Doubly periodic textile patterns

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Abstract

Knitted and woven textile structures are examples of doubly periodic structures in a thickened plane made out of intertwining strands of yarn. Factoring out the group of translation symmetries of such a structure gives rise to a link diagram in a thickened torus, as in [2]. We convert such a diagram on a standard torus in S^3 into a classical link by including two auxiliary components which form the cores of the complementary solid tori. The resulting link, called a *kernel* for the structure, is determined by a choice of generators u, v for the group of symmetries.

We show how a normalised form of the multi-variable Alexander polynomial of a kernel can be used to provide polynomial invariants of the original structure which are essentially independent of the choice of generators u and v.

The multi-variable Alexander polynomial of a kernel gives immediate information about the existence of closed curves and other topological features in the original textile structure. Because of its natural algebraic properties under coverings we can recover the multi-variable Alexander polynomial for kernels based on a proper subgroup of the translation group for the structure from the polynomial dervied from the full symmetry group. This allows us in principle to compare two structures at similar scales, even when one has a much smaller minimal repeating cell than the other.

Examples of simple traditional structures are given, and their Alexander data polynomials are presented to illustrate the techniques and results.

1 Introduction

Textiles represent a diverse class of commonly used materials with specific structural properties which impose a number of restrictions on the mutual position of constituting threads and the way in which they are intermeshed. Unlike general knots and links, textile structures cannot contain closed components or knots tied on their threads; they must be structurally coherent, i.e. fabrics cannot contain non-interlaced threads, disconnected strips or layers.

The development of specific tools to identify such forbidden elements in the fabric structure forms the focus of this paper.

1.1 Representing a fabric by a link

In [2] Grishanov, Meshkov and Omelchenko introduced the idea of representing a fabric with a repeating (doubly periodic) pattern by a knot diagram on a torus, having made a choice of a unit cell for the repeat of the pattern. Algebraic invariants of this diagram based on the Jones polynomial were used to associate a polynomial to the fabric which was independent of the choice of unit cell, so long as a minimal choice of repeating cell was made. In this paper we enhance the nature of the diagram used to represent the fabric by including two further auxiliary curves, to produce a link in the 3-dimensional sphere S^3 from which the original fabric can be recovered. We use the multivariable Alexander polynomial of the resulting link so as to strengthen the information available about topological properties of the fabric, and remove the need to work with a minimal choice of repeating cell.

We use the term *fabric* to mean a doubly periodic oriented plane knot diagram, consisting of coloured strands with at worst simple double point crossings, up to the classic Reidemeister moves. A fabric gives rise to a link diagram on the torus $T^2 \cong S^1 \times S^1$ by choosing a repeating cell in the pattern and splicing together the strands where they cross corresponding edges to form the diagram on the torus. A link in S^3 with two further auxiliary components X and Y is constructed by placing the torus in S^3 as a standard torus and including the core curves on each side of the torus in addition to the curves forming the diagram on the torus. We make the convention that the curve X lies on the side of the torus towards the *face* of the original fabric, and the curve Y lies on the side of the torus towards the *back* of the fabric. The resulting link, with the distinguished choice of curves X and Y, will be called a *kernel* for the fabric.

We assume that our fabric lies in a thickened plane, and is invariant under a discrete group G generated by two independent translations. The quotient of the thickened plane by the action of G is then a thickened torus $T^2 \times I$, bounded by two tori corresponding to the face and the back of the fabric. In forming a kernel of the fabric we have made a choice of embedding of this thickened torus in S^3, determined by an explicit choice of two generators u and v for G along the edges of our chosen unit cell.

A schematic view of a fabric, with its face uppermost, and a kernel for it, are shown in Figure 1.

1.2 Some examples

Some traditional fabrics are shown in the following figures, with a choice of repeating cell indicated, and the resulting kernel in each case. A wide variety of fabrics can be found in the books by Watson, [11, 12], and Spencer, [14], and some of the primary structural elements are described in [13].
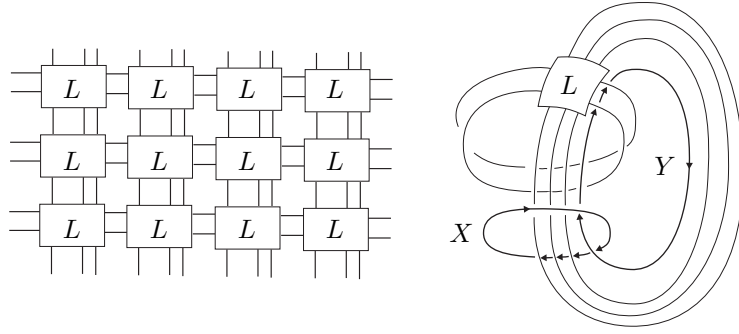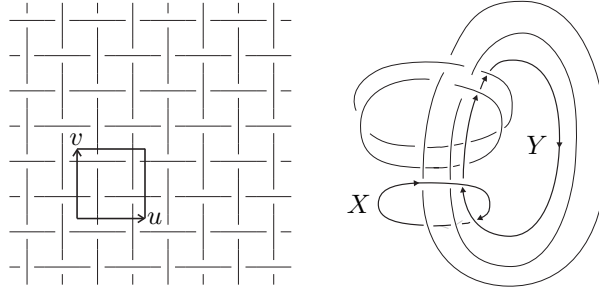
Figure 1: A kernel for a fabric
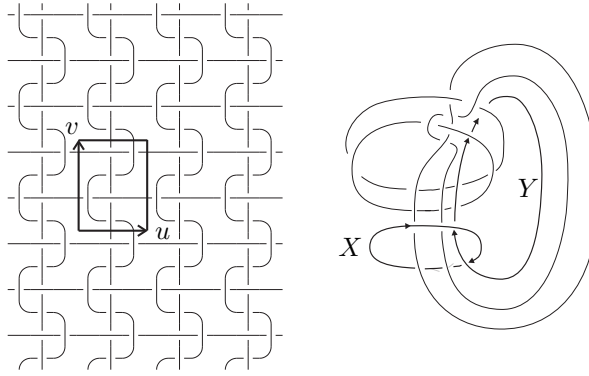

Figure 2: Plain Weave

Figure 3: Leno Weave

Different choices of repeating cell for a given fabric will give rise to different kernels. For example, choosing the cell shown in Figure 6 for the single jersey fabric gives the kernel shown.
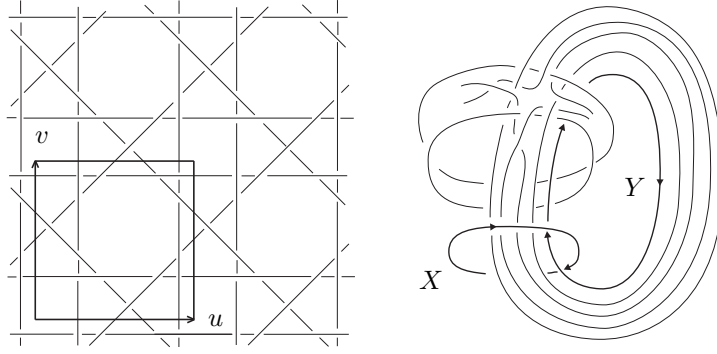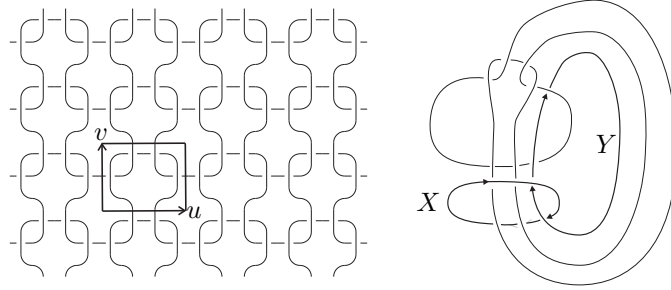
Figure 4: Multiaxial Weave

Figure 5: Single Jersey

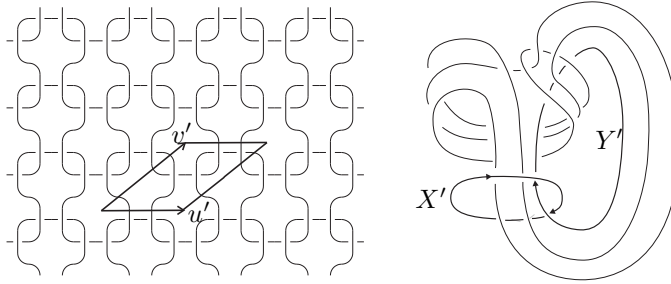Figure 6: Single Jersey with a different repeating cell

1.3   Fabric kernels

The original fabric can be recovered from any one of its kernels. Since the region between the two auxiliary components X and Y is topologically a thickened torus this means that the diagram on the torus can be recovered. The whole fabric can be reconstructed as a doubly periodic plane pattern by unwrapping

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the torus, - in effect constructing the inverse image of the diagram on the torus under the universal cover of the torus by the plane.

We adopt the name *fabric kernel* for such links.

Definition. A *fabric kernel* is a link consisting of two distinguished unknotted components X and Y which form a Hopf link and one or more further components representing the fabric strands.

Any fabric kernel determines a fabric as above. Many fabric kernels will give rise to fabrics which decompose into disconnected layers or strips, or can prove physically difficult to make. The problem of identification of such structures which are not textiles from the traditional point of view is a part of a wider problem of enumeration of all possible textile structures. Some restricted subclasses corresponding to variants of traditional woven or knitted material will be of particular interest to us, but much of our theoretical work will apply to very general fabric links.

We show in the corollary to Theorem 2 how to use the classical multivariable Alexander polynomial of a fabric kernel to tell whether the resulting fabric contains any closed components. Such *chain mail* type of fabric is impossible to make using traditional textile technology, and it is useful to be able to identify readily the corresponding fabric kernels.

In section 3.3 we show how to compare the polynomials for different kernels for the same fabric. The *Alexander data*, which can be displayed as described in section 3.3 on a diagram of the fabric, can be found from any given kernel, and allows the polynomial of any other kernel to be readily calculated.

The Alexander data is then enough to allow comparison of different fabrics. While two different fabrics can give rise to the same Alexander data the size of the kernels must be sufficiently large for this to be possible, and an inventory of small fabric kernels using their polynomials will give a good means of listing the corresponding fabrics. We expect that more detailed comparisons of two fabrics will be best conducted by using kernels of approximately the same size - measured for example by the number of crossings in a repeating cell.

The Alexander data of a layered fabric has some characteristic features, described in section 3.7. Assuming that the layered fabric has no closed components then the multivariable Alexander polynomial of any of its kernels must factorise non-trivially in a certain way, described in Theorem 9. This leads to a quick check for the maximum number of potential layers in a fabric.

2 The axial type of a strand in a fabric

Suppose that we are given a fabric lying in a thickened plane, with a discrete group G of invariant translations generated by two independent elements u and v. We shall assume that the normal direction $u \times v$ lies to the face side of the fabric.

Any strand of yarn in the fabric is invariant under some subgroup of G. This subgroup is either trivial, when the strand is a closed curve, or is infinite cyclic, generated by some $w \in G$. The element w can be recognised, up to sign, as the

smallest translation in G that carries one point of the chosen strand to another point of the same strand. If the strand is oriented then we select w so as to translate the strand in its preferred direction.

Definition. The *axial type* of a strand in a fabric is 0 if the strand is closed and is otherwise a generator w of the subgroup of translations which leaves the strand invariant.

Remark 1. Once we have chosen generators u and v for G each axial type has the form $w = \alpha u + \beta v$ for some integers (α, β), determined up to an overeall sign if the strand is unoriented.

For example, in the single jersey fabric shown in Figure 5 all fabric strands have axial type u. In fact all strands are equivalent under translations in G, and yield a single fabric component in the kernel.

In general there may be two or more inequivalent strands with the same axial type. For example in Leno weave (Figure 3) there are two inequivalent strands with axial type u and two with axial type v, while the multiaxial fabric in Figure 4 has strands of type $u, v, u + v$ and $u - v$.

2.1 Axial type and kernels

We can identify the axial type of the strands in a fabric by calculating certain linking numbers in a kernel for the fabric.

Assume that the kernel has been constructed using a choice of generators u, v for the group G of invariant translations. In the torus $T^2 = \mathbf{R}^2/G$ the lines in the directions of u and v become closed curves, U and V say, which generate the homology group $H_1(T^2)$. A strand in the fabric with axial type $w = \alpha u + \beta v$ will become a closed curve W in the thickened torus $T^2 \times I$ which represents $\alpha U + \beta V$ in its homology group. Now the curves X and Y in the fabric kernel lie parallel to U and V respectively on either side of the standard thickened torus. They are oriented so that $lk(X, U) = 0, lk(X, V) = 1, lk(Y, U) = 1, lk(Y, U) = 0$. Since $W = \alpha U + \beta V$ in the homology of $T^2 \times I \cong S^3 - (X \cup Y)$ we get $lk(X, W) = \alpha lk(X, U) + \beta lk(X, V) = \beta$ and similarly $lk(Y, W) = \alpha$.

Consequently in a fabric kernel $L = X \cup Y \cup T_1 \cup \ldots \cup T_k$ each oriented component T_i represents a family of translation-equivalent strands in the fabric of axial type $b_i u + a_i v$, where $a_i = lk(X, T_i)$ and $b_i = lk(Y, T_i)$. In particular the strands in the fabric corresponding to T_i are *closed curves* if and only if $a_i = b_i = 0$.

3 The multivariable Alexander polynomial

For a link $L \subset S^3$ with $r > 1$ oriented components $X_1 \cup \ldots \cup X_r$ the group $H_1(S^3 - L)$ is free abelian of rank r. It has distinguished generators $x_1, \ldots, x_r$ represented by oriented meridians of the components. The multivariable Alexander polynomial Δ_L is an element of the integer group ring $\mathbf{Z}[H_1(S^3 - L)]$ and is

thus a Laurent polynomial in $\mathbf{Z}[x_1^{\pm 1}, \ldots, x_r^{\pm 1}]$. It is defined up to a unit in this ring, and thus up to multiplication by a signed monomial $\pm x_1^{a_1} \cdots x_r^{a_r}$.

With some care an absolute version of the polynomial can be defined, as in [7]. The Torres symmetry condition (1) shows that

$$\Delta_L(x_1^{-1}, \ldots, x_r^{-1}) = (-1)^r x_1^{m_1} \ldots x_r^{m_r} \Delta_L(x_1, \ldots, x_r),$$

for some integers $m_1, \ldots, m_r$. Multiplying Δ_L by the monomial $x_1^{\frac{1}{2}m_1} \ldots x_r^{\frac{1}{2}m_r}$ gives Murakami's absolute version, up to sign, with the property that

$$\Delta_L(x_1^{-1}, \ldots, x_r^{-1}) = (-1)^r \Delta_L(x_1, \ldots, x_r).$$

In this format we need to use half-integer powers $x_i^{\frac{1}{2}}$ of some of the variables $\{x_i\}$.

We make use of the absolute version in presenting the Alexander data, but for many of the properties and calculations it is enough to allow a monomial multiple, so as to avoid negative powers of the variables.

In the case of a single component link, in other words a knot K, we write $\overline{\Delta}_K \in \mathbf{Z}[t^{\pm 1}]$ for its classical Alexander polynomial, and in this paper we use the non-standard notation Δ_K to denote the rational function $\Delta_K = \frac{1}{1-t} \overline{\Delta}_K$. In this way we can give formulae for the behaviour of the Alexander polynomials of related knots and links in a uniform way, without having to treat the single component case, $r = 1$, separately. The formulae can be most consistently handled in the context of Reidemeister torsion, where an excellent recent account is given in [10]. Here we give a summary of the properties needed, following the constructions of Fox and Torres based on Fox's free differential calculus [9, 1].

3.1 Deletion of components

For an oriented link L with components $X_1 \cup \ldots \cup X_r$ the curve X_i represents the monomial $\prod_{j \neq i} x_j^{l_{ij}}$ as an element of $H_1(S^3 - (L - X_i))$, where $l_{ij} = lk(X_i, X_j) \in \mathbf{Z}$ is the linking number of X_i and X_j. We write

$$< X_i > = \prod_{j \neq i} x_j^{l_{ij}} \tag{1}$$

for this monomial. A special case of the Torres-Fox satellite formula relates the invariants Δ_L and Δ_{L-X_i}.

Theorem 1 (Torres)**.**

$$\Delta_L|_{x_i=1} = (1- < X_i >)\Delta_{L-X_i}.$$

Remark 2. If the component X_i has a non-zero linking number with *at least one* of the other components then $< X_i > \neq 1$. It is thus possible to recover the Alexander polynomial of the sublink $L - X_i$ where X_i is removed, starting from

the polynomial Δ_L of the whole link, unless X_i has linking number 0 with *all* the other components of L.

For the absolute versions of the invariants in Theorem 1 the factor $< X_i >^{\frac{1}{2}} - < X_i >^{-\frac{1}{2}}$ appears in place of $1- < X_i >$.

3.2 Closed components in a fabric

We start from a fabric kernel L. Then $L = X \cup Y \cup T_1 \cup \ldots \cup T_k$, where X and Y are the distinguished face and back curves forming a Hopf link. The complement of $X \cup Y$ forms a standard thickened torus containing the remaining components $T_1, \ldots, T_k$, termed the *fabric components* of L. These curves correspond to the strands of yarn in the fabric resulting from unwrapping L.

The curve T_i in the fabric kernel unwraps to give closed components in the covering fabric if and only if it has linking number 0 with both of the auxiliary curves X and Y.

Theorem 2. *Let* $L = X \cup Y \cup T_1 \cup \ldots \cup T_k$ *be a fabric kernel. Write* $a_i = lk(T_i, X)$ *and* $b_i = lk(T_i, Y)$. *Then its Alexander polynomial* $\Delta_L(x, y, t_1, \ldots, t_k)$ *satisfies*

$$\Delta_L(x, y, 1, \ldots, 1) = \prod_{i=1}^{k}(1 - x^{a_i} y^{b_i}) \in \mathbf{Z}[x^{\pm 1}, y^{\pm 1}].$$

Proof. After setting all $t_j = 1$ we have $< T_i >= x^{a_i} y^{b_i}$ by equation (1). Repeated use of theorem 1, suppressing the components $T_1, \ldots, T_k$ in turn, gives

$$\Delta_L(x, y, 1, \ldots, 1) = \prod_{i=1}^{k}(1 - x^{a_i} y^{b_i}) \Delta_{X \cup Y}.$$

Since the remaining link $X \cup Y$ is the Hopf link, whose Alexander polynomial is 1, the result follows. $\square$

Corollary 3. *There is a closed component in the covering fabric of L if and only if* $\Delta_L(x, y, 1, \ldots, 1) = 0$.

Proof. The Laurent polynomial $\prod_{i=1}^{k}(1 - x^{a_i} y^{b_i})$ is equal to 0 in $\mathbf{Z}[x^{\pm 1}, y^{\pm 1}]$ if and only if $a_i = b_i = 0$ for some i. $\square$

Remark 3. The element $< T_i >= x^{a_i} y^{b_i}$ represents the homology class of T_i in the thickened torus $S^3 - (X \cup Y)$. Written additively this is $b_i U + a_i V$, where U and V are determined by the generators u and v as in subsection 2.1. The axial type $b_i u + a_i v$ of the corresponding strands in the fabric, and indeed the number of inequivalent strands of each axial type, can then be read off immediately from Δ_L, so long as there are no closed strands in the fabric.

To determine the axial types using Theorem 2 it is enough to be given the polynomial Δ_L as a function of x, y and t only, where all the yarn variables t_i have been set equal to t. If there are no closed components in the fabric we can then recover the number k of yarn components in the kernel, and the axial

types of the corresponding fabric strands, from the factors in the evaluation with $t = 1$.

In a traditional woven fabric there are just two axial types, corresponding to the warp and weft directions, while a traditional knitted fabric has only one type.

More sophisticated woven fabrics with multiaxial types have recently been introduced, [15], such as the fabrics shown in Figure 4 and 15. Knitted fabrics too may have warp and weft type inserts, as in Figure 7, leading to a multiaxial fabric.
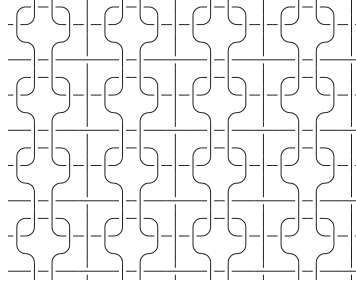

Figure 7: Single Jersey with warp and weft inlays

From this basic analysis of the polynomial of a kernel we can identify immediately the axial types, although a detailed discussion of whether a fabric is woven or knitted in the appropriate sense may not be available from the polynomial alone. Of course if the kernel has more than 2 different axial types then we can certainly conclude that the fabric is not a traditional woven structure, while equally a traditional knitted structure can be excluded unless there is a single axial type.

We give here the polynomials for the kernels of several fabrics.

Example 1. Single jersey with a closed thread around the fabric thread

$$\Delta_L = (tx - t - x)(t + x - 1)(t - 1)(y - 1).$$

Example 2. Single jersey with a trefoil added to the fabric thread

$$\Delta_L = (tx - t - x)(t + x - 1)(t^2 - t + 1)(y - 1).$$

Example 3. Single Jersey with warp and weft inlays, as in Figure 7.

$$\Delta_L = (t + x - 1)(tx - t - x)(px - 1)(ey - 1)(y - 1),$$

where the jersey strand has variable t, the warp strand has variable e and the weft has variable p.
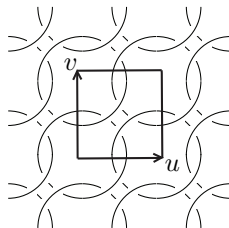
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Figure 8: Chain Mail

Example 4. By way of contrast, the Alexander polynomial for the kernel of the chain mail pattern shown in Figure 8 is

$$\Delta_L = (x-y)(1-xy)(1-t). \tag{2}$$

This is one of the fabric kernels which does not correspond to a physically constructible textile fabric, on account of the closed components. The substitution $t = 1$ for the fabric variables leads to the value 0, as required by Theorem 2.

3.3 The Alexander data for a fabric

We now present a normalised form of the Alexander polynomial of a kernel giving rise to an array of polynomials in the fabric variables which are independent of the choice of generators u and v. The result is a set of invariants for the fabric which readily determine the Alexander polynomial for *any* choice of kernel.

Start from the Alexander polynomial $\Delta_L(x, y, t_1, \ldots, t_k)$ of a kernel L determined by a choice of generators u and v of the group of translations of the fabric.

Write $t_i = s_i^2$ and set

$$U = y \prod s_i^{a_i}, V = x \prod s_i^{b_i} \tag{3}$$

where $a_i = lk(T_i, X)$ and $b_i = lk(T_i, Y)$. The integers a_i and b_i can be easily read off from the unit cell determined by u and v using the fact that T_i crosses the edge u algebraically a_i times in the direction of v and crosses the edge v algebraically b_i times in the direction of u.

The *Alexander data* of the fabric consists of the coefficients of Δ_L when written as a polynomial in U and V. When written in this way we say that the Alexander polynomial Δ_L is in *data form*.

The Alexander data coefficients are Laurent polynomials in the fabric variables $\{s_i\}$. In Theorem 4 we show how they depend essentially on the fabric only and not on the choice of kernel.

Write

$$\Delta_L = \sum L_{\alpha,\beta} U^\alpha V^\beta$$

and set $T_w = L_{\alpha,\beta}$ for each $w = \alpha u + \beta v \in G$. In this way we associate a Laurent polynomial $T_w(s_1, \ldots, s_k)$ to each element w in the translation group G.

Since the Alexander polynomial Δ_L is often defined only up to multiplication by a signed monomial in $x, y, t_1, \ldots, t_k$ there is a measure of ambiguity in the definition of T_w. Apart from this ambiguity, which can be eliminated up to an overall sign by use of the Torres symmetry conditions, the elements T_w are independent on the choice of kernel, and are termed collectively the *Alexander data* of the fabric. The independence is made precise in the following invariance theorem.

Theorem 4. *The Alexander data of a fabric is independent of the choice of generators for G, up to an overall multiplication by a monomial in $s_1, \ldots, s_k$ and translation by an element of G. When we use generators u' and v' in place of u and v to form a kernel L', then the resulting polynomials $T'_w, w \in G$ satisfy the equation*

$$T'_w = \pm m T_{w+g},$$

for some $g \in G$ and monomial m in $s_1, \ldots, s_k$ independent of w.

Proof. Our choice u, v of generators for the group of translations G determines the repeating cell used to construct the kernel. Lines in the plane in the directions of u and v become closed curves U and V in the quotient torus $T^2 = \mathbf{R}^2/G$. The face torus is embedded as the boundary of a neighbourhood of the face curve X and the back torus as the boundary of a neighbourhood of the back curve Y, where X and Y are parallel to the closed curves U and V respectively. In this embedding the face torus contains curves m_X and l_X, which are respectively the meridian and longitude of the curve X. The back torus correspondingly contains m_Y and l_Y, giving the longitude and meridian curves as $l_X = U \times \{1\}$, $l_Y = V \times \{0\}$, while $m_X = V \times \{1\}$ and $m_Y = U \times \{0\}$.

In terms of the generators $x, y, t_1, \ldots, t_k$ for the homology $H_1(S^3 - L)$ we have

$$U \times \{0\} = y, \qquad U \times \{1\} = <X> = y \prod t_i^{a_i},$$
$$V \times \{1\} = x, \qquad V \times \{0\} = <Y> = x \prod t_i^{b_i},$$

where $a_i = lk(T_i, X)$ and $b_i = lk(T_i, Y)$. The variables $U = y \prod s_i^{a_i}$ and $V = x \prod s_i^{b_i}$ introduced in equation (3) thus represent an 'average' of the homology of the curves U and V on the front and back faces of the torus. When the homology $H_1(S^3 - L)$ is written additively we have

$$
\begin{aligned}
2U &= U \times \{0\} + U \times \{1\} \\
2V &= V \times \{0\} + V \times \{1\}.
\end{aligned}
$$

A different choice of generators u', v' for G gives rise to a different repeating cell, and a different kernel for the fabric. Now the complement of a kernel in S^3 is the complement of the fabric quotient in the thickened torus $T^2 \times I$.

This complement is unchanged by the new choice of generators for G, so that $S^3 - L' \cong S^3 - L$. What *does* change is the curves on the face and back tori that correspond to the longitudes and meridians of the face and back curves in the two kernels. We then know that $\Delta'_L = \Delta_L$, when the variables are changed appropriately. We need to compare the elements $x, y, t_1, \ldots, t_k$ and $x', y', t_1, \ldots, t_k$ as elements of $H_1(S^3 - L) = H_1(S^3 - L')$. The comparison is very straightforward when we use the data form of the two polynomials.

Lemma 5. *We can suppose that*

$$u' = pu + qv, v' = ru + sv$$

with $p, q, r, s \in \mathbf{Z}$ and $ps - qr = 1$. Then

$$U' = pU + qV, V' = rU + sV,$$

when written additively as elements of $H_1(S^3 - L)$.

Proof. The curves U', V' on $T^2 = \mathbf{R}^2/G$ corresponding to the vectors u', v' satisfy the equations

$$U' = pU + qV, \quad V' = rU + sV,$$

as elements of $H_1(T^2)$.

Hence on the boundary of the thickened torus we have

$$U' \times \{i\} = pU \times \{i\} + qV \times \{i\}, \quad V' \times \{i\} = rU \times \{i\} + sV \times \{i\},$$

for $i = 0, 1$.

Adding these equations for $i = 0$ and $i = 1$ in $H_1(S^3 - L)$ we then have

$$2U' = 2(pU + qV), 2V' = 2(rU + sV).$$

$\square$

The multivariable Alexander polynomial Δ_L of a fabric kernel $L = X \cup Y \cup T_1 \cup \ldots \cup T_k$ is an element of the integer group ring $\mathbf{Z}[H_1(S^3 - L)]$, and hence is a Laurent polynomial in variables $x, y, t_1, \ldots, t_k$, since the homology group can be generated by meridians of the components. The element Δ_L depends only on the fundamental group of $S^3 - L$, which is also the fundamental group of the complement of the fabric quotient in the thickened torus $T^2 \times I$. Hence if L' is another kernel for the same fabric it will have the same Alexander polynomial, since $\pi_1(S^3 - L') = \pi_1(S^3 - L)$, written now in terms of different generators of the group $H_1(S^3 - L)$.

In terms of the Alexander polynomial we can relate $\Delta_{L'}$ to Δ_L, up to multiplication by a signed monomial, using the multiplicative version $U' = U^p V^q$, $V' = U^r V^s$ of Lemma 5.

The Alexander data polynomials T'_w defined using the kernel L' are given by the equation $\Delta_{L'} = \sum T'_w(U')^{\alpha'}(V')^{\beta'}$, where $w = \alpha'u' + \beta'v'$. Making the substitution for U' and V' gives

$$\Delta_L = \Delta_{L'} = \sum T'_w U^{(\alpha'p+\beta'r)}V^{(\alpha'q+\beta's)} = \sum T'_w U^{\alpha}V^{\beta},$$

with $\alpha = \alpha'p+\beta'r$, $\beta = \alpha'q+\beta's$. Hence, in terms of the Alexander data defined using L we have $T'_w = T_g$ with $g = \alpha u + \beta v$. Now $g = (\alpha'p+\beta'r)u+(\alpha'q+\beta's)v = \alpha'u' + \beta'v' = w$, giving the invariance result that $T'_w = T_w$, up to the overall ambiguity of the definition of the Alexander polynomial. $\square$

3.4   Changing the choice of the repeating cell

Given the Alexander data of a fabric, calculated using any one choice of generators for the translation group, Theorem 4 shows how the multivariable Alexander polynomial for any other kernel of the same fabric can be found.

We can now compare the Alexander data from two fabrics quite readily. We just need one choice of unit cell for each to provide the Alexander data. Having put the two sets of data onto a plane we can see that if the fabrics themselves are affinely equivalent, in other words related by a translation and a linear transformation, then one set of data will transform to the other by a similar affine transformation φ carrying the coefficient polynomials $\{T_w\}$ for one fabric to the polynomials $\{T_{\varphi(w)}\}$ of the other. Fabrics with inequivalent Alexander data are then inequivalent.

In particular the number and affine location of the non-zero polynomials in the Alexander data is a simple invariant of the fabric. This could be refined, without presenting the whole data, to include the degrees of each fabric variable at the plane locations.

3.5   Presenting the Alexander data

Superimpose a grid on the fabric by choosing a point O as origin in the plane, and label the translates of O by elements of the group G. At the translate of O by $w \in G$ place the polynomial T_w. This display of the polynomials T_w is uniquely determined by the choice of origin, where the polynomials themselves have been normalised, up to an overall sign, by multiplying all polynomials by a monomial in the variables $\{s_i = t_i^{\frac{1}{2}}\}$ so as to respect the Torres symmetry conditions.

Thus to find the Alexander data polynomials T_w we choose any kernel L and put Δ_L into data form by setting $x = V/\prod s_i^{b_i}$ and $y = U/\prod s_i^{a_i}$ as in equation (3). Then $T_{\alpha u+\beta v}$ is the coefficient of $U^{\alpha}V^{\beta}$ in the resulting polynomial.

Conversely, given u and v we can recover the data form of Δ_L in terms of U and V immediately from the Alexander data. Equation (3), using the numbers a_i and b_i calculated from the choice of u and v, then gives Δ_L in terms of x and y.

Example 5. We may compare the single jersey data derived from the different choices of unit cell shown in Figures 5 and 6.

From a braid presentation of the single jersey kernel L shown in Figure 5 we can calculate its multivariable Alexander polynomial

$$\Delta_L = (1-y)(1-x-t)(x+t-tx),$$

where the variable $t = s^2$ is the single fabric variable. Setting $t = 1$ gives $y - 1$ up to monomial multiples, confirming that the fabric has a single axial direction u. From the intersections with the cell edges we have the Alexander data substitution $U = y$ and $V = xs$.

A similar braid based calculation for the polynomial $\Delta_{L'}$ of the kernel in Figure 6 gives

$$\Delta_{L'} = (1-y')(x'-tx'+ty')(x'-y'+ty')$$

up to a signed monomial multiple. Again the intersections with the cell edges show that the data substitution is $U' = y', V' = x's$. Now this kernel is related to L by the choice of generators $u' = u, v' = u+v$. Consequently the substitution $U' = U, V' = UV$ will convert the data form for L' into the data form for L, up to a monomial multiple.

We can exhibit the Alexander data starting from either data form. On the one hand we have

$$\Delta_L = (1-U)(1-s^2-V/s)(s^2+(s^{-1}-s)V),$$

while

$$\Delta_{L'} = (1-U')((s^{-1}-s)V'+s^2U')(V'/s-(1-s^2)U') = U^2\Delta_L$$

after the substitution.

Multiplication by s^{-1} gives the Torres symmetric form of the data displayed in matrix form below.

$$
\begin{matrix}
s^{-3}-s^{-1} & s^{-1}-s^{-3} \\
-s^2+3-s^{-2} & s^2-3+s^{-2} \\
s^3-s & s-s^3
\end{matrix}
$$

Figure 9: Single Jersey (face) data

Here and in the examples below we present the Alexander data as a matrix of polynomials based on the stated choice of u and v. The rows of the matrix correspond to translations by multiples of u, read from left to right, and the columns to multiples of v, read from the bottom up.
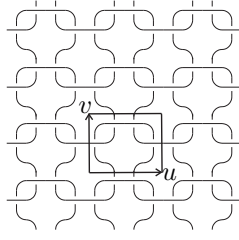
14

Figure 10: Single Jersey, from the back

$$s^3 - s \qquad\qquad s - s^3$$

$$-s^2 + 3 - s^{-2} \quad s^2 - 3 + s^{-2}$$

$$s^{-3} - s^{-1} \qquad s^{-1} - s^{-3}$$

Figure 11: Single Jersey (back) data

Example 6. In the previous example we have used the single jersey fabric viewed from its traditional face side. When viewed from the back, as in Figure 10, the polynomial for the kernel is $\Delta_L = (1-y)(t^2x - tx + 1)(t^2x - t + 1)$. This gives the normalised Alexander data shown in Figure 11.

Example 7. The Alexander polynomial for the kernel L of plain weave corresponding to the choices of u and v in Figure 2 is

$$\begin{aligned}
\Delta_L &= -1 + 2px + 2ey - p^2x^2 - e^2y^2 + ((1-e)^2(1-p)^2 - 4ep)xy \\
&\quad + 2p^2ex^2y + 2pe^2xy^2 - e^2p^2x^2y^2,
\end{aligned}$$

where the two fabric strands in the direction of u have the meridian variable p and those in the direction v have variable e. Write $e = a^2$ and $p = b^2$. Then setting $U = y/a^2 = y/e$ and $V = x/b^2 = x/p$, as required by equation (4), gives the data form

$$\Delta_L = -1 + 2V + 2U - V^2 - U^2 + ((1-e)^2(1-p)^2 - 4ep)UV/ep + 2UV^2 + 2U^2V - U^2V^2.$$

This leads to the data shown in Figure 12.

Example 8. Leno weave, which was presented in Figure 3, has four threads in the minimal unit cell; two non-equivalent threads are running in the u direction and two in the v direction.

The Alexander data for Leno weave is presented in Figure 13, where

$$\begin{aligned}
P &= (a^{-2} - 1 + a^2)(b^{-1} - b)^2 + 2 \\
Q &= 2(a^{-1} - a)^2(b^{-1} - b)^2 + 2(a^{-2} - a^2)^2 - (a^{-2}b^{-1} + a^2b)^2 + 2.
\end{aligned}$$

15

$$
\begin{array}{ccc}
-1 & 2 & -1 \\
2 & (a-a^{-1})^2(b-b^{-1})^2-4 & 2 \\
-1 & 2 & -1
\end{array}
$$

Figure 12: Plain Weave data

$$
\begin{array}{ccc}
-1 & 2 & -1 \\
P & Q & P \\
-1 & 2 & -1
\end{array}
$$

Figure 13: Leno weave data

Example 9. The Alexander data for chain mail is shown in Figure 14. It is based on the choice of generators in Figure 8, and the corresponding Alexander polynomial in equation (2). In this case the data form is given simply by $U = y, V = x$.

$$
\begin{array}{ccc}
0 & s-s^{-1} & 0 \\
s^{-1}-s & 0 & s^{-1}-s \\
0 & s-s^{-1} & 0
\end{array}
$$

Figure 14: Chain Mail data

Example 10. The triaxial weave shown in Figure 15 has a lattice of symmetries with a 3-fold symmetry when the strings are oriented in the three axial directions, u, v and $w = -u - v$ as shown.
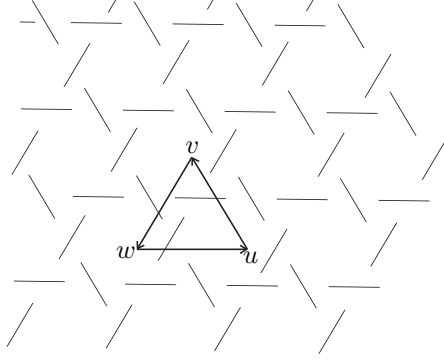

Figure 15: Triaxial weave

The Alexander data, when presented on this lattice, exhibits the same symmetry very nicely. This is shown in Figure 16, where the meridian variables of the strings are a^2, b^2 and c^2 respectively, and $P = (a - a^{-1})(b - b^{-1})(c - c^{-1})$.
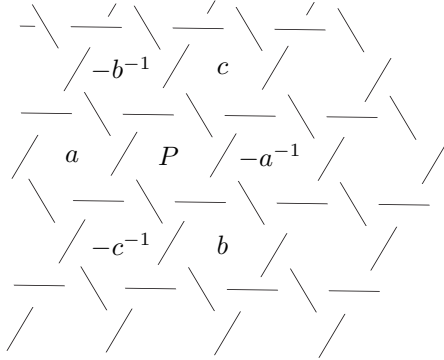

Figure 16: Triaxial weave

3.6 Multiple repeating cells

For comparison purposes it can be useful to look not just at a minimal area repeat of a pattern in a fabric. This means that we use a repeating cell whose edges are determined by two vectors u', v' in the group G of translations which generate a proper subgroup of G. It is always possible to choose generators u, v for G and u', v' for a subgroup H of G so that u' and v' are multiples of u and v respectively. We can insist further that $u' = su$ and $v' = rv$ where r is a

multiple of s. Then every vector w' in H is a multiple $w' = sw$ of a vector w in G, and in many cases we will have $s = 1$.

The relation between the Alexander data for the fabric using the full group G, corresponding to a minimal choice of repeating cell, and the data derived from using the translations in H only (and thus only considering a larger repeating area) can be determined by comparing the Alexander polynomials of the kernels L and $L^{(r)}$ which come from the choice of translations u, v and $u, w = rv$ respectively.

These polynomials are related by a more general result of Salkeld [8] connecting the multivariable polynomials for a link L with a distinguished unknotted component X and the link $L^{(r)}$ given by taking the r-fold cyclic cover of L branched over X. The two kernels above are related in this way, so Salkeld's result applies.

There is an r-fold covering map p from the complement of $L^{(r)}$ to the complement of L, inducing a map p^* on the first homology group. Salkeld's theorem expresses $p^*(\Delta_{L^{(r)}})$ in terms of Δ_L as follows.

Theorem 6 (Salkeld). *Write x for the meridian variable corresponding to the component X in L. Then*

$$p^*(\Delta_{L^{(r)}}) = \prod \Delta_L(\zeta x),$$

as ζ runs through all rth roots of 1.

Corollary 7. *The data form of $\Delta_{L^{(r)}}$, in terms of U and W, corresponding to the translations u and $w = rv$, satisfies the equation*

$$p^*(\Delta_{L^{(r)}}) = \prod \Delta_L(\zeta V),$$

as ζ runs through all rth roots of 1, where $W = V^r$.

Proof. In the data form for Δ_L we have $V = x \prod s_i^{a_i}$, while in $\Delta_{L^{(r)}}$ we have $W = X(\prod s_i^{a_i})^r$, and $p^*(X) = x^r$. Now $\Delta_L(\zeta x)$, given by replacing x with ζx, can be found from the data form by replacing V with ζV. $\qquad\blacksquare$

Salkeld's theorem does not give the complete polynomial for the covering link, in the case when there is more than one component of $L^{(r)}$ covering some component of L, since $p^*(\Delta_{L^{(r)}})$ has the same variable for all components which project to the same component in L. In the case of fabric kernels we would expect to make this restriction in any case, as the different components of $L^{(r)}$ with the same image must be equivalent strands in the fabric under the full group G of translations. We would only use a different variable if they were to be considered different in the fabric, and in this case a translation carrying one to the other would not be part of the group G. The meridian for the branch curve in $L^{(r)}$ projects to x^r in L, and so the polynomial $\Delta_{L^{(r)}}$, with equivalent components in the fabric having the same labels, can be recovered from Δ_L. Conversely, given $\Delta_{L^{(r)}}$ we can recover Δ_L as one factor, corresponding to the root $\zeta = 1$, when the branch curve meridian is replaced by x^r.

If Δ_L factorises, then $\Delta_{L^{(r)}}$ can be found as a product using the same operation on each factor. Indeed the operation of passing from Δ_L to $p^*(\Delta_{L^{(r)}})$ can be regarded formally as replacing the roots (for x) of Δ_L by their r-th powers; the coefficients of powers of x in $p^*(\Delta_{L^{(r)}})$ are integer polynomials in the coefficients of Δ_L.

Example 11. We can find the data form of the Alexander polynomial for single jersey doubled horizontally, using the cell with sides $w = 2u$ and v in terms of the generators u and v shown in Figure 5 for the translation group. This is given from Salkeld's theorem by taking the 2-fold cover of the kernel branched over the curve Y. The result, as a polynomial in W and V, will be the polynomial $P(W, V, t) = \Delta_L(U, V, t)\Delta_L(-U, V, t)$ with $W = U^2$. Using the data form $\Delta_L = (1 - U)(1 - s^2 - V/s)(s^2 + (s^{-1} - s)V)$ we get

$$
\begin{aligned}
P &= (1 - U)(1 + U)(1 - s^2 - V/s)^2(s^2 + (s^{-1} - s)V)^2 \\
&= (1 - W)(1 - s^2 - V/s)^2(s^2 + (s^{-1} - s)V)^2.
\end{aligned}
$$

The polynomial Q for single jersey doubled vertically, corresponding to u and $w = 2v$ has $Q(U, W, t) = \Delta_L(U, V, t)\Delta_L(U, -V, t)$ with $W = V^2$, giving the data form

$$
Q = (1 - U)^2((1 - t)^2 - W/t)(t^2 - (1 - t)^2W/t),
$$

where both fabric strands have the variable t. This converts to the Alexander polynomial by setting $U = y, W = xs^2 = xt$, to get

$$
Q = (1 - y)^2((1 - t)^2 - x)(t^2 - (1 - t)^2x).
$$

A more refined version of the Alexander polynomial, using different variables for the two strands, can be calculated directly from a kernel as

$$
(1 - y)^2((1 - t_1)(1 - t_2) - x)(t_1t_2 - (1 - t_1)(1 - t_2)x).
$$

Example 12. The Alexander polynomial for 1×1 rib, which alternates face and back loops in the u direction, is

$$
\Delta_L = (tx - t - x)(t + x - 1)(t^2x - tx + 1)(t^2x - t + 1)(y - 1).
$$

This polynomial is a mix of the polynomials for the face and back kernels of single jersey, and can be compared with the multiple cell case of single jersey in the previous example, where two face loops are repeated in the u direction. This was calculated above to be

$$
\Delta_L = (tx - t - x)^2(t + x - 1)^2(y - 1).
$$

A repeat of the back loop in the u direction gives

$$
\Delta_L = (t^2x - tx + 1)^2(t^2x - t + 1)^2(y - 1).
$$

For the general case of $m \times n$ rib, which is a combination of $m \geq 0$ face and $n \geq 0$ back loops in the u direction, we have

$$\Delta_L = ((tx - t - x)(t + x - 1))^m((t^2x - tx + 1)(t^2x - t + 1))^n(y - 1).$$

Indeed this polynomial is the same irrespective of the order in which the face and back loops occur, so that the standard 2×2 rib gives the same data as the 1×1 rib repeated twice in the u direction. We have tried comparing the polynomials in which alternate rows have variables t_1 and t_2, to see if this is enough to distinguish these fabrics, but again the results are the same.

Example 13. Alternate knit and purl rows, sometimes known as *garter stitch*. This is a combination of one face and one back loop in the v direction, and has Alexander polynomial

$$\Delta_L = (1 + (1 - t_1)(1 - t_2)x)((1 - t_1)(1 - t_2) + t_1t_2x)(y - 1)^2.$$

It can be compared with the polynomial for plain knit or purl fabric, calculated with different variables t_1 and t_2 for alternate rows.
For two face loops in the v direction the polynomial is

$$\Delta_L = (t_1t_2 - (1 - t_1)(1 - t_2)x)((1 - t_1)(1 - t_2) - x)(y - 1)^2.$$

For two back loops in the v direction the polynomial is

$$\Delta_L = (t_1^2t_2^2x - t_1t_2 + t_1 + t_2 - 1)(t_1^2t_2^2x - t_1^2t_2x - t_2^2t_1x - t_1t_2x - 1)(y - 1)^2.$$

Replacing x by $1/t_1t_2x$ converts the purl to the plain version here, and leaves the polynomial of the mixed fabric unchanged. It is possible to use the different signs in the brackets to distinguish the mixed fabric with alternately knit and purl rows from the plain fabric viewed from either side, by setting $t_1 = t_2 = -1$.

Example 14. We can use Salkeld's theorem as in Example 11 to calculate the polynomials for plain weave when doubled in either the u or the v directions. We start from the data form of the polynomial $\Delta_L(U, V, e, p)$ given in Example 12, and in each case repeat the same variable p for all the weft strands (in the u direction), and e for all the warp strands, (in the v direction) to arrive at a formula for the multiple cell polynomial.

When doubled in the v direction, using the cell determined by u and $w = 2v$ this gives the polynomial $Q(U, W, e, p) = \Delta_L(U, V, e, p)\Delta(U, -V, e, p)$ with $W = V^2$.

Here

$$\Delta_L = -(1 - U)^2(1 + V^2) + [2 + (1 - e)^2(1 - p)^2U/ep + 2U^2]V,$$

giving

$$Q = (1 - U)^4(1 + W)^2 - [2 + (1 - e)^2(1 - p)^2U/ep + 2U^2]^2W.$$

The Alexander data can be presented on a translation lattice on the fabric, after expansion in terms of U and W. The Salkeld factorisation characteristic of the multiple cell repeat is not immediately obvious in the lattice, but shows up from the Alexander polynomial when the variable W is replaced by V^2.

In fact the plain weave itself as given traditionally is invariant under a larger group of translations than those generated by u and v. The translations by $\frac{1}{2}(u+v)$ and $\frac{1}{2}(u-v)$ generate the full group of invariant translations, and give rise to half-size repeating cells. The kernel based on the choice of $u, v' = \frac{1}{2}(u+v)$ can be used to calculate the Alexander data for this larger group of translations. If we use the translations $\frac{1}{2}u$ and $\frac{1}{2}v$ to index rows and columns of a matrix then the Alexander data has the simple form shown, where as before we write $e = a^2$ and $p = b^2$.

$$
\begin{array}{ccc}
-1 & . & 1 \\[2ex]
. & (a - a^{-1})(b - b^{-1}) & . \\[2ex]
1 & . & -1
\end{array}
$$

Figure 17: Plain Weave minimal cell data

In this matrix view of the Alexander data the entries indicated by . correspond to translations which do not belong to the group. The Salkeld factorisation of the traditional polynomial arising from the fact that the repeat lattice is not minimal can be seen from its data form in terms of U and V by putting $UV = V'^2$.

Example 15. We can also see examples of the multiple cell repeat when we study twills. Figure 18 shows a picture of a 1/2 twill, with the traditional repeating cell based on vectors u and v.
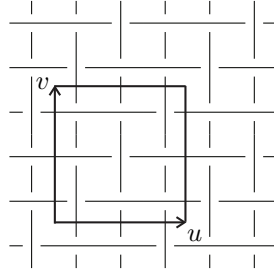

Figure 18: 1/2 twill in traditional representation

The complete translation symmetry group is generated by u and $v' = \frac{1}{3}(u+v)$, giving a repeat cell of one third the size, shown in Figure 19.

The data for the kernel based on u and v' is given here, with u drawn horizontally and v' vertically.
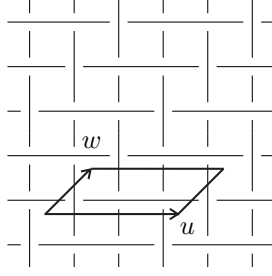
Figure 19: Minimal unit cell of 1/2 twill

$$
\begin{array}{ccc}
\sqrt{ab} & -\sqrt{b/a} & 0 \\[2mm]
0 & \sqrt{b/a}(a^{-1}-a)(b^{-1}-b) & 0 \\[2mm]
0 & \sqrt{a/b}(a^{-1}-a)(b^{-1}-b) & 0 \\[2mm]
0 & -\sqrt{a/b} & 1/\sqrt{ab}
\end{array}
$$

When placed on the plane of the fabric the data, drawn with $\frac{1}{3}u$ horizontally and $\frac{1}{3}v$ vertically, appears as shown in Figure 20.

$$
\begin{array}{cccc}
\sqrt{ab} & . & . & -\sqrt{b/a} \\[2mm]
. & . & \sqrt{b/a}(a^{-1}-a)(b^{-1}-b) & \\[2mm]
. & \sqrt{a/b}(a^{-1}-a)(b^{-1}-b) & . & . \\[2mm]
-\sqrt{a/b} & . & . & 1/\sqrt{ab}
\end{array}
$$

Figure 20: 1/2 twill minimal cell data

In this array a dot . is used for entries which do not correspond to any translation symmetry, as in the minimal cell display for traditional weave shown in Figure 17.

As in the case of the minimal cell repeat for traditional weave, discussed above, Salkeld's theorem shows that the data form of the Alexander polynomial for the kernel based on u and v will factorise when we write $UV = V'^3$, corresponding to the relation $u + v = 3v'$. One of the resulting factors, when written in terms of U and V', gives the data form for the minimal cell based on u and v'. A direct check starting from the Alexander polynomial of the kernel based on u and v does indeed confirm this.

More generally, an m/n twill consists of a weave in which there are m warp overlaps and n weft overlaps on any warp or weft thread within the repeat,

with an offset of one thread in each successive row. Take u and v as horizontal and vertical translations by $m + n$ threads. These are in the symmetry group of the fabric, but the complete symmetry group can be generated by u and $v' = \frac{1}{m+n}(u + v)$, giving a minimal repeat cell of $\frac{1}{m+n}$ times the obvious cell generated by u and v.

3.7 Layered fabrics

If a fabric decomposes into two separate layers then the plane separating the two layers becomes a torus separating a kernel N for the fabric into some curves $W_1, \ldots, W_r$ on one side of the thickened torus, coming from the lower layer of the fabric, and others $T_1, \ldots, T_k$ on the other side coming from the top layer. The face curve X lies on the side of the torus containing the curves T, while the back curve Y lies on the side containing the curves W. Because of the torus separating the components of N there is a decomposition of the polynomial Δ_N as a product. This arises as a special case of a general result of Fox, best described in the following 'Fox gluing formula'.

Suppose that we have two links $L = T_1 \cup \ldots \cup T_k \cup A$ and $M = W_1 \cup \ldots \cup W_r \cup B$. Remove a neighbourhood V_A of A and V_B of B and glue $S^3 - V_A$ to $S^3 - V_B$, matching the meridian a of V_A to the longitude of V_B and the meridian b of V_B to the longitude of V_A. The curves $T_1 \cup \ldots \cup T_k \cup W_1 \cup \ldots \cup W_r$ form a link N in the resulting manifold, which is again S^3 if one of A or B is unknotted.

Theorem 8 (Fox gluing formula)**.** *([1]) The Alexander polynomial of the link N resulting from this gluing is given by*

$$\Delta_N = \Delta_L \Delta_M$$

after substituting $a = \langle B \rangle$ in $\mathbf{Z}[w_j^{\pm 1}]$ and $b = \langle A \rangle$ in $\mathbf{Z}[t_i^{\pm 1}]$.

Remark 4. Theorem 1 is a corollary of this formula. Take the link M to be the unknot B with no further components. Then $\langle B \rangle = 1$ and $\Delta_M = \dfrac{1}{1 - b}$. The link N is the link $L - A$, and Theorem 8 shows that $(1 - \langle A \rangle)\Delta_{L-A} = \Delta_L|_{a=1}$.

Theorem 9. *If a fabric kernel N arises from a layered fabric as above, then its Alexander polynomial factorises as*

$$\Delta_N = \Delta_L(x_L, y_M \prod w_j^{a_j}, t_1, \ldots, t_k)\Delta_M(x_L \prod t_i^{b_i}, y_M, w_1, \ldots, w_r),$$

where the numbers a_j are the linking numbers of the fabric strands in the bottom layer with the face curve, while the numbers b_j are the linking numbers of the strands in the top layer with the back curve.

Proof. If a fabric kernel N arises from a layered fabric then N comes from a gluing construction to which Theorem 8 applies. Define two other links which correspond to the two layers of the fabric. These are the kernel for the top layer, $L = X_L \cup Y_L \cup T_1 \cup \ldots \cup T_k$, given by deleting the curves W from N, and the

kernel for the bottom layer, $M = X_M \cup Y_M \cup W_1 \cup \ldots \cup W_r$, given by deleting the curves T. Then the link N comes from L and M by the gluing construction, taking $A = Y_L$ and $B = X_M$.

Now $< Y_L >= x_L \prod t_i^{b_i}$ and $< X_M >= y_M \prod w_j^{a_j}$. In N the auxiliary curves are $X = X_L$, the face curve of the top layer, and $Y = Y_M$, the back curve of the bottom layer. Hence the Alexander polynomial factorises as

$$\Delta_N = \Delta_L(x_L, y_M \prod w_j^{a_j}, t_1, \ldots, t_k)\Delta_M(x_L \prod t_i^{b_i}, y_M, w_1, \ldots, w_r).$$

Assuming that the fabric has no closed components this factorisation is non-trivial, since evaluation of either factor when $t_i = w_j = 1$ for all i and j gives a polynomial which is not 0 or 1. $\qquad\blacksquare$

Corollary 10. *If a fabric decomposes into r layers then the polynomial of any kernel factorises into r factors corresponding to the kernels of the layers.*

The number of non-trivial factors in the Alexander polynomial of any kernel is then an upper bound for the number of layers. It follows that if the polynomial Δ_N does not factorise then the fabric does not decompose into layers.

The converse does not hold, since some fabrics which do not decompose into layers can have kernels whose polynomials do factorise non-trivially.

3.8 Strip-like fabrics

A fabric may decompose into disconnected parallel strips. When this happens there will be only one axial type for the strands. Assume that one of the generators u is chosen in this direction. In this case the polynomial for the kernel will be independent of the variable x and must have a factor of $1-y$ even before setting all the fabric variables to 1. It is worth testing a fabric with only one axial type for the possibility that it may fall into strips. While a knitted fabric will only have one axial type it may be possible to confirm that it is a genuinely 2-dimensional fabric if its polynomial has no factor of $1-y$, or has a non-trivial dependence on x. The knitted fabrics studied in examples 5, 6, 11, 12 and 13, with u as their axial type, do indeed exhibit a factor $1-y$, but the polynomials depend non-trivially on x, confirming that the fabrics do not fall into strips.

The warp-knitted chain illustrated in Figure 21 shows a linear chain repeated so as to give a doubly periodic fabric, in this case presented with a single axial type v . The Alexander polynomial of its kernel,

$$(x-1)(xt+1-t)(xt-x+1),$$

depends on x and t only, and has a factor of $1-x$, both of which are pointers to the strip-like nature of the fabric. Its data form then depends on V only and the data appears as a linear array of polynomials in the v direction.

An example of a fabric which is not a traditional knit or weave is the fishing net, shown in Figure 22. It has one axial type, $u+v$, relative to the choice of generating translations indicated, but it does not have the geometric characteristics of a knitted fabric with this as the thread direction.
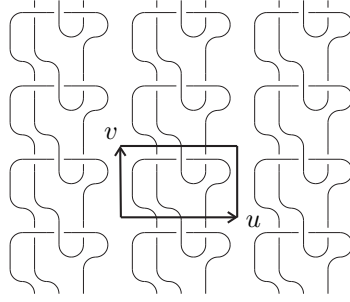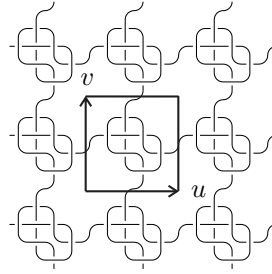
Figure 21: Warp-knitted chain

Figure 22: Fishnet

The Alexander polynomial of the kernel based on this choice of generators is

$$t^2x - t^3x + t^2y - t^3y - 4t^3xy + 2t^2xy - txy - t^5xy + 3t^4xy + t^4 - 2t^3 + 4t^2 - 3t + 1.$$

To convert to data form we set $U = ys, V = xs$, with $t = s^2$. This leads to the display of Alexander data,

$$\begin{array}{cc} s^{-1} - s & -s^4 + 3s^2 - 4 + 2s^{-2} - s^{-4} \\ s^4 - 2s^2 + 4 - 3s^{-2} + s^{-4} & s^{-1} - s, \end{array}$$

confirming that this is not a strip-like fabric.

3.9   Vassiliev invariants

In a recent paper, [3], Grishanov, Meshkov and Vassiliev have looked at the use of Vassiliev invariants of a curve in the thickened torus or Klein bottle as a tool for distinguishing textile patterns.

Following the paper of [5] in relation to Fiedler's invariant we believe that the Alexander polynomial of a kernel can be used to find some of the Vassiliev invariants for a fabric with a single fabric curve.

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Suppose that a kernel L has been constructed from the fabric, with polynomial $\Delta_L(x, y, t)$, normalised to have Torres symmetry. Expand $\Delta_L(x, y, e^h)$ as a power series $\sum a_r(x, y)h^r$. The coefficients can be regarded as elements of the integer group ring of $H_1(T^2)$, where x and y are represented by the curves V and U respectively in the torus T^2. In this setting the polynomial a_r appears to be a Vassiliev invariant of degree r in the sense of [3]. The constant term a_0 is, by Theorem 2, the polynomial $1 - x^a y^b$ up to normalisation, and the term $x^a y^b$ represents the homotopy class of the fabric component. This is constant over all homotopic curves in the torus, as for the degree 0 invariant of [3].

The invariants of the fabric curve of degree 1 should also arise in this way. A check on the examples above with only one fabric component give the following results for a_1.

Chain mail	$a_0 = 0$	$a_1 = x + x^{-1} - y - y^{-1}$
Single jersey (face)	$a_0 = 1 - y$	$a_1 = a_0(x^{-1} - x)$
Single jersey (back)	$a_0 = 1 - y$	$a_1 = a_0(x - x^{-1})$
$m \times n$ rib	$a_0 = 1 - y$	$a_1 = a_0(n - m)(x - x^{-1})$
Fish net	$a_0 = 1 - xy$	$a_1 = \frac{1}{2} - x - y + \frac{1}{2}xy$
Warp knitted chain	$a_0 = 1 - x$	$a_1 = a_0(x + 1 - x^{-1})$

4   Further investigations

Most of our calculations have made use of a Maple program based on [6] for calculating the multi-variable Alexander polynomial of a closed braid. This has required work to present a kernel for the fabric in closed braid form. We hope to develop a similar program to handle links presented as plats, or even just from the diagram of the fabric inside a unit cell. This approach looks likely to relate to a presentation of the diagram on the torus as a genus 1 virtual knot, [4].

Acknowledgments

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