Mutual braiding and the band presentation of braid groups

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Abstract

This work is concerned with detecting when a closed braid and its axis are 'mutually braided' in the sense of Rudolph [7]. It deals with closed braids which are fibred links, the simplest case being closed braids which present the unknot. The geometric condition for mutual braiding refers to the existence of a close control on the way in which the whole family of fibre surfaces meet the family of discs spanning the braid axis. We show how such a braid can be presented naturally as a word in the 'band generators' of the braid group discussed by Birman, Ko and Lee [1] in their recent account of the band presentation of the braid groups. In this context we are able to convert the conditions for mutual braiding into the existence of a suitable sequence of band relations and other moves on the braid word, and thus derive a combinatorial method for deciding whether a braid is mutually braided.

1 Introduction

This is an account of part of the second author's doctoral dissertation [4]. It is an extension of the work of the first author on exchangeable braids dating from about 1981, reported at the Sussex Low-dimensional Topology meeting of 1982, [3]. Among its antecedents are a problem of Stallings on representing the unknot as a closed braid, and some constructions of fibred knots by Goldsmith using closed braids. Subsequent work has been done by Rudolph [7], and by John Salkeld [8] who showed that a braid β is exchangeable if and only if β^2 closes to a fibred link. A more extended account has been written, giving further details of the techniques and their applications to exchangeable braids [5].

2 Exchangeable braids

The original idea arose from the principle that many features of a braid β are best seen by looking at its closure $\hat{\beta}$ *along with its axis* A as a link $A \cup B$ with a distinguished unknotted component A, as in figure 1.

$\beta = \sigma_1$ $\qquad\qquad$ $\beta = \sigma_1 \sigma_2^{-1}$

Figure 1.

Suppose conversely that we are given a link $A \cup B$ with A unknotted. Then there is a fibre projection $p_A : S^3 - A \to S^1$ whose fibres $A_\theta = p_A^{-1}(e^{i\theta})$ are discs.

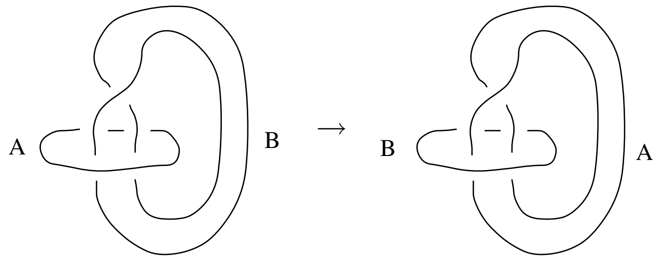
Definition. We say that B is *braided rel* A if it can be isotoped, avoiding A, to meet all fibres transversely.

Then $A \cup B$ can be viewed as a closed braid $B = \hat{\beta}$ with axis A; the braid β is determined up to conjugacy in B_n where $n = |\mathrm{lk}(A, B)|$. The link $A \cup B$ is thus a good way to capture the conjugacy class of a braid β.

Exchangeable braids arise naturally when we look at such a link and ask whether it is also possible for B to be the axis and A the closed braid, in other words whether we can have B unknotted and $A = \hat{\alpha}$ braided rel B.

Definition. We say that $A \cup B$ is *exchangeably braided* if A and B are unknotted, B is braided rel A and A is braided rel B.

The examples of figure 1 can be redrawn to see that both are exchangeably braided, as in figure 2. In each of these cases there is even an isotopy of the 2-component link which interchanges A and B, so that the braids α and β are the same up to conjugacy.



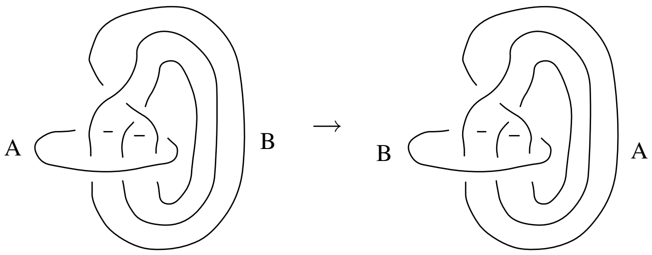


Figure 2.

This is not true in general. For example the braid β shown in figure 3, which is built as a satellite of $\sigma_1\sigma_2^{-1}$ with pattern σ_1, is exchangeable and the exchanged braid α, shown also in figure 3, is built as a satellite of σ_1 with pattern $\sigma_1\sigma_2^{-1}$. However there can not be an isotopy of the link $A \cup B$ interchanging the components, since its 2-variable Alexander polynomial $\Delta_{A\cup B}(a,b) = \sum c_{rs}a^r b^s$, defined up to a power of a and b, has coefficient matrix

$$C = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Now C is not symmetric, and consequently α is not conjugate to β.

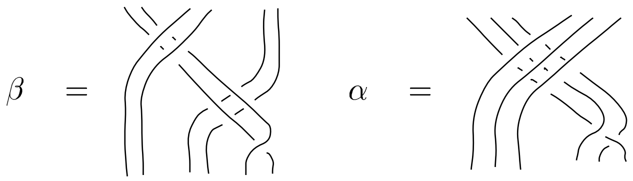


Figure 3.

3 Bands

The basic question of when a given braid β is exchangeable is most readily answered in terms of the elementary n-braids a_{ij} known as *embedded bands* in [6], which we shall simply call *bands* in this paper.

Definition. The *band* $a_{ij} = a_{ji}$ is the braid illustrated in figure 4, in which the strings i and j interchange in the positive sense in front of the other strings.

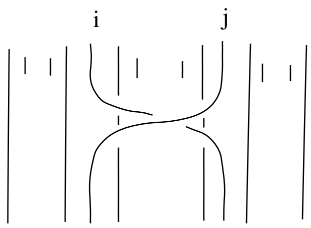


Figure 4.

Definition. A *Stallings* n-braid is a product of $n-1$ bands or their inverses whose closure is connected.

The closure of such a braid is readily seen to be spanned by a disc meeting the axis in n points made up of one disc for each string, joined by $n-1$ half-twisted bands. It is clearly necessary for an exchangeable n-braid β to have unknotted closure $B = \hat{\beta}$, and for B to be spanned by a disc which meets the axis A in n points. This property was shown in [3] to characterise Stallings braids up to conjugacy. It follows that any exchangeable braid is conjugate to a Stallings braid.

This condition alone is not enough, however; there has to be not just one spanning disc, but a whole family $\{B_\varphi\}, \varphi \in [0, 2\pi]$ which all meet A in n points. Examples of non-exchangeable Stallings braids are readily available for $n \geq 4$. While the Stallings 4-braids $a_{13}(a_{23})^{-1}a_{24}$, illustrated in figure 5, and $a_{13}a_{23}a_{24}$ are both exchangeable, the Stallings braid $\beta = a_{24}a_{23}a_{13}$ is *not*. This can be detected from the nature of the Alexander polynomial $\Delta_{A \cup \hat{\beta}}(a, b) = \sum d_{rs} a^r b^s$ of any closed braid and axis. Such a polynomial always has the form $\det(aI - B(b))$ where $B(b)$ is the $(n-1) \times (n-1)$ reduced Burau matrix of β, [3]. The extreme coefficients of powers of a are then monomials in b with coefficient ± 1. A similar constraint must apply to the extreme coefficients of powers of b when the braid is exchangeable. In the case of β above, the matrix D of coefficients of the Alexander polynomial is

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 2 & -1 \\ -1 & 2 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix};$$

although the first and last rows have each a single entry ± 1, this does not hold for the first and last columns, so in this case A can not be braided rel $\hat{\beta}$.

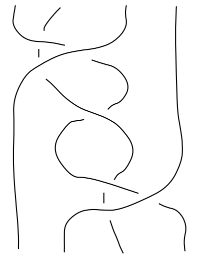


Figure 5.

Birman, Ko and Lee [1] have recently looked at the presentation of the braid groups using bands as generators. The relations in this presentation turn out to give a very good combinatorial method to help in determining when a braid is exchangeable.

The relations themselves can be stated very simply as follows.

- For $i > j > k$ we have $a_{ij}a_{jk} = a_{jk}a_{ki} = a_{ki}a_{ij}$.

  The common product of these pairs of bands can be visualised as a positive twist on the strings i, j and k through one-third of a full turn.

- The bands a_{pq} and a_{rs} commute if p, q, r and s are all different, and the pairs (p, q) and (r, s) do not interlace.

  Thus $a_{14}a_{23} = a_{23}a_{14}$ but $a_{13}a_{24} \neq a_{24}a_{13}$.

4 Generalised exchangeable braids

In trying to determine which Stallings braids are exchangeable we follow Goldsmith [2] in using the term *generalised axis* for an oriented fibred link A, not necessarily the unknot, and extending the definition of a closed braid to include a curve B which meets all the fibres A_θ of the fibration $p_A : S^3 - A \to S^1$ transversely. In this case we also extend our previous terminology to say that B is braided rel A when A is fibred and the fibration can be chosen so that B meets all the fibres transversely, and thus meets each fibre in $n = |\mathrm{lk}(A, B)|$ points.

Definition. Call $A \cup B$ a *(generalised) exchangeable link* if A and B are both fibred, B is braided rel A *and* A is braided rel B.

In the most general case, when both A and B are knotted, our analysis is still rather sketchy, even in the case of braid index 1.

In what follows we shall restrict attention to the case in which one component, A say, is unknotted, when B will determine a classical braid β up to conjugacy. We call β a *generalised exchangeable braid* if the axis A is braided rel $B = \hat{\beta}$ as above.

Theorem 1 *Every generalised exchangeable braid β is conjugate to a product of k bands, such that the resulting banded surface made up from one disc for each braid string joined by k half-twisted bands forms a fibre surface for $B = \hat{\beta}$.*

Proof : We may suppose that β is an n-braid such that $B = \hat{\beta}$ is fibred and whose axis A is braided rel B. Then A meets each fibre B_φ in n points, with all intersections in the same sense. Concentrate on just one of these fibre surfaces B_0 and its intersection with all the discs A_θ which span the axis A, or equivalently look at the function $p_A|B_0 - A \to S^1$. We can assume that the intersections are transverse except for finitely many values of θ where there is either a saddle or a centre. Now B_0 has minimal genus among spanning surfaces of B and all its intersections with A are in the same sense. Hence, following standard arguments as in [6], we can isotop it to eliminate centres. There are then $n - \chi(B_0)$ saddles, each of which lies on a component of $A_\theta \cap B_0$ as in figure 6.

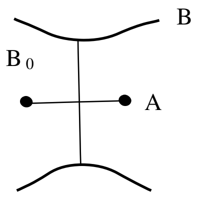


Figure 6.

The remainder of B_0 is foliated by the arcs of $A_\theta \cap B_0$ in a way which is determined up to isotopy by the position of the saddles. Each saddle determines two crossing arcs in B_0, one which joins two points of the boundary B and one which connects two of the n points of A lying in B_0. Within B_0 take a neighbourhood N of the points of A and the cross arcs of the $k = n - \chi(B_0)$ saddles which join them, whose boundary meets the foliation transversely. The resulting subsurface differs from B_0 only by a collar of the boundary B, so that B can be isotoped through B_0, avoiding A, to form the boundary of N. We may choose a sufficiently close neighbourhood N as a surface in S^3 made up of n disc neighbourhoods of the intersections with the axis together with neighbourhoods of the k cross-arcs each at a different level of θ. The close neighbourhood of each such arc within N lies as a half-twisted band when viewed with A as axis, while its boundary still lies as a closed braid. As a result, B_0 is isotopic to a banded surface N whose boundary can replace the original curve B without loss. When the points of intersection of B_0 with the axis are labelled $1, \ldots, n$ in order around A then the neighbourhood of a saddle arc joining points i and j becomes a band $(a_{ij})^{\pm 1}$, whose sign depends on the direction of tangency of the surfaces B_0 and A_θ at that saddle. The boundary can be presented as the closure of a braid consisting of the product of these $k = n - \chi(B_0)$ bands in the order of the value of θ at each saddle. $\square$

We may thus assume that any exchangeable braid is, up to conjugacy, a product of bands and that a fibre surface for its closure is the resulting banded surface. In the case when B is unknotted this is the result from [3] about Stallings braids.

Corollary 1 *There are only a finite number of possible exchangeable links $A \cup B$ with unknotted A, and a given genus for B and linking number n.*

Proof : The component B can be written as the closure of an n-braid β with axis A. By the theorem we may assume that β is the product of k bands, where k is determined by the genus of B and the linking number n. There are only finitely many products of k bands. $\square$

To help decide exactly which products of bands *are* exchangeable, we must look at the singular foliations induced on *every one* of the family of fibre surfaces B_φ. At the same time we will see singular foliations of each disc A_θ by the curves of $A_\theta \cap B_\varphi$.

We can best keep track of the foliations by plotting the values of $(\theta, \varphi) \in S^1 \times S^1$ for which A_θ and B_φ do not meet transversely. After a small isotopy we may assume that the singularities of any intersections are generic, so that the graphic of singular values may look something like figure 7.

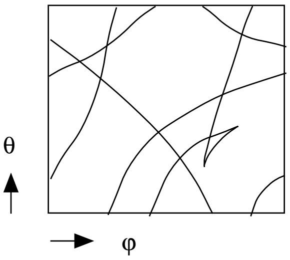


Figure 7.

At a general singular point (θ, φ) there is either a single saddle or centre tangency between A_θ and B_φ. For any φ we can calculate the Euler characteristic of the n-punctured surface $B_\varphi - A$ as the number of centres minus the number of saddles in the foliation of B_φ, which we can read from the graphic on the line of the chosen value of φ. Since this number must be constant $(= \chi(B_0) - n)$ as φ varies we see that this sum remains fixed as the vertical line sweeps through the graphic. Similarly each horizontal line meets the graphic so as to contribute $\chi(A_\theta) - n = 1 - n$ to the Euler characteristic for the n-punctured disc $A_\theta - B$.

Thus in the generic case no smooth local maxima or minima can occur either vertically or horizontally on the lines of the graphic, which must consist of a number of monotone increasing or decreasing lines with some simple crossings and cusps. Each monotone line represents either a persistent saddle or centre, by constancy of the Euler characteristic. At a crossing, two tangencies generically occur at distinct places and the saddles or centres continue through the crossings, while at each cusp a line of centres and a line of saddles meet. The slope of a line is positive when the surfaces A_θ and B_φ have the same orientation at the point of tangency, and negative when they have reverse orientations.

5 Mutual braiding and labelled graphics

To simplify further analysis we now impose Rudolph's condition that the two fibrations form *mutually braided open books*, [7]. Equivalently we require all the local tangencies between fibres of the two families to be saddles, so that the graphic of singular values has no centres, and hence no cusps. Rampichini proves in [4, 5] that, for an exchangeable link with one unknotted component, this condition can be assumed without loss of generality.

Definition. A link $A \cup B$ with unknotted A which satisfies Rudolph's condition is called *mutually braided.*

The graphic of a mutually braided link then consists only of a number of monotone lines, with simple crossings. We can add some simple combinatorial information to the graphic of a mutually braided link to describe the positions of the saddles in each disc A_θ. The boundary of each disc A_θ is the fixed curve A which meets every fibre B_φ in n points. The n points of A where $\varphi = 0$ as $1, \dots, n$ dissect A into n half-open segments. Label these $1, \dots, n$ in order around A.

Where $A_\theta \cap B_\varphi$ contains a single saddle tangency the cross-arc of the saddle joins two points of A lying in different segments i and j. Label the corresponding point (θ, φ) of the graphic of singular values with the pair ij. Along any line of the graphic the label remains constant until we reach a point where two lines of saddles cross, or we get to the transition where φ increases through the level $\varphi = 2\pi$ to return to $\varphi = 0$. At this transition the corresponding arcs of the saddles will move from segments i and j to segments $i+1$ and $j+1$.

Before describing the labelling of the graphic at a crossing it is worth visualising the singular foliation of the disc A_θ for fixed θ. A typical configuration with $n = 4$ is shown in figure 8.

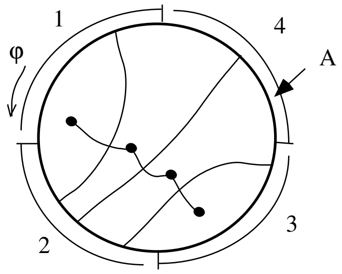


Figure 8.

The general line $\theta =$ constant meets the graphic in $n-1$ points, each labelled by some pair ij. These correspond to the cross arms of $n-1$ saddles, and the labelling tells us in which segments of the circle A their ends lie. They divide the disc A_θ into n regions, each of which contains one point of B; a choice of this point and the rest of the foliation is determined up to isotopy by the saddle arms. As θ increases the position of the saddles, and the foliation changes, initially by an isotopy in which the end points of the positive saddle arms rotate in one direction while those of the negative arms rotate in the other.

This evolution is viewed by Rampichini as a sort of cinematic film, in which the screen is a disc with fixed boundary on which is projected the foliation of A_θ with the parameter θ in the role of a time variable. The main part of the action is the movement of the saddle arms. This is punctuated by isolated instants where two lines of saddles cross in the graphic. In the film at each such critical time θ there are two saddles having the same value of φ and thus two singular points in the set $A_\theta \cap B_\varphi$. These singular points either belong to two disjoint components of

$A_\theta \cap B_\varphi$, each consisting of simple saddle arms connecting non-interlocking pairs of segments p, q and r, s say, or the two saddles belong to the same component of $A_\theta \cap B_\varphi$ which lies in A_θ as shown in figure 9.

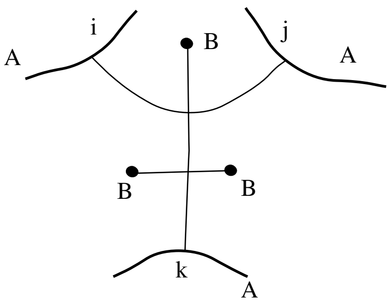


Figure 9.

Nothing very dramatic happens in the film when disjoint saddle arms are involved; they simply interchange levels of the value φ, while in the graphic the labels pq and rs continue through the crossing. Where the two saddles belong to the same component, however, we see the saddle arms in the film coming together at one end and then reforming with one of the two arms remaining essentially the same and the other changing radically to emerge from the other end of the unchanged saddle arm. The whole process is determined by the 'Christmas tree' configuration of the intermediate position as in figure 9. The essential information is the labelling of the three segments i, j and k of the end points, and the knowledge of which segment lies at the base of the tree. The labelling of the saddles on the graphic before and after the crossing depends on this combinatorial information. As a summary of the film itself it is useful to think of simply drawing the sequence of shots at the critical levels, with each Christmas tree truncated to form a 'T' and the signs of each saddle indicated so that the direction of its evolution in the film is known. Rampichini presents such data, with some extension, in what she terms a 'film script', with these shots to dictate the changes of scene.

The labelling on the graphic is then quite constrained in its nature at the crossing points. It is easier to describe the exact transitions of labelling at the crossings when we change our viewpoint and consider what happens as we change φ.

For a fixed φ the graphic provides a complete list of all the saddles in B_φ which arise in the foliation by the intersections with the level discs A_θ, and the labelling shows which of the segments of A are connected by their cross-arms. The labelling on the graphic, read in order of increasing θ, determines a word, w_φ, as a product of bands, using a_{ij} or its inverse according to the slope of the line on the graphic with label ij.

Theorem 2 *The band word w_φ determined by a labelled graphic arising from a mutually braided link changes only*

- *when two lines of the graphic cross, or*

- *when a line on the graphic passes between $\theta = 0$ and $\theta = 2\pi$.*

In the first case w_φ changes by the application of a band relation on passing the crossing. In the second case, which occurs exactly $n-1$ times, either a final band a_{ij} is transferred to the beginning of the band word w_φ, or an initial band a_{ij}^{-1} to the end, as φ increases.

Proof : The first case is shown by examining the possible generic behaviour close to the configuration in figure 9. Full details are given in [4, 5]. The second case is immediate from the monotonicity of the lines and the fact that there are $n-1$ saddles at $\theta = 0$. □

As a result, the labelled graphic arising from a mutally braided closed braid B determines a sequence of band words w_φ, each representing B and a spanning surface B_φ. The sequence starts from the band word w_0, which can be regarded as a given band word for the proposed exchangeable braid B along with its spanning surface, and finishes with the word $w_{2\pi}$, which is the original word w_0 with all its indices i, j reduced by 1 mod n. They are connected by a sequence of moves consisting of an unspecified number of applications of the band relations along with exactly $n-1$ cyclings as specified in the theorem.

By way of example, a labelled graphic for the exchangeable 4-braid with band word $a_{34}(a_{12})^{-1}a_{23}$ is shown in figure 10.

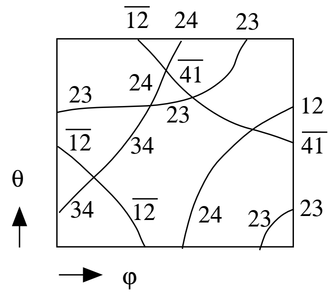


Figure 10.

The resulting sequence of band words, given simply by their indices, is listed below.

$$(34)\overline{(12)}(23) \to \overline{(12)}(34)(23) \to (34)(23)\overline{(12)}$$
$$\to (23)(24)\overline{(12)} \to (23)\overline{(14)}(24) \to (24)(23)\overline{(14)}$$
$$\to (24)\overline{(14)}(23) \to (23)(24)\overline{(14)} \to (23)\overline{(41)}(12).$$

6 Band algorithms for mutual braiding

The previous section shows that if a braid determines a mutually braided open book then there must be a labelled graphic giving rise to a suitably restricted sequence of band words. It can be shown, with a little more argument, that there are only finitely many possible sequences which need to be considered when asking if a given band word determines a mutually braided link, and if none of them fulfil all the necessary conditions for arising from a labelled graphic then the link is definitely not mutually braided.

The key to the converse of this result, which also covers Rampichini's extension to exchangeable braids, lies in the reconstruction of a family of spanning surfaces from an 'admissibly' labelled graphic, [4, 5].

Definition. A graphic of monotone increasing and decreasing lines in $S^1 \times S^1$, labelled by pairs of numbers i, j between 1 and n, is *admissibly labelled* if

- the labelled lines cross one basic circle $\theta = 0$ in $n-1$ points,
- the band words w_φ read from the graphic following the basic circles $\varphi =$ constant of the other family satisfy the band relations at every crossing,
- the indices in the bands all change by 1 at the reference circle $\varphi = 0$,
- the $n-1$ labels on the circle $\theta = 0$ contain no interlocking pairs.

Theorem 3 *Every admissibly labelled graphic arises from some exchangeable braid.*

Proof : Use the last condition in the definition to draw a set of saddles in the disc A_0 which correspond with the labelling when $\theta = 0$. The evolution of the graphic as θ increases then provides a film script, which is converted into a film by filling in suitably the non-singular parts of the foliation at each level. Use of standard models around the singular levels where there are crossings in the graphic, coupled with the monotonicity of the movement of the saddle arms with increasing θ, ensures that the family of surfaces in the solid torus generated in this way are non-singular everywhere, [4, 5]. $\square$

Consequently an admissibly labelled graphic with a given band word at $\varphi = 0$ will give rise to a mutually braided fibration for the corresponding closed braid.

Rampichini also shows how the graphic for a general exchangeable braid, which includes cusps and lines of centres, can be used to construct an admissibly labelled graphic, and hence deduce that any exchangeable braid is mutually braided [4, 5].

We thus have an algorithm to decide whether a band word w_0 determines a mutually braided (and indeed an exchangeable) braid.

Algorithm. Look for a chain of moves to convert the word w_0 to the same word but with all indices reduced by 1 mod n. The moves must consist of a number of band relations and exactly $n-1$ cyclings in which a positive band is moved from back to front or a negative band from front to back of the word. It can be shown that no repetitions of words between cyclings need occur in an admissibly labelled graphic, so there are only a finite number of possible chains to consider [4, 5].

If no chains satisfy these conditions then the braid is *not* mutually braided.

Given a chain satisfying these conditions try to realise it by an admissibly labelled graphic. It may not be possible to do this, bearing in mind that the lines in the graphic, which are determined combinatorially by the chain as φ increases, are required to move strictly monotonically in θ, and must end at the same level that they begin. If any chain can be realised then the braid *is* mutually braided, otherwise again it is not.

As an example we apply this method to the Stallings braid $w = a_{24}a_{23}a_{13}$ mentioned earlier to show that its closure is not exchangeable.

We must consider possible chains of moves which lead to the word $w' = a_{13}a_{12}a_{24}$, and include exactly 3 cyclings. Now no band relations are possible in w so the first move in any chain must be a cycling, to get $a_{13}a_{24}a_{23}$. Again no relations are possible, so we must cycle to $a_{23}a_{13}a_{24}$. Once more there are no relations and we must cycle, returning to w. Since we have used all 3 cyclings there is nothing more we can do; as we have not reached w' we conclude that the braid is not exchangeable.

In contrast we *can* find a chain and an admissibly labelled graphic leading from the word $v = a_{13}a_{23}a_{24}$ to $v' = a_{24}a_{12}a_{13}$, by making use of band relations such as $a_{23}a_{24} = a_{24}a_{34}$.

Acknowledgments

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