(by Lemma 3.11)}\\
&= \beta_1\beta_2'f_1g_1\cdots g_{\mu-1}g_\mu g_{\mu+1}^{-1}g_{-1}^{-1}\cdots g_{-(\mu-1)}^{-1}\underline{g_{-\mu}^{-1}f_j}B_1'B_2'\\
&\qquad \text{(by (1))}\\
&= \beta_1\beta_2'f_1g_1\cdots g_{\mu-1}g_\mu g_{\mu+1}^{-1}g_{-1}^{-1}\cdots g_{-(\mu-1)}^{-1}\underline{g_\mu^{-1}}f_jB_1'B_2'\\
&\qquad \text{(by Lemma 3.11)}\\
&= \beta_1\beta_2'f_1g_1\cdots g_{\mu-1}\underline{g_\mu g_{\mu+1}^{-1}g_\mu^{-1}}g_{-1}^{-1}\cdots g_{-(\mu-1)}^{-1}f_jB_1'B_2'\\
&\qquad \text{(by (1))}\\
&= \beta_1\beta_2'f_1g_1\cdots g_{\mu-1}\underline{g_{\mu+1}^{-1}}g_\mu^{-1}\underline{g_{\mu+1}}g_{-1}^{-1}\cdots g_{-(\mu-1)}^{-1}f_jB_1'B_2'\\
&\qquad \text{(by (2))}\\
&= \beta_1\beta_2'g_{\mu+1}^{-1}f_1g_1\cdots g_{\mu-1}g_\mu^{-1}g_{-1}^{-1}\cdots g_{-(\mu-1)}^{-1}\underline{g_{\mu+1}f_j}B_1'B_2'\\
&\qquad \text{(by (1))}\\
&= \beta_1\beta_2'g_{\mu+1}^{-1}f_1g_1\cdots g_{\mu-1}g_\mu^{-1}g_{-1}^{-1}\cdots g_{-(\mu-1)}^{-1}\underline{g_{-(\mu+1)}}f_jB_1'B_2'\\
&\qquad \text{(by Lemma 3.11)}\\
&= \beta_1g_{-(\mu+1)}\beta_2'g_{\mu+1}^{-1}f_1g_1\cdots g_{\mu-1}\underline{g_\mu^{-1}}g_{-1}^{-1}\cdots g_{-(\mu-1)}^{-1}f_jB_1'B_2'\\
&\qquad \text{(by (1))}\\
&= \beta_1g_{-(\mu+1)}\beta_2'g_{\mu+1}^{-1}f_1g_1\cdots g_{\mu-1}g_{-1}^{-1}\cdots g_{-(\mu-1)}^{-1}\underline{g_\mu^{-1}f_j}B_1'B_2'\\
&\qquad \text{(by (1))}\\
&= \beta_1g_{-(\mu+1)}\beta_2'g_{\mu+1}^{-1}f_1g_1\cdots g_{\mu-1}g_{-1}^{-1}\cdots g_{-(\mu-1)}^{-1}g_{-\mu}^{-1}f_jB_1'B_2'\\
&\qquad \text{(by Lemma 3.11)}\\
&\in W_j.\\
&\qquad \text{(by inductive hypothesis)}
\end{aligned}$$

This completes the proof of the Lemma. $\square$

We use Lemma 3.11 on the algebra $A_{n,p}$, applying an exactly analogous procedure to that described in the earlier section finding an explicit spanning set for $M_{n,p}$ using Proposition 3.3. Such an argument clearly is valid since we already know that a tangle argument holds for $A_{n,0}$ and $A_{0,p}$.

We can therefore reduce the spanning set for $A_{n,p}$ found in Lemma 3.12 to $(n+p)!$ elements giving an upper bound on the dimension, the object of the above strategy. Hence ι is an isomorphism and therefore the above presentation is a valid presentation for $M_{n,p}$. □

Chapter 4

Conclusion

4.1 Some notes on the string algebra, $S_{n,n}$

In this section we will give a conjecture (with some ideas on a possible proof) about the nature of the string algebra $S_{n,n}$ defined in the previous chapter.

We shall adopt the convention of having tangles in $S_{n,n}$ being defined on $2n$ strings with the orientation on the strings alternating. We shall label the points at the top with $1^+, 1^-, 2^+, 2^-, \ldots, n^+, n^-$, where $+$ indicates an input and $-$ an output. An example of a tangle in $S_{2,2}$ is shown in Figure 4.1.

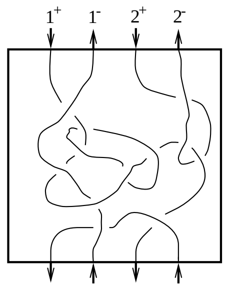


Figure 4.1: A tangle in $S_{2,2}$.

We now depict in Figure 4.2 some elementary tangles, including those labelled m_{i^+} and m_{i^-} which is a particular form of braid that has been used in various contexts, including the study of the conjugacy question for braids (see for example [BKL98]). It is generally referred to as a *band generator*.

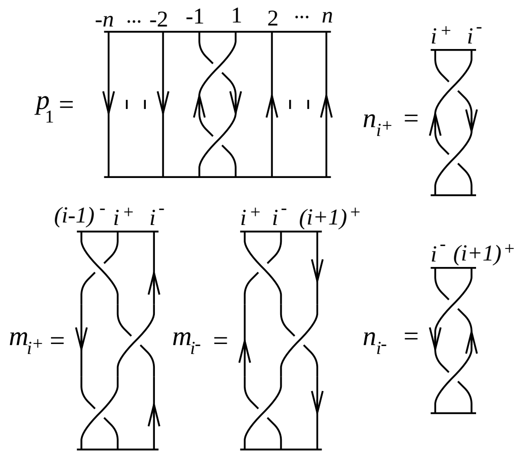


Figure 4.2: Some elementary tangles.

Remark. We have already seen that the algebra $S_{n,n}$ is isomorphic to the algebra $M_{n,n}$. It is also clear that we can replace the generator referred to as f_1 by p_1 which is shown in Figure 4.2 since:

$$p_1 = \mathrm{id}_{2n} - z\lambda f_1.$$

Conjecture. *The string algebra $S_{n,n}$ can be generated by the elements $n_{i^\pm}$ and $m_{i^\pm}$ for $1 \le i^+ \le n$ and $1 \le i^- < n$.*

Our interest in the string algebra is mainly caused by the possibility of creating some conputer implementation of an algorithm to allow us to calculate the Homfly polynomial of a knot when it is presented by a braid with such an alternating orientation on its strings. Such circumstances may arise if one were to consider a knot with its reverse parallel. In such a case, to present this as a braid with all orientations running from top to bottom, the result would be a significant increase in the braid index.

This increase has been studied by various parties including E. Beltrami, J.S. Birman, P.R. Cromwell, W.W. Menasco and I.J. Nutt. Therefore, it is clear that presenting such a construction as a braid with alternating orientations would not cause this increase to take place.

To make such a program possible we would require, as with the Morton-Short program [MS90], some hopefully "well-behaved" explicit spanning set for the algebra.

How would one prove the above conjecture? One possibility may be to consider a suitable isomorphism, α_* say, which maps $M_{n,n}$ to $S_{n,n}$, given by $T \mapsto \alpha T \alpha^{-1}$ for $T \in M_{n,n}$ and α some 'braid type' tangle. One could then consider the effect of α_* upon the known generators of $M_{n,n}$.

4.2 Other related work

There have been other developments made from considering tangles of the form described in Chapter 3. The two main areas known to the author will be given a very brief description below; the first is work that has actually been done, and the second is an area that may yield some developments if considered in this context.

4.2.1 The Kosuda and Murakami approach

M. Kosuda and J. Murakami construct an algebra $H^n_{N,M}(q)$ with complex parameter q. It is then shown that the Homfly polynomial of links is equal to a trace of $H^n_{N,M}(q)$. They then obtain a formula for the Homfly polynomial of satellite links. A summary of some of this work is published in [KM92] with a more detailed account found in [KM93].

This work begins with letting $G = GL_n(\mathbb{C})$ be the group of linear transformations of the n-dimensional $\mathbb{C}$-vector space V_n. Then V_n^* is the dual space of V_n, and define $V_n^{(N,M)} := V_n^{\otimes N} \otimes (V_n^*)^{\otimes M}$ as the (N, M)-mixed tensor power of V_n.

The main result of these papers then shows that for a generic q, the algebra $H^n_{N,M}(q)$ is semisimple and isomorphic to the centralizer algebra $C_n^{(N,M)}(q)$ of the natural representation of the quantum algebra $U_q(gl(n, \mathbb{C}))$ on $V_n^{(N,M)}$ if n is sufficiently large.

In fact, the presentation given for the algebra $H^n_{N,M}$ is equivalent to the presentation given for $M_{n,p}$ in the Main Theorem of Chapter 3. The relationship between the two theorems can be seen by the following identifications:

$$\begin{aligned} N &= p; \\ M &= n; \\ T_i &= g_i, \text{ with } 0 < i < p; \\ T_i^* &= g_{-(n-i)}, \text{ with } 0 < i < n; \end{aligned}$$

$$E = f_1.$$

In other related work by M. Kosuda, the Homfly polynomial is shown to be found from irreducible representations of the Hecke category which is seen to define isotopy invariants of oriented tangles. Then method uses the fact that the set of oriented tangles (up to isotopy) forms a category denoted by $\mathcal{OTA}$.

4.2.2 Kazhdan-Lusztig bases and the Hecke algebra

We end this report with a brief discussion of an alternative basis for the Hecke algebra, H_n, discovered by D. Kazhdan and G. Lusztig in [KL79]. This basis may prove worthy of further research in the context of the tangle algebras described here as it may yield an algebraic view of the geometrically defined string algebra.

We shall summarise the Kazhdan-Lusztig basis in terms of the basis already known to correspond to the positive permutation braids. We have that the positive permutation braids are a basis for H_n over $\mathbb{Z}[z]$ and we shall denote this by $\{b_\pi\}_{\pi\in S_n}$. Now put $z = s - s^{-1}$, then the Kazhdan-Lusztig basis is a basis for H_n when considered over $\mathbb{Z}[s^{\pm 1}]$ and will be denoted here by $\{c_\pi\}_{\pi\in S_n}$.

The Kazhdan-Lusztig basis is such that:

1. each c_π is invariant under the *mirror map* (where the mirror map is an isomorphism from H_n to itself which maps braids to their mirror image, and is defined by $\sigma_i^\varepsilon \mapsto \sigma_i^{-\varepsilon}$ and $s^\varepsilon \mapsto s^{-\varepsilon}$ where $\varepsilon = \pm 1$);

2. each c_π can be expressed as:
$$c_\pi = b_\pi + \sum_{\rho<\pi} P_{\pi,\rho} b_\rho, \tag{4.1}$$
where $P_{\pi,\rho} = P_{\pi,\rho}(s)$ is an integer polynomial in s (without a constant term) which can be found from a triangular change of basis matrix.

Theorem 4.1 (Kazhdan-Lusztig [KL79]) *There is a unique basis of the form $\{c_\pi\}_{\pi\in S_n}$.*

In a similar way we can also define what can be thought of as a *conjugate* basis where the coefficients are now polynomials in s^{-1}. Such a basis is also unique.

Now we address the question of what is meant in equation (4.1) by $\rho < \pi$. This ordering is a partial ordering on the permutations and is described in

the following way: draw the positive permutation braid b_π and smooth any non-empty subset of its crossings. If the result is a positive permutation braid b_ρ, then $\rho < \pi$. It is a direct consequence that:

$$\rho < \pi \Rightarrow l(\rho) < l(\pi),$$

where the function l is the standard word-length of the braid.

Remark. We have referred to the positive permutation braid basis of $H_n(z)$ by $\{b_\pi\}_{\pi\in S_n}$. Algebraists would usually adopt the notation $\{T_w\}_{w\in W}$ where W would be the Weyl group of the Lie algebra, so indeed, for our purposes, $W = S_n$.

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