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The coloured Jones function and Alexander polynomial for torus knots

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Abstract

In [**2**] it was conjectured that the coloured Jones function of a framed knot K, or equivalently the Jones polynomials of all parallels of K, is sufficient to determine the Alexander polynomial of K. An explicit formula was proposed in terms of the power series expansion $J_{K,k}(h)=\sum_{d=0}^{\infty}a_d(k)\,h^d$, where $J_{K,k}(h)$ is the $SU(2)_q$ quantum invariant of K when coloured by the irreducible module of dimension k, and $q=e^h$ is the quantum group parameter.

In this paper I show that the explicit formula does give the Alexander polynomial when K is any torus knot.

1. *Introduction*

Invariants for a framed knot K defined using a quantum group $\mathscr{G}$ have been described [**5**] in terms of 'colouring' the knot K with a $\mathscr{G}$-module. Any choice V_λ of $\mathscr{G}$-module determines a power series $J(K;V_\lambda)\in\mathbf{Q}[[h]]$, which can generally be rewritten as a Laurent polynomial with integer coefficients in $q=e^h$. The invariant is additive under sums of modules, while using the tensor product of two modules on a knot K gives the same invariant as that of the link $K^{(2)}$ with two parallel strands, when each strand is coloured by one of the two modules. It is thus usual to interpret the whole collection of invariants, for all $\mathscr{G}$-modules, as a linear function $J(K)$ from the representation ring $\mathscr{R}$ of $\mathscr{G}$ to $\mathbf{Q}[[h]]$, where $\mathscr{R}$ is taken as linear combinations of irreducible $\mathscr{G}$-modules, and the coefficients in $\mathscr{R}$ are drawn from $\mathbf{Q}[[h]]$, [**3**].

The coloured Jones function $J_{K,k}(h)$, which is the subject of this paper, refers to the quantum group $\mathscr{G}=SU(2)_q$, and is given in the notation above by $J_{K,k}(h)=J(K;V_k)$, where V_k is the unique k-dimensional $\mathscr{G}$-module. Thus $J_{K,k}(h)$ is a power series in h,

$$J_{K,k}(h)=\sum_{d=0}^{\infty}a_d(k)\,h^d.$$

The coefficients $a_d(k)$ have been shown in [**2**] to be odd polynomials in k of degree at most $2d+1$. In the power series for the normalized function $J_{K,k}(h)/[k]$, where

$$[k]=\frac{\exp(hk/2)-\exp(-hk/2)}{\exp(h/2)-\exp(-h/2)}$$

is the function $J_{O,k}(h)$ for the unknot O with zero framing, the coefficient of h^d is then an even polynomial in k of degree at most $2d$.

Denote by $J^u_{K,k}(h)$ the coloured Jones function where the framing of K is altered to the zero framing. It is shown in [2] that the degree of the coefficients of h^d as polynomials in k then reduces by at least 2. It is conjectured there that the degree in the normalized form $J^u_{K,k}(h)/[k]$ reduces from $2d$, when the framing is non-zero, to at most d in the case of zero framing. Thus when written as

$$J^u_{K,k}(h)/[k] = \sum_{l,d=0}^{\infty} b_{ld}\, k^l h^d,$$

this first conjecture in [2] is that $b_{ld} = 0$ for $l > d$.

The second conjecture is that the terms in $k^d h^d$ give the Alexander polynomial of K in the form

$$\sum_{d=0}^{\infty} b_{dd}(kh)^d = 1/\Delta_K(e^{kh}).$$

The aim of this paper is to show that the formula is correct in the case where K is any torus knot. The main effort is needed in showing, by explicit calculations, that $J^u_{K,k}(h)/[k]$, for a torus knot K, satisfies the bounds on degrees in the first conjecture, and then identifying the terms in $(kh)^d$. For this I draw on more general results about quantum invariants of cables, which I summarize in the next section, before specializing to the case of $SU(2)_q$ and torus knots.

2. *Invariants of cables*

Explicit details of how to calculate the $\mathscr{G}$-invariants for a cable about a framed knot K in terms of the invariants of K are given by Rosso and Jones [4]; a similar description by Strickland appears in [6]. I shall give a brief summary of these results.

Write $K_{(m,p)}$ for the (m,p) cable about K, where m and p are co-prime and $K_{(m,p)}$ is best described, as a framed knot, in terms of the (m,m) tangle T illustrated below (Fig. 1), by decorating a correctly framed diagram of K with the closure in the annulus of the diagram T^p. Further details of this terminology can be found in [3]. As so defined, the (m,p) cable has m strands, making p/m full twists relative to the framing of K; the choice of framing corresponds to a choice of parallel which lies on the surface of the torus neighbourhood of K, alongside the cable itself. The notation is consistent with the description of the 2-parallel of K as the $(2,0)$ cable about K; in their paper, Rosso and Jones use the reverse order for m and p. I have reluctantly avoided using (p,q) cables in view of the other meaning for q in a quantum group.

There is a relation between the functions $J(K_{(m,p)})$ and $J(K)$ given, independently of K, in terms of two linear maps $F:\mathscr{R}\to\mathscr{R}$ and $\psi_m:\mathscr{R}\to\mathscr{R}$. The map F gives the effect on the $\mathscr{G}$-invariant of a framing change on K. When an extra positive curl is added to the framed knot K to make K' then $J(K') = J(K)\circ F$, as a function on $\mathscr{R}$. In terms of the notation above, $K' = K_{(1,1)}$ and we have more generally that

$$J(K_{(1,p)}) = J(K)\circ F^p.$$

Every irreducible $V_\lambda \in \mathscr{R}$ is an eigenvector for F, whose eigenvalue $f_\lambda \in \mathbf{Q}[[h]]$ has the form $f_\lambda = e^{hp_\lambda}$, where p_λ is independent of h and can be found explicitly in terms of the Killing form for the classical Lie algebra corresponding to $\mathscr{G}$. For example, in the case of $SU(2)_q$, with $V_\lambda = V_k$ taken to be the k-dimensional irreducible then $f_k = e^{h(k^2-1)/4}$.

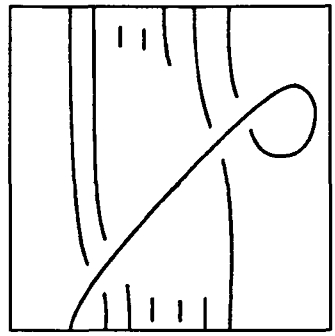
$T =$ 

Fig. 1.

For each $t \in \mathbf{Q}$ we can define a linear map $F^t : \mathscr{R} \to \mathscr{R}$ by setting $F^t(V_\lambda) = e^{thp_\lambda} V_\lambda$ for each irreducible V_λ.

The second map $\psi_m : \mathscr{R} \to \mathscr{R}$ which features in the description of the cable invariants is a ring homomorphism which is also known as the mth Adams operation. An account of the Adams operations is given by Atiyah in [**1**]. The element $\psi_m(V_\lambda)$ can be defined on the representation ring of the corresponding classical Lie algebra, which is isomorphic to the ring $\mathscr{R}$, in terms of the permutation action of the cyclic group C_m on the tensor product $V_\lambda^{\otimes m}$. The element $\psi_m(V_\lambda)$ is an integer linear combination of irreducibles in $\mathscr{R}$, which may be calculated by classical means.

In the case of $SU(N)$ the ring $\mathscr{R}$ may be identified with the ring of symmetric polynomials in N indeterminates $x_1, \ldots, x_N$, with $x_1 x_2 \ldots x_N = 1$, and thus with the ring of polynomials in the elementary symmetric functions $c_1 = x_1 + \cdots + x_N, c_2, \ldots, c_{N-1}$. In this case the map $\psi_m : \mathscr{R} \to \mathscr{R}$ is induced by $\psi_m(x_i) = (x_i)^m$.

There is an extensive literature on the description of the representation ring $\mathscr{R}$ for $SU(N)$ in which irreducible representations V_λ are indexed by Young diagrams; for example, calculations in Weyl [**7**] give determinantal formulae for V_λ as a polynomial in $\{c_j\}$ which can be found readily from the Young diagram of V_λ. Conventionally c_j is represented by the Young diagram with a single column of j cells, and corresponds to the jth exterior power of the 'fundamental' N-dimensional representation c_1 of $SU(N)$.

The description for the invariant of a cable in terms of the invariant of the original knot, for any quantum group $\mathscr{G}$, can be summarized in the following theorem, which appears in [**4**] and [**6**].

THEOREM (Rosso-Jones, Strickland). *The quantum invariant $J(K_{(m,p)})$ for the (m, p) cable about K is given by*

$$J(K_{(m,p)}) = J(K) \circ F^{p/m} \circ \psi_m$$

as a function on the representation ring $\mathscr{R}$ of the quantum group. ∎

Thus, when we find $\psi_m(V_\mu) = \Sigma a_\lambda V_\lambda$, with $a_\lambda \in \mathbf{Z}$ and V_λ irreducible, we have $J(K_{(m,p)}; V_\mu) = \Sigma (f_\lambda)^{p/m} a_\lambda J(K; V_\lambda)$.

Now in the case of the quantum group $SU(2)_q$ the ring $\mathscr{R}$ is isomorphic to the polynomial ring in one variable $c_1 = x + x^{-1}$, or equally to the symmetric part of the Laurent polynomial ring in x. We have a basis of irreducibles in $\mathscr{R}$ consisting of

$$V_k = \frac{x^k - x^{-k}}{x - x^{-1}}, \quad \text{for} \quad k > 0 \in \mathbf{N}.$$

The map $\psi_m:\mathscr{R}\to\mathscr{R}$ is the ring homomorphism given by $\psi_m(x)=x^m$. Thus

$$\psi_m(V_k)=\frac{x^{km}-x^{-km}}{x^m-x^{-m}}=\sum_{\lambda\in\mathbf{N}} a_\lambda V_\lambda, \quad \text{say.}$$

To identify the coefficients a_λ we may write

$$\begin{aligned}(x-x^{-1})\,\psi_m(V_k) &= \sum a_\lambda(x^\lambda-x^{-\lambda})\\ &= (x^{(k-1)m}+x^{(k-3)m}+\cdots+x^{-(k-1)m})\,(x-x^{-1})\end{aligned}$$

in the full Laurent polynomial ring on x. For $\lambda<0$ set $a_\lambda=-a_{-\lambda}$ and $f_\lambda=f_{-\lambda}=e^{h((\pm\lambda)^2-1)/4}$. Then

$$\begin{aligned}(x-x^{-1})\,(F^{p/m}(\psi_m(V_k))) &= \sum_{\lambda\in\mathbf{N}}(f_\lambda)^{p/m}a_\lambda(x^\lambda-x^{-\lambda})\\ &= \sum_{\lambda\in\mathbf{Z}}(f_\lambda)^{p/m}a_\lambda x^\lambda,\end{aligned}$$

where a_λ is found from the equation

$$\begin{aligned}\sum_{\lambda\in\mathbf{Z}} a_\lambda x^\lambda &= (x^{(k-1)m}+x^{(k-3)m}+\cdots+x^{-(k-1)m})\,(x-x^{-1})\\ &= \sum_{r=-(k-1)/2}^{(k-1)/2}(x^{2rm+1}-x^{2rm-1}).\end{aligned}$$

Now for $\lambda=2rm\pm1$ we have $f_\lambda=s^{(\lambda^2-1)/2}=s^{2r^2m^2\pm2rm}$, with $s=e^{h/2}$. Thus

$$(x-x^{-1})\,(F^{p/m}(\psi_m(V_k)))=\sum_{r=-(k-1)/2}^{(k-1)/2}(s^{2r^2mp+2rp}x^{2rm+1}-s^{2r^2mp-2rp}x^{2rm-1}).$$

3. *Calculations for torus knots*

The goal is to calculate $J_{L,k}(h)=J(L;V_k)$, where L is the (m,p) torus knot. Then $L=K(m,p)$, where $K=O$ is the unknot with zero framing. Now the invariant $J(O):\mathscr{R}\to\mathbf{Q}[[h]]$ is a ring homomorphism defined on the full Laurent polynomial ring by $x\mapsto s=e^{h/2}$. Using the theorem above, we can write $J_{L,k}(h)=J(O_{(m,p)};V_k)=J(O;F^{p/m}(\psi_m(V_k)))$. We thus have

$$(s-s^{-1})\,J_{L,k}(h)=\sum_{r=-(k-1)/2}^{(k-1)/2}(s^{2r^2mp+2rp}s^{2rm+1}-s^{2r^2mp-2rp}s^{2rm-1}).$$

To calculate the formula proposed for the Alexander polynomial we must first normalize the Jones function to find $J^u_{L,k}(h)/[k]$, where

$$[k]=\frac{s^k-s^{-k}}{s-s^{-1}}$$

and $J^u_{L,k}$ is the Jones function when the framing of L is altered to zero. Our calculation of $J_{L,k}(h)$ above has been made from a diagram with writhe mp, so that $J^u_{L,k}(h)=f_k^{-mp}J_{L,k}(h)=s^{-2mp(c^2+c)}J_{L,k}(h)$, where we set $c=(k-1)/2$. This gives an explicit expression

$$\begin{aligned}(s^k-s^{-k})\frac{J^u_{L,k}(h)}{[k]} &= s^{-2mp(c^2+c)}\sum_{r=-c}^{c}(s^{2r^2mp+2rp+2rm+1}-s^{2r^2mp-2rp+2rm-1})\\ &= I(s,c), \quad \text{say.}\end{aligned}$$

The first part of the conjecture concerns $I(s,c)/(s^k - s^{-k})$ as a function of $k = 2c+1$ and h, and says that the coefficient of h^d in this function is a polynomial of degree no more than d in k. Since $s^k - s^{-k}$ is a power series in (kh) the conjecture holds if and only if it holds for $I(s,c)$ as a function of h and k, and thus equally as a function of h and c.

The second part of the conjecture states that the terms in $k^d h^d$ in $J^u_{L,k}(h)/[k]$ are given by $1/\Delta_L(e^{kh})$. For the torus knot L the Alexander polynomial is given by

$$\Delta_L(e^h) = \frac{(s^{mp} - s^{-mp})(s - s^{-1})}{(s^m - s^{-m})(s^p - s^{-p})}.$$

Now write $I_{\max}(s,c)$ for the sum of the terms in $k^d h^d$ in $I(s,c)$. The second conjecture then becomes

$$I_{\max}(s,c)/(s^k - s^{-k}) = 1/\Delta_L(e^{kh}),$$

or equivalently

$$I_{\max}(s,c) = \frac{(s^{mk} - s^{-mk})(s^{pk} - s^{-pk})}{s^{mpk} - s^{-mpk}}.$$

I shall now complete the analysis of $I(s,c) = (s^k - s^{-k})\, J^u_{L,k}(h)/[k] = (s - s^{-1})\, J^u_{L,k}(h)$ using two propositions, the first of which proves the first conjecture for L, while the second, after a short argument, proves that the Alexander polynomial for torus knots is given by the formula above from the coloured Jones function.

Proposition 1. *The coefficient of h^d in the function $I(s,c)$ is a polynomial of degree $\leqslant d$ in c.*

Proposition 2. *The terms in $c^d h^d$ in $I(s,c)$ can be written as*

$$\frac{(s^{2mc} - s^{-2mc})(s^{2pc} - s^{-2pc})}{s^{2mpc} - s^{-2mpc}}.$$

Proof of proposition 1. Write $I(s,c)$ in terms of $H = mph$ and set $e^H = s^{2mp} = Q$. It is enough to show that the coefficient of H^d has degree $\leqslant d$ in c.

Write $a = 1/2p$ and $b = 1/2m$. Then

$$\begin{aligned} I(s,c) &= Q^{-(c^2+c)} \sum_{r=-c}^{c} (Q^{r^2+2rb+2ra+2ab} - Q^{r^2-2rb+2ra-2ab}) \\ &= \sum_{r=-c}^{c} Q^{\phi(r)} - Q^{\phi(r-2b)} \end{aligned}$$

where $\phi(r) = r^2 + 2r(a+b) + 2ab - c^2 - c = (r+a+b)^2 - a^2 - b^2 - c^2 - c$. We then have $I(s,c) = \sum P_d(a,b,c) H^d$, with

$$P_d(a,b,c) = \frac{1}{d!} \sum_{r=-c}^{c} (\phi(r))^d - (\phi(r-2b))^d.$$

We have to establish that $P_d(a,b,c)$ has degree $\leqslant d$ in c for all a and b. Now $P_d(a,b,c)$ is clearly a polynomial in b and so if we can show that the coefficient of c^l for $l > d$ is zero for all *positive integer* values of b it must then be identically zero.

It is thus enough to prove that $P_d(a,b,c)$ has degree $\leqslant d$ in c under the assumption that $b \in \mathbf{N}$. In this case

$$\begin{aligned} I(s,c) = \sum_{r=-c}^{c} Q^{\phi(r)} - Q^{\phi(r-2b)} &= \sum_{r=-c}^{c} Q^{\phi(r)} - \sum_{r=-c-2b}^{c-2b} Q^{\phi(r)} \\ &= \sum_{r=c-2b+1}^{c} Q^{\phi(r)} - \sum_{r=-c-2b}^{-c-1} Q^{\phi(r)} \\ &= \sum_{r=-b+1}^{b} Q^{\phi(r+c-b)} - \sum_{r=-b}^{b-1} Q^{\phi(r-c-b)}, \end{aligned}$$

and so

$$d!\,P_d(a,b,c) = \sum_{r=-b+1}^{b} (\phi(r-b+c))^d - \sum_{r=-b}^{b-1} (\phi(r-b-c))^d.$$

Each of the summands is a polynomial of degree $\leqslant d$ in c, since $\phi(r-b\pm c) = (r+a\pm c)^2 - a^2 - b^2 - c^2 - c = (r+a)^2 \pm 2c(r+a) - c - a^2 - b^2$ is linear in c. The limits in these sums do not involve c, and hence $P_d(a,b,c)$ has degree $\leqslant d$ in c. ∎

Proof of proposition 2. To find the terms in $c^d h^d$ in $I(s,c)$ it is enough to find the term in c^d in $P_d(a,b,c)$. We need only isolate the term in c^d in each of the summands $(\phi(r-b\pm c))^d$. Now from the calculation above this will be $(\pm 2c(r+a)-c)^d$ and so the term in c^d in $P_d(a,b,c)$ is

$$\frac{1}{d!}\left(\sum_{r=-b+1}^{b} (2c(r+a)-c)^d - \sum_{r=-b}^{b-1} (-2c(r+a)-c)^d\right).$$

This is also the coefficient of H^d in

$$\begin{aligned} J(s,c) &= \sum_{r=-b+1}^{b} Q^{2c(r+a)-c} - \sum_{r=-b}^{b-1} Q^{-2c(r+a)-c} \\ &= Q^{2ca}\left(\sum_{r=-b+1}^{b} Q^{2cr-c}\right) - Q^{-2ca}\left(\sum_{r=-b}^{b-1} Q^{-2cr-c}\right) \\ &= (Q^{2ca} - Q^{-2ca})\left(\sum_{r=-b+1}^{b} Q^{2cr-c}\right) \\ &= (Q^{2ca} - Q^{-2ca})\frac{Q^{2bc} - Q^{-2bc}}{Q^c - Q^{-c}}. \end{aligned}$$

Thus $J(s,c)$ gives the terms in $c^d H^d$, and hence those in $c^d h^d$, in the function $I(s,c)$. These may be rewritten in terms of s, m and p, by putting $Q = s^{2mp}$, and recalling that $2pa = 1$ and $2mb = 1$, to give

$$J(s,c) = \frac{(s^{2mc} - s^{-2mc})(s^{2pc} - s^{-2pc})}{s^{2mpc} - s^{-2mpc}}.$$

This completes the proof of Proposition 2. ∎

To finish the proof of the Alexander polynomial formula for torus knots, as stated in terms of k, observe that $I_{\max}(s,c)$, which was defined to be the terms in $k^d h^d$ in $I(s,c)$, can be found from $J(s,c)$, the terms in $c^d h^d$ in $I(s,c)$, by putting $k = 2c$. This follows, since by Proposition 1 the highest degree coefficient of h^d in $I(s,c)$ is k^d, and the substitution $k = 2c+1$ in $I(s,c)$ will have the same effect on the terms in $c^d h^d$ as would the substitution $k = 2c$. Thus

$$I_{\max}(s,c) = \frac{(s^{mk}-s^{-mk})(s^{pk}-s^{-pk})}{s^{mpk}-s^{-mpk}},$$

and the check on the formula for torus knots is complete.

Remark. It is interesting that the proof of proposition 1 was most easily carried out by assuming that b was an integer, whereas in the actual application $b = (2m)^{-1}$ and m is an integer.

I am grateful to Paul Melvin, who encouraged some of my early explicit computations, and to Paul Strickland, whose further computational checks in low degree gave me enough confirmation to persist in trying to construct an analytical solution. Finally I must thank my colleague, Kit Nair, for the helpful discussions which led to the solution given here.

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