The Coloured Jones Function

P. M. Melvin[1] **and H. R. Morton**[2]

[1] Department of Mathematics, Bryn Mawr College, Bryn Mawr, PA 19010, USA

[2] Department of Pure Mathematics, University of Liverpool, Liverpool L69 3BX, UK

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Abstract. The invariants $J_{K,k}$ of a framed knot K coloured by the irreducible $SU(2)_q$-module of dimension k are studied as a function of k by means of the universal R-matrix. It is shown that when $J_{K,k}$ is written as a power series in h with $q = e^h$, the coefficient of h^d is an odd polynomial in k of degree at most $2d + 1$. This coefficient is a Vassiliev invariant of K. In the second part of the paper it is shown that as k varies, these invariants span a d-dimensional subspace of the space of all Vassiliev invariants of degree d for framed knots. The analogous questions for unframed knots are also studied.

Introduction

A framed knot K in the 3-sphere determines an $SU(2)_q$ invariant $J_{K,k}$ for each positive integer k by using the irreducible $SU(2)_q$-module of dimension k to 'colour' the knot. These invariants, sometimes called the *coloured Jones invariants* of K, are Laurent polynomials in $q^{1/4}$ with integer coefficients. Setting $q = e^h$, each coloured Jones invariant can be expanded as a rational power series

$$J_{K,k}(h) = \sum_{d=0}^{\infty} J_d(k) h^d$$

in the variable h. Together they form a single function of h and the colour k, the *coloured Jones function* of K. We shall study the dependence of this function on k.

Our main result, Theorem 1.6, is that the coefficient $J_d(k)$ of h^d in the expansion of $J_{K,k}$ is an odd polynomial in k of degree at most $2d + 1$. Furthermore, if K has the zero framing then the term in k^{2d+1} vanishes, and so in this case $J_d(k)$ is of degree at most $2d-1$. An extension to the case of framed links is given in Theorem 1.7. These results have proved fruitful in our study with Kirby [7] of algebraic properties of the $SU(2)$-quantum invariants of 3-manifolds introduced by Witten [14] and Reshetikhin-Turaev [12].

In the spirit of Vassiliev's finite type invariants, we note that for each k the coefficient $J_d(k)$ of h^d in $J_{K,k}$ is an invariant of K of *degree* d, that is of type d but not of type $d-1$. By varying k we can find, by the result above, at most $d+1$ independent invariants. In fact there is always a relation among the coefficients of the polynomial J_d, since $J_d(1) = 0$ for $d > 0$, and so $J_d(k)$ can provide at most d independent invariants. By considering the values of these invariants on certain 'chord diagrams' in the sense of Bar-Natan [2] (corresponding to linear combinations of knots), we show that $J_d(k)$ does in fact determine d independent framed Vassiliev invariants of degree d (Corollary 2.4).

If K has the zero framing, then $J_d(k)$ is of degree at most $2d-1$ in k, and so provides at most $d-1$ independent unframed invariants of degree d as k varies. Evidence points toward a much lower bound of $d+1$ for the degree of J_d, and a consequent reduction to $\lfloor d/2 \rfloor$ in the number of independent invariants. We show by another explicit calculation on chord diagrams that there are in general at least this number of independent unframed invariants of degree d arising as values of the coloured Jones function (Corollary 2.9).

We conclude with a conjecture about determining the Alexander polynomial from the coloured Jones function.

§1. Calculations from the universal R-matrix

The coloured Jones invariants $J_{K,k}$ of a framed knot K can be calculated from a closed braid representation of K using Drinfeld's universal R-matrix for $SU(2)_q$ [4], as described in Reshetikhin and Turaev [11]. We shall use this approach to produce a state sum for $J_{K,k}$ which will be seen to reduce to a finite sum when calculating $J_{K,k}$ up to terms in h^d, where $q = \exp(h)$.

Recall that the R-matrix is an invertible element of the topological tensor product $\mathcal{G} \otimes \mathcal{G}$, where $\mathcal{G}$ is the deformed universal enveloping algebra $U_h(\mathrm{su}(2) \otimes \mathbf{C})$. It can be written, following Kirby and Melvin [6], as

$$R = \sum_{n=0}^{\infty} s_n(h)\, X^n \otimes Y^n \, \exp(\tfrac{1}{4}h(H+nI) \otimes (H-nI))$$

where

$$s_n(h) = \frac{(s-s^{-1})^n}{s^{n/2}\,[n]!}\ , \qquad s = \exp(\tfrac{1}{2}h)\ , \qquad [n] = \frac{\sinh(\frac{1}{2}hn)}{\sinh(\frac{1}{2}h)} = \frac{s^n - s^{-n}}{s - s^{-1}}$$

and X, Y and H are generators of $\mathcal{G}$ satisfying the relations

$$[H,X] = 2X\ , \qquad [H,Y] = -2Y\ , \qquad [X,Y] = [H] = \frac{\sinh(\frac{1}{2}hH)}{\sinh(\frac{1}{2}h)}.$$

We shall also make use of the element $\mu = \exp(\frac{1}{2}hH)$ in $\mathcal{G}$ which is sometimes called the *enhancement* of R.

As is shown below, the elements $R^{\pm 1}$ and μ can all be expressed as sums of 'bounded degree' in the following sense. Any element in $\mathcal{G}^{\otimes n}$ can be written as a power series in h with coefficients in $\mathcal{F}^{\otimes n}$, where $\mathcal{F}$ is the algebra of complex polynomials in the noncommuting variables X, Y and H. Rearranging the terms in this series produces a sum of the form

$$\sum_{j \in J} c_j(h)\, \alpha_{j1} \otimes \cdots \otimes \alpha_{jn}$$

in which the α_{jk} are monomials in X, Y and H, and the coefficients $c_j(h)$ are complex power series in h. The index set J may be infinite, but we allow only finitely many coefficients c_j of any given order, where the *order* $\mathrm{ord}\,(c)$ of a power series $c(h) = \sum_{i=0}^{\infty} c_i h^i$ is the smallest i for which $c_i \neq 0$. Such a sum is said to have *bounded degree* if

$$\deg\,(\alpha_{jk}) \leq \mathrm{ord}\,(c_j)$$

for each j in J and all $k = 1, \ldots, n$. Equivalently, the coefficient of h^d is a linear combination of tensor products of monomials of degree at most d. Observe that sums, products and exponentials of bounded degree sums are again of bounded degree.

It is clear that $\mu = \exp(hH/2)$ has bounded degree. Indeed

$$\mu = \sum_{m=0}^{\infty} c_m(h)\mu_m$$

with $c_m(h) = h^m/(2^m m!)$ and $\mu_m = H^m$. The same is true of $R^{\pm 1}$.

Proposition 1.1. *The universal R-matrix for $SU(2)_q$ and its inverse can be written as sums of bounded degree,*

$$R^{\pm 1} = \sum_{j \in J} c_j^{\pm}(h)\, \alpha_j^{\pm} \otimes \beta_j^{\pm}$$

where $\alpha_j^{\pm}$ and $\beta_j^{\pm}$ are monomials of degree not exceeding the order of c_j.

Proof. Write $R = \sum_{n=0}^{\infty} S_n \exp(T_n)$, where $S_n = s_n(h)\, X^n \otimes Y^n$ (for s_n as above) and $T_n = \frac{1}{4}h\,(H+nI)\otimes(H-nI)$. Evidently T_n is of bounded degree, as is S_n since $\mathrm{ord}\,(s_n) = n$. Thus the product $S_n \exp(T_n)$ is a sum of bounded degree in which all the monomials which appear are of degree at least n, and it follows that R is of bounded degree. Using the formula $R^{-1} = (S \otimes I)R$ where S is the antiautomorphism of $\mathcal{G}$ defined by $S(H) = -H$, $S(X) = -sX$, $S(Y) = -s^{-1}Y$ (see §3.1.6 in [11]), it is not hard to show that $R^{-1} = R^{-1}(h) = R(-h)$, and so R^{-1} is of bounded degree as well. □

Remark. The index set J can be chosen explicitly to be the set of triples (n, a, b) of non-negative integers, with $\alpha_{nab}^{\pm} \otimes \beta_{nab}^{\pm} = X^n H^a \otimes Y^n H^b$, and

$$c_{nab}^{\pm}(h) = s_n(\pm h) \sum_{d=\max\,(a,b)}^{\infty} (-1)^{d+b} \binom{d}{a} \binom{d}{b} \frac{n^{2d-a-b}}{4^d d!} (\pm h)^d.$$

This formula is not essential for what follows, although it can be useful to note that X and Y occur in $\alpha_{nab}^{\pm}$ and $\beta_{nab}^{\pm}$ with the same degree.

Now suppose that a framed knot K has been presented as the closure of a braid B on n strings. The universal R-matrix and its inverse can be used to represent B by an automorphism B_k of the tensor product $V_k^{\otimes n}$ for each irreducible $\mathcal{G}$-module V_k. In particular, if B is written as a word in the braid generators σ_i^{ε}, for $1 \leq i < n$ and $\varepsilon = \pm 1$, then B_k is the corresponding composition of automorphisms $\check{R}_i^{\varepsilon} = (P_i R_i)^{\varepsilon}$, where P_i is the interchange of the i^{th} and $(i+1)^{\text{st}}$ factors of $V_k^{\otimes n}$, and R_i is the action of R in the same factors:

$$B = \sigma_{i_1}^{\varepsilon_1} \cdots \sigma_{i_c}^{\varepsilon_c} \qquad\qquad B_k = \check{R}_{i_1}^{\varepsilon_1} \cdots \check{R}_{i_c}^{\varepsilon_c}.$$

The invariant $J_{K,k}$ can then be calculated as a weighted trace of B_k, namely

$$J_{K,k} = \mathrm{Tr}\,(\mu^{\otimes n} B_k),$$

where μ is the element $\exp(\frac{1}{2}hH)$ regarded as an endomorphism of the module V_k by the action of H on V_k. In other words, we must compose B_k with $\mu \otimes \cdots \otimes \mu$ and then take the ordinary trace on $V_k^{\otimes n}$, as described in [11]. (For more general quantum group invariants, an orientation is required on K; the preceeding construction is for the downward orientation through the braid, and a dual construction is needed for the opposite orientation. This distinction disappears for $SU(2)_q$ since the modules V_k are self dual.)

We next produce a states sum for $J_{K,k}$. By the previous proposition, each automorphism $\breve{R}_i^\varepsilon$ can be written as an infinite linear combination $\sum_{j\in J} c_j^\varepsilon \breve{R}_{ij}^\varepsilon$ of endomorphisms $\breve{R}_{ij}^\varepsilon$. Explicitly $\breve{R}_{ij}^+$ maps $x \otimes y$ (in the i^{th} and $(i+1)^{\text{st}}$ factors of $V_k^{\otimes n}$) to $\beta_j^+(y) \otimes \alpha_j^+(x)$ while $\breve{R}_{ij}^-$ maps it to $\alpha_j^-(y) \otimes \beta_j^-(x)$. The effect of this endomorphism at a crossing is illustrated (suppressing the subscript j) in Figure 1 by showing the crossing with $\alpha^\pm$ on the overcrossing string and $\beta^\pm$ on the undercrossing string.

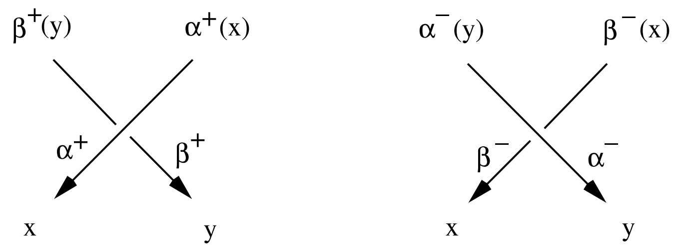


Figure 1

This yields the state sum $B_k = \sum_{j_1,\ldots,j_c\in J}(c_{j_1}^{\varepsilon_1} R_{i_1j_1}^{\varepsilon_1} \cdots c_{j_c}^{\varepsilon_c} R_{i_cj_c}^{\varepsilon_c})$, where each choice of indices $j_1, \ldots, j_c$ in J, corresponding to a choice of one term in $R^{\pm 1}$ for each of the crossings of B, is to be thought of as a state. To obtain $J_{K,k}$ we must compose B_k with $\mu^{\otimes n}$, and so we extend the state to include a choice of non-negative integers $m_1, \ldots, m_n$ specifying one term in $\mu = \sum c_m \mu_m$ at the top of each braid string. Thus a *state* S of the braid B consists of a choice of indices $j_1, \ldots, j_c$ in J for the crossings and of non-negative integers $m_1, \ldots, m_n$ for the tops of the strings. Setting $c_S = c_{j_1}^{\varepsilon_1} \cdots c_{j_c}^{\varepsilon_c} c_{m_1} \cdots c_{m_n}$, we have

$$J_{K,k} = \sum_S c_S \,\mathrm{Tr}\,(B_S)$$

where B_S is the endomorphism $(H^{m_1} \otimes \cdots \otimes H^{m_n})\,(R_{i_1j_1}^{\varepsilon_1} \cdots R_{i_cj_c}^{\varepsilon_c})$ of $V_k^{\otimes n}$ determined by S. Observe that the coefficient c_S does not depend on the colour k, whereas the endomorphism B_S does.

We now show how to replace B_S by a monomial M_S in $\mathcal{G}$, regarded as an endomorphism of V_k, with $\mathrm{Tr}\,(B_S) = \mathrm{Tr}\,(M_S)$. This method has been discussed by Lawrence [8] and developed in a diagrammatic form by Kauffman [5].

To define M_S, recall that the endomorphism B_S is built up from endomorphisms $\alpha_j^\pm \otimes \beta_j^\pm$ of $V_k \otimes V_k$ at each crossing of B, as shown in Figure 1, together with endomorphisms

μ_m of V_k at the top of each string. Following Kauffman, view the monomials $\alpha_j^{\pm}$, $\beta_j^{\pm}$ and μ_m as 'beads' which are free to move along the strings past the individual crossings, and may be multiplied when they occur next to one another. Thus the endomorphism B_S will take a vector of the form $x_1 \otimes \cdots \otimes x_n$ to a tensor product $y_1 \otimes \cdots \otimes y_n$ of vectors, each of which is the result of operating on one of the vectors x_i by the beads which it has passed on its way from the bottom of the braid to the top. In particular, if B induces the permutation π in the sense that the string at position i at the top of the braid is joined to position $\pi(i)$ at the bottom, then $y_i = \psi_i x_{\pi(i)}$ where ψ_i is the product (from top to bottom) of the beads on the i^{th} string. Now define

$$M_S = \psi_1 \cdot \psi_{\pi(1)} \cdots \psi_{\pi^{n-1}(1)}$$

which is just the product of all the beads on the single string K obtained by closing the braid B. For example, in the state of the diagram for the figure-eight knot in Figure 2, the monomial M_S is $\mu_1 \beta_2^+ \alpha_3^- \mu_3 \beta_1^- \alpha_2^+ \beta_4^+ \mu_2 \alpha_1^- \beta_3^- \alpha_4^+$.

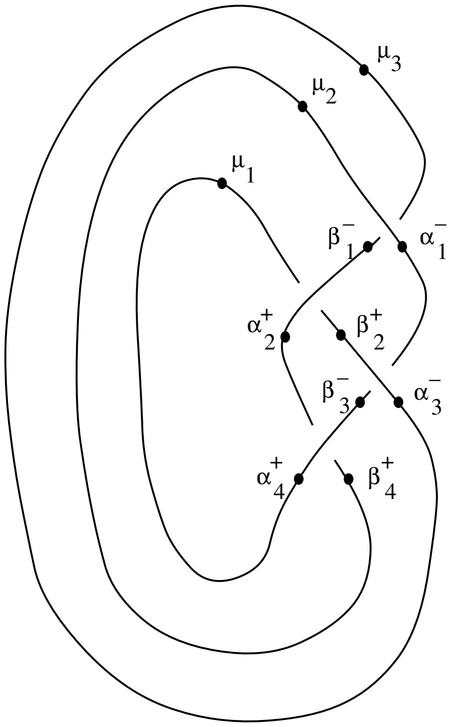


Figure 2

Proposition 1.2. *For each state S, the trace of the endomorphism B_S of $V_k^{\otimes n}$ is equal to the trace of the monomial M_S on V_k, and so*

$$J_{K,k}(h) = \sum_S c_S(h)\,\mathrm{Tr}\,(M_S).$$

summed over all states.

Proof. Choose a basis $e_1, \dots, e_k$ for V_k and denote the associated matrix for any endomorphism ψ of V_k by (ψ^i_j), so that $\psi e_j = \sum \psi^i_j e_i$. Then B_S maps $e_{j_1} \otimes \cdots \otimes e_{j_n}$ to

$$\psi_1 e_{j_{\pi(1)}} \otimes \cdots \otimes \psi_n e_{j_{\pi(n)}} = \sum_{i_1,\dots,i_n=1}^{k} \psi^{i_1}_{1,j_{\pi(1)}} e_{i_1} \otimes \cdots \otimes \psi^{i_n}_{n,j_{\pi(n)}} e_{i_n}.$$

and so

$$\begin{aligned}\mathrm{Tr}\,(B_S) &= \sum_{j_1,\dots,j_n=1}^{k} \psi^{j_1}_{1,j_{\pi(1)}} \cdots \psi^{j_n}_{n,j_{\pi(n)}} \\ &= \sum_{j=1}^{k} (\psi_1 \cdot \psi_{\pi(1)} \cdots \psi_{\pi^{n-1}(1)})^j_j \;=\; \mathrm{Tr}\,(M_S). \qquad \square\end{aligned}$$

Each state S thus makes a contribution $c_S \,\mathrm{Tr}\,(M_S)$ to the invariant $J_{K,k}$. The coefficient c_S does not depend on the colour k, nor does the monomial M_S. Dependence on k arises only on taking the trace of M_S in V_k. The polynomial nature of the dependence on k will already appear in the contribution of each individual state, and will be determined by a calculation of $\mathrm{Tr}\,(M_S)$ in terms of k. Before making this calculation, we note a restriction on the degree of M_S which arises from the bounded degree of the terms in the universal R-matrix. This will eventually give the desired control on the degree of k relative to that of h in $J_{K,k}(h)$.

Proposition 1.3. $\deg\,(M_S) \le 2\,\mathrm{ord}\,(c_S)$ *for each state* S.

Proof. The coefficient c_S is the product $\prod_{i=1}^{c} c^{\varepsilon_i}_{j_i} \prod_{i=1}^{n} c_{m_i}$ and so

$$\mathrm{ord}\,(c_S) = \sum_{i=1}^{c} \mathrm{ord}\,(c^{\varepsilon_i}_{j_i}) + \sum_{i=1}^{n} \mathrm{ord}\,(c_{m_i})$$

since order is additive on products. Now each term $c^{\varepsilon}_j\, \alpha^{\varepsilon}_j \otimes \beta^{\varepsilon}_j$ and $c_m \mu_m$ in the state S is chosen from a sum of bounded degree, so $\deg\,(\alpha^{\varepsilon}_j) \le \mathrm{ord}\,(c^{\varepsilon}_j)$, $\deg\,(\beta^{\varepsilon}_j) \le \mathrm{ord}\,(c^{\varepsilon}_j)$ and $\deg\,(\mu_m) \le \mathrm{ord}\,(c_m)$. The monomial M_S is the product in some order of the monomials α^{ε}_j, β^{ε}_j and μ_m chosen by the state S and so

$$\begin{aligned}\deg M_S &= \sum_{i=1}^{c} \deg \alpha^{\varepsilon_i}_{j_i} + \sum_{i=1}^{c} \deg \beta^{\varepsilon_i}_{j_i} + \sum_{i=1}^{n} \deg \mu_{m_i} \\ &\le 2 \sum_{i=1}^{c} \mathrm{ord}\,(c^{\varepsilon_i}_{j_i}) + \sum_{i=1}^{n} \mathrm{ord}\,(c_{m_i}) \;\le\; 2\,\mathrm{ord}\,(c_S). \qquad \square\end{aligned}$$

We now analyse the dependence on k of the trace of an arbitrary monomial M in X, Y and H, when operating on the $\mathcal{G}$-module V_k.

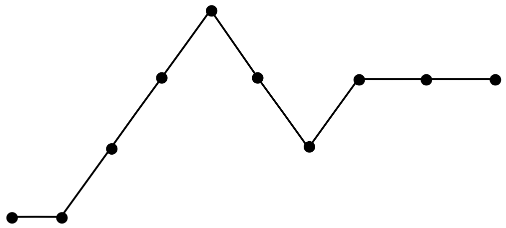
Proposition 1.4. *Let M be a monomial in X, Y and H, and consider the trace of M on the irreducible $\mathcal{G}$-module V_k of dimension k, expanded as a power series* $\mathrm{Tr}\,(M) = \sum_{i=0}^{\infty} M_l(k)\,h^l$ *in h. Then the coefficient $M_l(k)$ of h^l is an odd polynomial in k of degree at most* $l + \deg(M) + 1$.

Proof. Following [6], but with slightly modified notation, set $m = k/2$ and choose a basis for V_k consisting of weight vectors $e_{-m+1}, e_{-m+2} \dots, e_m$ with the property that

$$\begin{aligned} Xe_j &= [m+j]\,e_{j+1} \\ Ye_j &= [m-j+1]\,e_{j-1} \\ He_j &= (2j-1)\,e_j, \end{aligned}$$

where $[n] = \sinh(\frac{1}{2}hn)/\sinh(\frac{1}{2}h)$. These relations extend to all $j \equiv m \pmod 1$ by setting $e_j = 0$ for $j \leq -m$ and for $j > m$. The definitions given in this way of Xe_{-m} and of Ye_{m+1} are consistent, since $[0] = 0$.

The monomial M can be represented diagramatically by its *profile*, consisting of a sequence of rising, falling and level edges corresponding to the sequence of appearances of X, Y and H in M, read from right to left. For example, when $M = H^2XY^2X^3H$ the profile is



Now set $d(M) = \deg_X(M) - \deg_Y(M)$, which is just the final level of the profile with initial level zero. We claim that $\mathrm{Tr}\,(M) = 0$ if $d(M) \neq 0$, and otherwise that the vectors e_j are eigenvectors for M on V_k with eigenvalues λ_j (depending on k), so that

$$\mathrm{Tr}\,(M) = \sum_{j=-m+1}^{m} \lambda_j.$$

Indeed, it is clear from the effect of X, Y and H on e_j that Me_j is a multiple of $e_{j+d(M)}$, and the claim follows immediately. Thus we need only consider those M for which $d(M) = 0$. (Note that all the monomials M_S defined from the states of a knot diagram have this property, because of the balance between the degrees of X and Y in each pair $\alpha \otimes \beta$.)

Let us then assume that $d(M) = 0$, and compute the eigenvalues λ_j defined by $Me_j = \lambda_j e_j$. Suppose that there are p rising edges in the profile, starting at levels $a_1, \dots, a_p$ and q horizontal edges at levels $b_1, \dots, b_q$. There must also be p falling edges finishing at levels $a_1, \dots, a_p$, since the net change of level is $d(M) = 0$. Now each horizontal edge at level b contributes $2(j+b) - 1$ to λ_j, as H then appears at level b, to feature as He_{j+b}. A rising edge from level a to level $a+1$ contributes $[m + (j+a)]$, from the appearance of

Xe_{j+a}, while a falling edge from level $a+1$ to level a contributes $[m-(j+a)]$, from the appearance of Ye_{j+a+1}. Thus

$$\lambda_j = \prod_{i=1}^{p} [m+(j+a_i)][m-(j+a_i)] \prod_{i=1}^{q} (2(j+b_i)-1).$$

Now it is an easy calculus exercise to show that the coefficient of h^l in the power series expansion of $[n]$ is a polynomial in n of degree at most $l+1$, and it follows that the corresponding coefficient λ_{jl} in the expansion

$$\lambda_j = \sum_{i=0}^{\infty} \lambda_{jl} h^l$$

is a polynomial in j and k of degree $l+2p+q = l+\deg(M)$. In fact, λ_{jl} is *even* in k. This is immediate from the fact that $[m+(j+a_i)][m-(j+a_i)]$ is an even function of $k=2m$, which follows from the identity $2\sinh(m+n)\sinh(m-n) = \cosh(2m)-\cosh(2n)$. Since

$$M_l(k) = \sum_{j=-m+1}^{m} \lambda_{jl},$$

the proposition is a consequence of the following lemma.

Lemma 1.5. *Let p be a polynomial of degree d. Then the function f defined on integers k by*

$$f(k) = \sum_{j=-m+1}^{m} p(j),$$

where $m=k/2$, is an odd polynomial of degree $d+1$.

Proof. First observe that there exists a polynomial P of degree $d+1$ such that $p(x) = P(x)-P(x-1)$, a 'discrete integral' of p. For example, for $p_d = x^{d+1}-(x-1)^{d+1}$ the polynomial $P_d = x^{d+1}$ will do. But p can be written as a linear combination $p = \sum_{n=0}^{d} a_n p_n$, since $p_0,\ldots,p_d$ clearly span the space of polynomials of degree $\leq d$, and so $P = \sum_{n=0}^{d} a_n P_n$ is the desired integral.

Now

$$\sum_{j=-m+1}^{m} p(j) \;=\; \sum_{j=-m+1}^{m} (P(j)-P(j-1)) \;=\; P(m)-P(-m),$$

which proves the lemma, and thus the proposition. $\square$

We now give the proof of our main theorem on the dependence of the coloured $SU(2)_q$ invariants $J_{K,k}$ of a framed knot K on the colour k, the dimension of the module V_k.

Theorem 1.6. *Write the coloured Jones invariant $J_{K,k}$ of a framed knot K as a power series*

$$J_{K,k}(h) = \sum_{d=0}^{\infty} J_d(k)\, h^d$$

in h, where $q = \exp(h)$. Then $J_d(k)$ is an odd polynomial in k of degree at most $2d+1$. Furthermore, the coefficient a_d of k^{2d+1} in $J_d(k)$ depends only on the framing a on K , namely $a_d = a^d/(4^d d!)$.

It suffices to construct, for each d, a series $J^d_{K,k}(h)$ which agrees with $J_{K,k}(h)$ up to degree d in h, and which is an odd polynomial in k of degree at most $2d+1$ with coefficient of $k^{2d+1}h^d$ equal to $a^d/(4^d d!)$. To accomplish this, consider the state sum $J_{K,k}(h) = \sum_S c_S(h) \mathrm{Tr}\,(M_S)$ given in Proposition 1.2. Expand each trace as a power series $\mathrm{Tr}\,(M_S) = \sum_{l=0}^{\infty} M_{Sl}(k)h^l$ in h, and write $\mathrm{Tr}\,(M_S)|_d$ for the partial sum $\sum_{l=0}^{d} M_{Sl}(k)h^l$. Now set

$$J^d_{K,k}(h) = \sum_{S, o_S \le d} c_S(h)\, \mathrm{Tr}\,(M_S)|_{(d-o_S)}$$

where for convenience we write o_S for $\mathrm{ord}\,(c_S)$. Observe that this sum is finite since $d_S = \deg\,(M_S) \le 2o_S \le 2d$, by Proposition 1.3, and there are only finitely many monomials of any given degree.

It is clear that $J_{K,k}(h)$ and $J^d_{K,k}(h)$ agree up to degree d in h. The last proposition shows that each $M_{Sl}(k)$ which appears in $J^d_{K,k}(h)$ is an odd polynomial in k with

$$\deg\,(M_{Sl}) \le l + d_S + 1 \le 2d+1,$$

where the last inequality follows from the inequalities $l \le d - o_S$, $d_S \le 2o_S$ (by Proposition 1.3) and $o_S \le d$. Thus $J^d_{K,k}(h)$ is an odd polynomial in k of degree at most $2d+1$.

It remains to compute the coefficient a_d of $k^{2d+1}h^d$ in $J^d_{K,k}(h)$. To get any contribution of degree $2d+1$ in k from a state S, all the inequalities above must become equalities, giving $o_S = d$, $d_S = 2d$ and $l = 0$. Thus

$$a_d = \sum_{S, d_S = 2o_S = 2d} a_S$$

where a_S denotes the coefficient of $k^{2d+1}h^0$ in the expansion of $\mathrm{Tr}\,(M_S)$. We will show that this sum depends only on the framing a of K

First observe that since $d_S = 2d$ for the states S in the sum, the coefficient a_S is independent of the order of the variables in the monomial M_S. Indeed, for any monomial M, write a_M for the coefficient of $k^{\deg\,(M)+1}h^0$ in the expansion of $\mathrm{Tr}\,(M)$. Now if M' is a reordering of M, then it follows from the commutation relations $[H,X] = 2X$, $[H,Y] = -2Y$ and $[X,Y] = [H] = H + \mathrm{O}(h)$ that $M = M' + N + \mathrm{O}(h)$, where N is a sum of monomials of degree less that $\deg\,(M)$. Since N does not contribute to a_M, by Proposition 1.4, we have $a_M = a_{M'}$.

Since $d_S = 2o_S$ for the states S under consideration, it follows from the proof of Proposition 1.3 that S must select the term $\mu_0 = 1$ from the sum μ for each of the strings of the braid representing K. Now let K' be any knot presented as a braid with the same number of positive and negative crossings as K. Fix a bijection φ from the crossings of K to the crossings of K', respecting sign. This induces a bijection φ between the states of K and of K' which select μ_0 from each appearance of μ, namely if S assigns $\alpha_j^\varepsilon \otimes \beta_j^\varepsilon$ to a crossing x of K, then $\varphi(S)$ makes the same assignment to $\varphi(x)$. The coefficients c_S and $c_{\varphi(S)}$ are then equal, but the monomials M_S and $M_{\varphi(S)}$ are in general different, because of the different order in which the crossings appear on the two knots. These two monomials are however the product of the same elements α_j^ε and β_j^ε and differ only in the order of these elements. It follows from the observation above that $a_S = a_{\varphi(S)}$, and so a_d is the same for K and for K'. Now it is clearly always possible to choose K' as the unknot with the same framing as K, given by the sum of the signs of the crossings, and so a_d depends only on the framing.

Thus it suffices to compute the coefficient a_d of k^{2d+1} for the a-framed unknot $\bigcirc$. It is well known that

$$J_{\bigcirc,k} = s^{a(k^2-1)/2}[k]$$

(cf. §3.27 in [6]). An easy exercise shows that the power series $s^{a(k^2-1)/2} = \sum s_i(k)h^i$ and $[k] = \sum b_i(k)h^i$ satisfy $\deg(s_i) = 2i$ and $\deg(b_i) \le i+1$. It follows that $a_d k^{2d+1}$ is the leading term in the product $s_d(k)b_0(k)$. Since $s_d(k) = a^d(k^2-1)^d/(4^d d!)$ and $b_0(k) = k$, we have $a_d = a^d/(4^d d!)$. □

Remark. The function $J_{K,k}(h)/[k]$, which is multiplicative under connected sums of knots, may be considered in place of $J_{K,k}(h)$. It follows from the analysis above that $J_{K,k}(h)/[k]$ can be written as a power series in h and k where the coefficient of h^d is an even polynomial in k of degree at most $2d$. The coefficients of $k^{2d}h^d$ are exactly the coefficients a_d calculated above, and hence vanish for the zero framing when $d > 0$.

Links

The methods used above can be readily extended to cover the case of framed links. Let L be a framed link with $|L|$ components $\{L_i\}$. Write $J_{L,\mathbf{k}}$ for the $SU(2)_q$ invariants of L in which the component L_i is coloured by the irreducible $\mathcal{G}$-module of dimension k_i, where $\mathbf{k} = (k_1, \ldots, k_{|L|})$.

Theorem 1.7. *Write*

$$J_{L,\mathbf{k}}(h) = \sum_{d=0}^{\infty} J_d(\mathbf{k})\, h^d$$

as a power series in h, where $q = \exp(h)$. Then $J_d(\mathbf{k}) = k_1 \ldots k_{|L|} P_d(\mathbf{k})$, where $P_d(\mathbf{k})$ is a polynomial in $\mathbf{k}$ of total degree at most $2d$ which is an even function of each variable k_i.

Proof. As in the case of a knot, there is a state sum

$$J_{L,\mathbf{k}}(h) = \sum_S c_S(h)\, \mathrm{Tr}_S$$

associated with a closed braid presentation for L. A state S is a choice for each crossing of a term $c_j^\pm(h)\alpha_j^\pm \otimes \beta_j^\pm$ from a bounded sum for the R-matrix or its inverse, and a choice for the top of each braid string of a term $c_m(h)\mu_m$ in a sum for the enhancement μ. The product of the coefficients defines $c_S(h)$, a power series in h of order o_S. The choice of monomials $\alpha_j^\pm \otimes \beta_j^\pm$ and $c_m(h)\mu_m$ defines an endomorphism B_S of a tensor product of modules V_{k_i} once a colouring $\mathbf{k}$ of L has been chosen. This is determined by colouring each braid string with the module chosen for its component of L. Now $J_{L,\mathbf{k}} = \sum_S c_S(h)\mathrm{Tr}\,(B_S)$. As before, we may picture the selection of terms made by a state S as a collection of individual monomials $\alpha_j^\pm$, $\beta_j^\pm$ and μ_m attached to the strings. Each component L_i of L determines a monomial M_S^i, of degree d_S^i, given by composing the monomials attached to that string in the order in which they occur. It follows that $\mathrm{Tr}\,(B_S) = \prod \mathrm{Tr}\,(M_S^i)$, by the same argument we used in the case of a knot, which gives the state sum above with

$$\mathrm{Tr}_S = \prod_{i=1}^{|L|} \mathrm{Tr}\,(M_S^i).$$

Proceeding as in the proof of Theorem 1.6, consider the trace of each monomial M_S^i on V_{k_i} as a power series $\mathrm{Tr}\,(M_S^i) = \sum_{l=0}^\infty M_{Sl}^i(k_i)h^l$. The product of these series gives an expansion $\sum_{l=0}^\infty M_{Sl}(\mathbf{k})h^l$ for Tr_S whose finite truncation up to degree d in h will be denoted by $\mathrm{Tr}_S|_d$. By Proposition 1.4, the coefficients $M_{Sl}^i(k_i)$ are odd polynomials in k_i of degree at most $l + d_S^i + 1$. Thus $M_{Sl}(\mathbf{k})$ is a sum of products of odd polynomials in the k_i of total degree at most $l + d_S + n$, where $d_S = \sum d_S^i$ is the total degree of S. It follows that $\mathrm{Tr}_S|_d \,/\, (k_1 \dots k_n)$ is a polynomial in $\mathbf{k}$ of total degree at most $d + d_S$ which is an even function of each k_i.

Now consider the state sum

$$J_{L,\mathbf{k}}^d(h) = \sum_{S, o_S \le d} c_S(h)\, \mathrm{Tr}_S|_{(d-o_S)},$$

which evidently agrees with $J_{L,\mathbf{k}}(h)$ to degree d in h. The use of bounded degree sums for $R^\pm$ and μ ensures that $d_S = \sum d_S^i \le 2o_S$, and so this sum is finite. The remarks in the last paragraph show that each $\mathrm{Tr}_S|_{(d-o_S)}$ is the product of $k_1 \dots k_n$ with a polynomial in $\mathbf{k}$, even in each k_i, of total degree at most $d - o_S + d_S \le d + o_S \le 2d$. Hence so is $J_{L,\mathbf{k}}^d$, which completes the proof. □

Remark. It is interesting to note that a further restriction arises on states S in this case in that the profile of each individual monomial M_S^i must return to level zero in order for the state to contribute anything to the sum. While the form of the R-matrix guarantees this for every state in the case of a knot, it is not always the case for states of a link, and many states may thus be immediately excluded from the sum.

Even for a knot K (or link) with few crossings and a small value of d, this state sum is not a practical method for calculating $J_{K,k}$. It does however give theoretical bounds on the information carried by $J_{K,k}$ for any knot, when we retain only terms up to h^d, as it is evident that knowledge of the coefficient $J_d(k)$ for $d+1$ values of k will determine $J_d(k)$ completely. We take up this theme in the next section, where we discuss the Vassiliev invariants of degree d which can arise from the coloured Jones invariants. Some calculations in this setting prove to be feasible using the states approach and lead to results about the independence of the invariants $J_d(k)$ as k varies.

§2. The space of Jones invariants of degree d

Let $\mathcal{K}$ denote the real vector space of formal linear combinations of oriented framed knots in the 3-sphere. Any real-valued invariant of oriented framed knots can be viewed as an element of the dual space $\mathcal{K}^*$. Most of the invariants considered here will be evaluations $J(a) = \mathrm{ev}_a \circ J$ of polynomial invariants J,

$$\mathcal{K} \xrightarrow{J} \mathcal{P} \xrightarrow{\mathrm{ev}_a} \mathbf{R},$$

where J is a linear map to the space $\mathcal{P}$ of all real polynomials, a is a real number and $\mathrm{ev}_a(p) = p(a)$. One may then consider the subspace $\mathcal{J}$ of $\mathcal{K}^*$ spanned by all real evaluations of J, and use the following standard result from linear algebra to compute its dimension when J is of finite rank. In particular $\dim \mathcal{J} = \mathrm{rk}\, J$.

Lemma 2.1. *Let $T : V \to \mathcal{P}_n$ be a linear map from a vector space V into the space $\mathcal{P}_n$ of polynomials of degree at most n, and let $\mathcal{T}$ be the subspace of V^* spanned by all real evaluations of T. Then $\dim \mathcal{T} = \mathrm{rk}\, T$.*

Proof. For any set F of $n+1$ real numbers, the evaluations ev_i for $i \in F$ generate the dual space of $\mathcal{P}_n$. Indeed, any polynomial p in $\mathcal{P}_n$ is determined by its values on F, and so $p = \sum_{i \in F} p(i)\, p_i$ where the p_i are the unique polynomials in $\mathcal{P}_n$ with $p_i(j) = \delta_{ij}$ for all i and j in F (explicitly $p_i(x) = \prod_{j \in F \setminus i} (x-j)/(i-j)$). Thus for e in $\mathcal{P}_n^*$ we have $e = \sum_{i \in F} e(p_i)\, \mathrm{ev}_i$. It follows that $\mathcal{T} = \mathrm{im}(T^*)$, and so $\dim \mathcal{T} = \mathrm{rk}\, T^* = \mathrm{rk}\, T$. $\square$

Now for each choice of d, consider the polynomial invariant $J_d : \mathcal{K} \to \mathcal{P}$ whose value on a framed knot K is the coefficient of h^d in the coloured Jones function of K, denoted $J_{d,K}$ (the subscript K was suppressed in the last section since we were not considering the knot as a variable). Our goal in this section is to study the space $\mathcal{J}_d$ spanned by the evaluations $J_d(k)$ for all k, whose elements will be called *framed Jones invariants* of *degree* d. In particular, we will compute the dimension of $\mathcal{J}_d$.

We shall also consider the 'unframed' invariants $J_d^u = J_d \circ \pi : \mathcal{K} \to \mathcal{P}$ and their evaluations $J_d^u(k)$, where $\pi : \mathcal{K} \to \mathcal{K}$ is the projection which changes all framings to the zero framing. In other words the value of $J_d^u(k)$ on a framed knot K is the coefficient $J_{d,K}^u(k)$ of h^d in the coloured Jones invariant $J_{K,k}^u(h)$ of the knot K with the zero framing. These invariants, which are insensitive to framings, span the space $\mathcal{J}_d^u$ of *unframed Jones invariants* of *degree* d.

The results of the last section provide the following upper bounds on the dimensions of $\mathcal{J}_d$ and $\mathcal{J}_d^u$.

Theorem 2.2 $\dim \mathcal{J}_0 = \dim \mathcal{J}_0^u = 1$. *If $d > 0$ then $\mathrm{im}(J_d) \subset \mathcal{P}_{2d+1}^0$ and $\mathrm{im}(J_d^u) \subset \mathcal{P}_{2d-1}^0$, where $\mathcal{P}_n^o$ denotes the space of odd polynomials of degree at most n with 1 as a root, and so $\dim \mathcal{J}_d \leq d$ and $\dim \mathcal{J}_d^u \leq d-1$.*

Proof. The last statement of Theorem 1.6 shows that $J_0(k) = J_0^u(k) = k$ for all knots, and so $\mathcal{J}_0 = \mathcal{J}_0^u$ is the 1-dimensional space of constant knot invariants.

Now assume $d > 0$. In Theorem 1.6 we showed that the values of the Jones invariants of degree d are odd polynomials of degree at most $2d+1$ in the framed case and of degree at most $2d-1$ in the unframed case. Furthermore $J_{K,1} = 1$ for any knot K since the R-matrix acts trivially on the 1-dimensional representation (cf. §4.14 in [6]), and so $J_d(1) = J_d^u(1) = 0$ for $d > 0$. The last statement follows from Lemma 2.1, since $\dim \mathcal{P}_{2d+1}^o = d$ and $\dim \mathcal{P}_{2d-1}^o = d-1$. □

Remarks. (1) To obtain an explicit formula for the dependency of the invariants $J_d(k)$ as k varies, apply the proof of Lemma 2.1 with $F = P \cup -P$ for any set P of $d+1$ positive integers. Since J_d is odd, this gives

$$J_d(k) = \sum_{i \in P} \left(\prod_{j \in P \setminus i} \frac{k(k^2 - j^2)}{i(i^2 - j^2)} \right) J_d(i).$$

Since $J_d(1) = 0$, we may choose P to contain 1 and obtain a sum over the d values in $P \setminus 1$.

(2) Evidence points to the polynomials J_d^u having degree at most $d+1$, which would imply $\dim \mathcal{J}_d^u \le \lfloor d/2 \rfloor$, where $\lfloor\ \rfloor$ is the greatest integer function (see the conjectures at the end of the paper).

We now turn to the question of independence of the coloured Jones invariants, in quest of lower bounds for the dimensions of $\mathcal{J}_d$ and $\mathcal{J}_d^u$.

It is known that the framed and unframed coloured Jones functions of a knot K with framing a differ by a phase, namely $J_{K,k}(h) = e^{axh} J_{K,k}^u(h)$ where $x = (k^2-1)/4$ (see for example §3.27 in [6]). It is instructive to expand these series to see the effect of the framing on the coefficients, and also to squeeze out a little more information about the spaces $\mathcal{J}_d$ for small d. First write $J_d^u(k) = kj_d(x)$, where $j_0 = 1$ and j_d (for $d > 0$) is a polynomial invariant of unframed knots of degree $< d$ with no constant term. One then computes

$$J_d(k) = k \sum_{n=0}^{d} \frac{a^n x^n}{n!} j_{d-n}(x)$$

where as above a is the framing and $x = (k^2-1)/4$.

For example $J_1(k) = kax = a\,k(k^2-1)/4$, and so $\mathcal{J}_1$ is 1-dimensional, generated by the framing. For $d = 2$ we have $j_2(x) = bx$ for some knot invariant b (independent of the framing), and so $J_2(k) = k(a^2x^2/2 + bx) = a^2\,k(k^2-1)^2/32 + b\,k(k^2-1)/4$. It is not hard to show that $6b = 1 - 24\lambda$, where λ is the Casson invariant of $+1$ surgery on the knot (or equivalently $\lambda = \frac{1}{2}\Delta''(1)$ where Δ is the Alexander polynomial of K). Hence $\mathcal{J}_2$ is 2-dimensional, generated by the Casson invariant and the square of the framing. For higher values of d, one must work a little harder to establish independence of the invariants which arise.

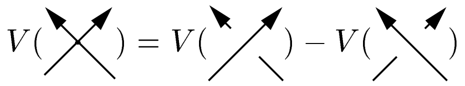
In general, we will show that $\dim \mathcal{J}_d \ge d$ by calculating the invariants J_d for d suitably chosen linear combinations of framed knots. A similar calculation will show $\dim \mathcal{J}_d^u \ge$

$\lfloor d/2 \rfloor$. The use of linear combinations of knots rather that single knots is a matter of convenience, encouraged by the behaviour of invariants such as J_d on certain alternating sums of knots derived from Vassiliev's theory of 'finite type' invariants. For the reader's convenience, we now give a brief review of this theory following the excellent account of Bar-Natan [2], which should be consulted for further details.

Vassiliev invariants

Consider the space $\mathcal{K}_\infty$ of linear combinations of *immersed curves*, that is framed immersions of the oriented circle in the 3-sphere with a finite number of transverse self-intersections or *nodes*. Write $\mathcal{K}_d$ for the subspace generated by immersed curves with exactly d nodes. Thus $\mathcal{K} = \mathcal{K}_0$, and $\mathcal{K}_\infty$ is the direct sum $\oplus \mathcal{K}_d$.

Now any framed knot invariant V can be extended to an invariant of immersed curves by defining



inductively on the number of nodes. This invariant on $\mathcal{K}_\infty$ can be thought of as a *restriction* of the original invariant. In particular $\mathcal{K}_d$ can be viewed as a subspace of $\mathcal{K}$ for each d by identifying any immersed curve in $\mathcal{K}_d$ with the alternating sum of the 2^d framed knots obtained by resolving each node, and the invariant on $\mathcal{K}_d$ is just the restriction of V to this subspace. Observe that these subspaces form a descending sequence, $\mathcal{K} = \mathcal{K}_0 \supset \mathcal{K}_1 \supset \mathcal{K}_2 \supset \cdots$.

A real valued framed knot invariant V will be called a (framed) *Vassiliev invariant* of *type* d if $V|\mathcal{K}_j = 0$ for all $j > d$, that is V is zero on all immersed curves with more that d nodes. The Vassiliev invariants of type d form a subspace $\mathcal{V}_d$ of $\mathcal{K}^*$, the annihilator of the subspace $\mathcal{K}_{d+1} \subset \mathcal{K}$, and clearly $\mathcal{V}_0 \subset \mathcal{V}_1 \subset \mathcal{V}_2 \subset \cdots$. The invariants in $\mathcal{V}_d \setminus \mathcal{V}_{d-1}$, that is of type d but not of type $d-1$, will be said to be of *degree d*.

Birman and Lin [3] have shown that the Jones invariants $J_d(k)$ and $J_d^u(k)$ are Vassiliev invariants of type d (in fact they are of degree d, as will be seen below), as is the coefficient of h^d in the power series expansion of any other quantum group invariant of framed knots. Thus the spaces $\mathcal{J}_d$ and $\mathcal{J}_d^u$ of Jones invariants of degree d are subspaces of $\mathcal{V}_d$. It will be shown below that $\mathcal{J}_d \bigcap \mathcal{V}_{d-1} = 0$, and so all non-trivial Jones invariants of degree d are of degree d in the Vassiliev sense.

The value of any invariant of type d on an immersed curve with d nodes can be readily shown to depend only on the way in which the $2d$ points in the pre-image of the d nodes are paired in the circle, and not on any other features of the immersion (including the framing). This information can be coded in a 'chord diagram', which consists of a circle with d chords indicating the pairs of points to be identified in the immersion. The chords are simply used as combinatorial guides to the eye; any intersections between chords are quite immaterial.

For example, the two immersed curves shown in Figure 3 both determine the same chord diagram. Any invariant of type 3 will have the same value on these two curves, although invariants of higher type may well differ.

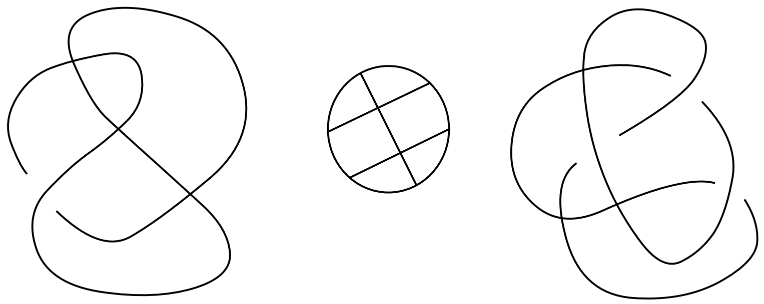


Figure 3

An invariant V of type d thus determines a linear functional on the space spanned by all chord diagrams with d chords, which in turn induces a functional $\mathcal{D}^d V$ on a certain quotient $\mathcal{A}_d$ of this space by a set of explicit linear relations, called the $4T$ relations by Bar-Natan. As the notation suggests, it is often helpful to think of V as a polynomial of degree d and of $\mathcal{D}^d V$ as the d^{th} derivative of V. Following Bar-Natan we write $\mathcal{W}_d$ for the dual space of $\mathcal{A}_d$, and call the elements of $\mathcal{W}_d$ (framed) *weight systems* of *degree d.* The function

$$\mathcal{D}^d : \mathcal{V}_d \to \mathcal{W}_d$$

is clearly linear with kernel $\mathcal{V}_{d-1}$. In fact $\mathcal{D}^d$ is onto by a result of Kontsevich (see [2] for a proof), and so $\mathcal{W}_d \cong \mathcal{V}_d/\mathcal{V}_{d-1}$. Indeed, by using an integral construction to find an element of $\mathcal{A}_d$ for every knot, Kontsevich produces a section $\mathcal{W}_d \to \mathcal{V}_d$ to $\mathcal{D}^d$ which is an isomorphism onto a complement of $\mathcal{V}_{d-1}$ in $\mathcal{V}_d$.

The preceding discussion can be understood in terms of the commutative diagrams

$$\begin{tikzcd} \mathcal{K}_d \arrow[r, "\mathrm{i}"] \arrow[d, "p"'] & \mathcal{K} \arrow[d, "V"] \\ \mathcal{A}_d \arrow[r, "\mathcal{D}^d V"'] & \mathbf{R} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{K}_d^* & \mathcal{K}^* \arrow[l, "i^*"'] \\ \mathcal{W}_d \arrow[u, "p^*"] & \mathcal{V}_d \arrow[l, "\mathcal{D}^d"] \arrow[u, "\text{inclusion}"'] \end{tikzcd}$$

where i is the inclusion and p is the projection sending any immersed curve to its chord diagram. Observe that p^* is $1-1$ and so the projection $\mathcal{D}^d : \mathcal{V}_d \to \mathcal{W}_d$ can be regarded as the restriction of i^* to $\mathcal{V}_d$.

The weight systems determined by the Jones function

We now investigate the weight systems $H_d(k) = \mathcal{D}^d J_d(k)$ of the Jones invariants $J_d(k)$, and the subspace $\mathcal{H}_d$ of $\mathcal{W}_d$ which they span. These weight systems can be regarded as evaluations of a polynomial valued weight system $H_d : \mathcal{A}_d \to \mathcal{P}$ for the polynomial Jones invariant J_d, and there is a commutative diagram

$$\begin{tikzcd} \mathcal{K}_d \arrow[r, "\mathrm{i}"] \arrow[d, "p"'] & \mathcal{K} \arrow[d, "J_d"] \\ \mathcal{A}_d \arrow[r, "H_d"'] & \mathcal{P} \end{tikzcd}$$

for each d. Observe that H_d is of finite rank since J_d is, by Theorem 2.2, and so $\dim \mathcal{H}_d = \operatorname{rk} H_d$ by Lemma 2.1.

Theorem 2.3. *The image of the polynomial weight system H_d contains the space $\mathcal{P}^o_{2d+1}$ of all odd polynomials of degree at most $2d+1$ with 1 as a root, and so $\dim \mathcal{H}_d \geq d$.*

Combining this result with the inequality $\dim \mathcal{J}_d \leq d$ of Theorem 2.2 shows that the projection

$$\mathcal{D}^d| : \mathcal{J}_d \to \mathcal{H}_d$$

is an isomorphism of rank d. Thus we have

Corollary 2.4. (a) *The space $\mathcal{J}_d$ of framed Jones invariants of degree $d > 0$ and the corresponding space $\mathcal{H}_d$ of weight systems are both of dimension d. Their associated maps J_d and H_d have common image equal to the space of all odd polynomials of degree at most $2d+1$ with 1 as a root.*

(b) $\mathcal{J}_d \bigcap \mathcal{V}_{d-1} = 0$ *(since $\ker \mathcal{D}^d = \mathcal{V}_{d-1}$). In other words, the non-trivial Jones invariants of degree d are of Vassiliev degree d.*

Proof of 2.3. It is enough to exhibit d linearly independent polynomials in the image of H_d, since $\operatorname{im}(H_d) \subset \operatorname{im}(J_d) \subset \mathcal{P}^o_{2d+1}$, by Theorem 2.2. It is convenient to allow d to vary, and we shall simply write $J(D)$ for the polynomial $H_d(D)$ when D has d chords. This defines a map

$$J : \mathcal{A} = \oplus \mathcal{A}_d \to \mathcal{P}$$

which encodes all the weight systems under consideration, with $J|\mathcal{A}_d = H_d$. For convenience we denote the variable in the polynomial $P(D)$ by k.

Thus we must exhibit chord diagrams $D_1, \ldots, D_d$, each with d chords, such that the polynomials $J(D_1), \ldots, J(D_d)$ are independent.

A simpler version of the state sum calculation for knots in the previous section allows calculation of $J(D)$ as a sum of traces of monomials. In its final form it is the special case of a more general result of Bar-Natan determining a weight system from any representation of a classical Lie algebra. Here we use a method based on the quantum group and the knot invariants.

Start with some immersed curve Γ with chord diagram D, presented as a closed braid with d nodes. If Γ is regarded as an alternating sum of knots by the embedding $\mathcal{K}_d \subset \mathcal{K}$, then $J(D)$ is the coefficient $J_{d,\Gamma}(k)$ of h^d in $J_{\Gamma,k}(h)$, the corresponding alternating sum of Jones functions. Instead of using this alternating sum, however, we can work directly with the braid presentation of the immersed curve Γ and represent each node by the endomorphism $\check{R} - \check{R}^{-1}$, as explained in the previous section. This endomorphism maps $x \otimes y$ to $\sum \beta_i(x) \otimes \alpha_i(y)$, where $\sum \alpha_i \otimes \beta_i$ is a bounded degree sum for the element $Q = R - P(R^{-1})$ in $\mathcal{G} \otimes \mathcal{G}$ (P is the interchange map). The invariant $J_{\Gamma,k}(h)$ can then be calculated by a state sum as before, and we are interested in the coefficient of h^d.

To compute this coefficient, observe that $R^{\pm 1} = I \otimes I \pm (X \otimes Y + \frac{1}{4} H \otimes H)h + \mathrm{O}(h^2)$, and so $Q = Q_1 h + \mathrm{O}(h^2)$ where $Q_1 = X \otimes Y + Y \otimes X + \frac{1}{2} H \otimes H$. Every state involves a choice of a term from Q for each node, together with terms for the crossings in the braid and the ends of the braid strings. Since each term in Q has a factor of at least h, the resulting invariant must have a factor of h^d. To get a non-zero contribution to the coefficient $J_{d,\Gamma}$ of h^d, we therefore need only consider states S which assign one term from Q_1 to each node, namely $X \otimes Y$, $Y \otimes X$ or $\frac{1}{2}H \otimes H$, and which make trivial assignments to every crossing or end of braid string. There is an associated monomial M_S of degree $2d$ in X, Y and $\frac{1}{\sqrt{2}}H$, obtained by reading round the immersed curve, and

$$J(D) = \sum_S \mathrm{Tr}_0(M_S)$$

where M_S operates on the module V_k and $\mathrm{Tr}_0(M_S)$ is the value of the trace when $h = 0$.

The state S can be indicated on the chord diagram by labelling the endpoints of each chord according to the term chosen for the corresponding node; for example if $X \otimes Y$ is assigned to the node then the endpoints are labelled X and Y. Then M_S is given by reading round the circle. This can be seen as a special case of Bar-Natan's prescription for finding a weight system from a representation of a semi-simple Lie algebra; the essential link with his work is that the linear term Q_1 in the quantum group is a multiple of the quadratic Casimir of the Lie algebra. This fact is used by Piunikhin [10] in identifying the weight systems arising from quantum group knot invariants with those found directly from the use of Lie algebras and chord diagrams, as it is a feature of general quantum groups and not just $SU(2)_q$.

We shall now make use of this state sum in calculating $J(D)$ explicitly for some diagrams D. The simplest diagram is the trivial diagram $\bigcirc$ with no chords, and evidently

$$J(\bigcirc) = \mathrm{Tr}_0(I) = k$$

since V_k is k-dimensional.

Next consider the diagram F with exactly one chord. The value of $J(F)$, which can be thought of as the 'framing contribution' for the coloured Jones invariants, can be calculated from the state sum on F as $\mathrm{Tr}_0(XY + YX + \frac{1}{2}HH)$ (the Casimir again). This trace can be determined by direct computation or by observing that $J(F)$ is the coefficient of h in the coloured Jones function for the planar immersed curve Γ with one node. Since $J_{\Gamma,k}(h) = (e^{xh} - e^{-xh})[k]$, where $x = \frac{1}{4}(k^2 - 1)$, we have

$$J(F) = \mathrm{Tr}_0(XY + YX + \tfrac{1}{2}HH) = \tfrac{1}{2}k(k^2 - 1).$$

More generally consider the diagrams T_i with i chords, one horizontal and the rest vertical, as shown in Figure 4.

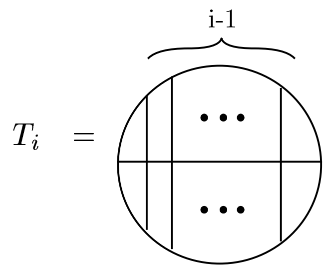


Figure 4

In particular $T_0 = \bigcirc$ and $T_1 = F$. We will show how to calculate $J(T_i)$ below, but first we describe an operation on chord diagrams that will be useful for building further examples.

Construction. From two chord diagrams D_1 and D_2, construct a *connected sum* $D = D_1 D_2$ by breaking each apart at some point on the circle, and then joining the two together (preserving orientations). The new diagram depends on the choice of breaking points, but it can be shown that any Vassiliev invariant will have the same value on all the connected sums of D_1 and D_2. In particular, for the Jones invariants we have:

Lemma 2.5. $J(D_1)J(D_2) = kJ(D_1 D_2)$.

Proof. Choose immersed curves Γ_1 and Γ_2 corresponding to the diagrams D_1 and D_2. Then a connected sum $\Gamma_1\Gamma_2$ in the obvious sense corresponds to a sum $D_1 D_2$. Now calculate $J_{\Gamma_1\Gamma_2,k}$ in terms of the alternating sum of knots, each of which will have the form K_1K_2, the connected sum of two knots. Since the coloured Jones function satisfies the relation $J_{K_1,k}J_{K_2,k} = [k]J_{K_1K_2,k}$ for framed knots, the lemma follows readily. □

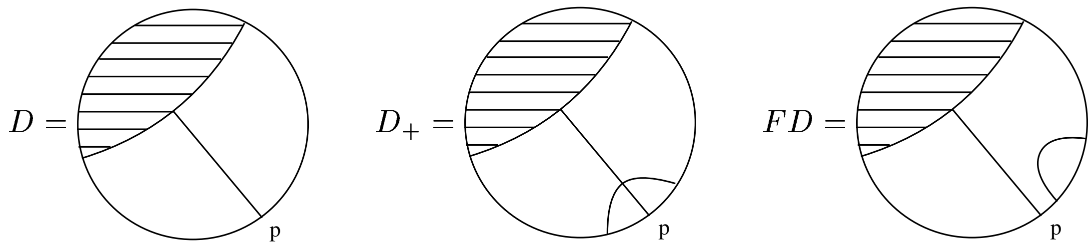
It follows from the lemma that adding a trivial chord to any diagram D has the effect of multiplying the polynomial $J(D)$ by the factor $J(F)/k$,

$$J(FD) = \tfrac{1}{2}(k^2 - 1)J(D).$$

There is a similar result, needed for the calculation of $J(T_i)$, when certain non-trivial chords are added to D.

Lemma 2.6 *Let D be a nontrivial chord diagram. Construct a new diagram D_+ by adding a chord that 'crosses' exactly one chord of D near one of its endpoints p. Then $J(D_+) = \frac{1}{2}(k^2 - 5)J(D)$.*

Proof. It is enough to show $J(FD) - J(D_+) = 2J(D)$, where the trivial chord in FD is added just to one side of the point p as shown below.



(Note: Bar-Natan proves a general result [1] that $V(FD) - V(D_+)$ is a multiple of $V(D)$ when V arises from a Lie algebra representation.)

Fix a state S on D, and suppose that S assigns the generator α to p. Then starting at p, the monomial M_S can be written as a product αM for some monomial M, and so S contributes $\mathrm{Tr}_0(\alpha M)$ to $J(D)$.

There are exactly three states on FD, and equally on D_+, which extend S, according to the three possible choices $X \otimes Y$, $Y \otimes X$ and $\frac{1}{2}H \otimes H$ for the extra chord. The sum of the monomials for these states is $(\alpha XY + \alpha YX + \frac{1}{2}\alpha HH)M$ on FD and $(X\alpha Y + Y\alpha X + \frac{1}{2}H\alpha H)M$ on D_+. The difference of their traces is $\mathrm{Tr}_0(\hat{\alpha}M)$, where $\hat{\alpha} = [\alpha, X]Y + [\alpha, Y]X + \frac{1}{2}[\alpha, H]H$. Since $\hat{\alpha} = 2\alpha + \mathrm{O}(h^2)$ for $\alpha = X$, Y and H, as is readily verified, these states contributute $2\mathrm{Tr}_0(\alpha M)$ to $J(FD) - J(D_+)$.

Summing over all states S on D now gives the lemma. □

Finally consider the diagrams $D_i = F^{i-1}T_{d-i+1}$ for $i = 1, \cdots, d$, each with d chords. Observe that D_i can be constructed from F^i by adding $d - i$ parallel chords as in the lemma, and so $J(D_i) = (\frac{1}{2}(k^2 - 5))^{d-i} J(F^i)$. Since $J(F^i) = k(\frac{1}{2}(k^2 - 1))^i$, it follows that

$$J(D_i) = k\left(\frac{k^2-5}{2}\right)^d \left(\frac{k^2-1}{k^2-5}\right)^i .$$

Any linear relation among these polynomials will give a relation $\sum a_i f^i$, where the a_i are real numbers and $f(k) = (k^2 - 1)/(k^2 - 5)$. This is impossible unless $a_i = 0$ for all i, since otherwise the polynomial $\sum a_i x^i$ has only a finite number of roots while $f(k)$ takes on infinitely many values as k varies. Thus the polynomials $J(D_i)$ are independent, and Theorem 2.3 is proved. $\square$

The unframed case

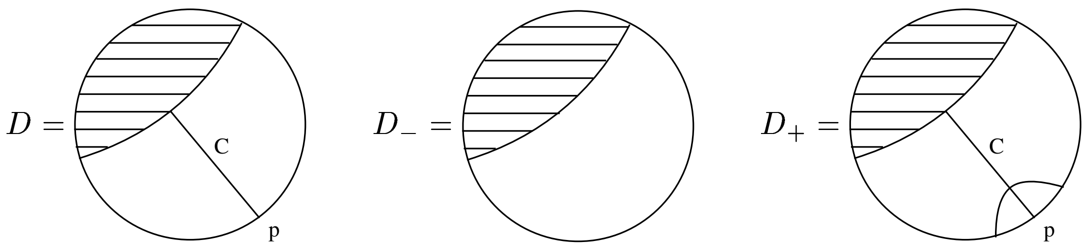
The weight systems $H_d^u(k) = \mathcal{D}^d J_d^u(k)$ for the unframed Jones invariants $J_d^u(k)$ span a subspace $\mathcal{H}_d^u$ of the space $\mathcal{W}_d$ of all degree d weight systems, whose dimension can be calculated as the rank of the corresponding polynomial weight system $H_d^u : \mathcal{A}_d \to \mathcal{P}$. As in the framed case, these weight systems can be encoded in a single linear map

$$J^u : \mathcal{A} = \oplus \mathcal{A}_d \to \mathcal{P},$$

defined on a chord diagram D with d chords to be the coefficient of h^d in the unframed Jones function $J_{\Gamma,k}^u(h)$, where Γ is any immersed curve with diagram D. To calculate $J^u(D)$ using a state sum, we must alter the matrix Q used for the nodes of Γ to incorporate a correction factor for the framing. Indeed the endomorphism associated with a node is now $e^{-xh}\breve{R} - e^{xh}\breve{R}^{-1}$, where $x = \frac{1}{4}(k^2 - 1)$, and so we replace Q by $Q^u = Q_1^u h + \mathrm{O}(h^2)$ where $Q_1^u = Q_1 - 2xI \otimes I$. Thus we must include states in which both ends of some chords are labelled with $\sqrt{-2x}\, I$.

We then have $J^u(\bigcirc) = k$ and $J^u(F) = 0$, as expected, and the same multiplicative formula for connected sums of diagrams as in the framed case. In particular $J^u(FD) = 0$ for any D. There is an analogue of Lemma 2.6 as well, giving the polynomial $J^u(D_+)$ in terms of polynomials for simpler diagrams. It should be observed, however, that this polynomial depends on which chord of D is 'crossed' by the new chord of D_+.

Lemma 2.7. *Let D be a nontrivial chord diagram with a chosen chord C. Construct two new diagrams D_- and D_+, where D_- is obtained by deleting the chord C, and D_+ is obtained by adding a new chord to D that 'crosses' C near one of its endpoints p, as shown below. Then $J^u(D_+) = -2J^u(D) - (k^2 - 1)J^u(D_-)$.*



Proof. It is enough to show $J^u(FD) - J^u(D_+) = 2J^u(D) + (k^2-1)J^u(D_-)$ since $J^u(FD) = 0$. As in the framed case, we fix a state S on D with label α on p, and write $M_S = \alpha M$.

The contribution to $J^u(FD) - J^u(D_+)$ of the corresponding states on $FD - D_+$ is $\mathrm{Tr}_0(\hat{\alpha} M)$, where $\hat{\alpha} = [\alpha, X]Y + [\alpha, Y]X + \frac{1}{2}[\alpha, H]H - 2x[\alpha, I]I$ (for $x = \frac{1}{4}(k^2-1)$ as usual). Now $\hat{\alpha}$ equals $2\alpha + \mathrm{O}(h^2)$ for $\alpha = X$, Y or $\frac{1}{\sqrt{2}} H$, but vanishes for $\alpha = \sqrt{-2x}\, I$, and so the contribution is $2\mathrm{Tr}_0(\alpha M)$ in the former case and nothing in the latter. Summing over all states gives $J^u(FD) - J^u(D_+) = 2(J^u(D) + 2xJ^u(D_-))$; the right hand side is the result of omitting those states for which C is labelled by $-2x\, I \otimes I$. Since $4x = k^2 - 1$, the proof is complete. $\square$

Using this lemma, we give a lower bound for the dimension of the space $\mathcal{H}^u_d$ of weight systems of unframed Jones invariants.

Theorem 2.8. *The image of the polynomial weight system H^u_d contains the space $\mathcal{P}^0_{d+1}$ of all odd polynomials of degree at most d+1 with 1 as a root, and so $\dim \mathcal{H}^u_d \geq \lfloor d/2 \rfloor$.*

Proof. Set $X = T_2$ and consider the d-chord diagrams $D_i = X^{i-1}T_{d-2(i-1)}$ for $i = 1, \ldots, \lfloor d/2 \rfloor$, where the chord diagrams T_i are defined above. Observe that D_i can be constructed from X^i by adding $d-2i$ parallel chords which 'cross' exactly one of the chords C in X^i. By the lemma we have $J^u(D_i) = (-2)^{d-i} J^u(X^i)$, since deleting C from X^i leaves one trivial chord. Applying the lemma again we have $J^u(X) = -2J^u(F) - (k^2-1)J^u(\bigcirc) = k(1-k^2)$, and so $J^u(X^i) = k(1-k^2)^i$ by the multiplicative property of J^u. This gives

$$J^u(D_i) = (-2)^{d-2i}\, k(1-k^2)^i,$$

and these polynomials clearly span $\mathcal{P}^0_{d+1}$ since they are of different degrees. $\square$

Combining Theorems 2.2 and 2.8 gives the following estimate for the dimension of the space $\mathcal{J}^u_d$ of unframed Jones invariants of degree d.

Corollary 2.9. $\lfloor d/2 \rfloor \leq \dim \mathcal{J}^u_d \leq d - 1$.

It is likely that in fact $\dim \mathcal{J}^u_d = \lfloor d/2 \rfloor$, which would be implied by Theorems 2.2 and 2.8, together with the following conjecture on the degrees of the coefficients in the unframed Jones function $J^u_{K,k}(h)$.

Conjecture 1. *The coefficient $J^u_d(k)$ of h^d in $J^u_{K,k}(h)$ has degree at most $d+1$ in k; equivalently, the coefficient of h^d in $J^u_{K,k}(h)/[k]$, which is an even polynomial in k, has degree at most d.*

Assuming Conjecture 1, extract the terms in $h^d k^d$ from $J^u_{K,k}(h)/[k]$ to write

$$J^u_{K,k}(h)/[k] = \sum b_d k^d h^d + \text{ terms in } k^l h^d,\ l < d,$$

and set $J_K(h) = \sum b_d h^d$.

Conjecture 2. *The Alexander polynomial $\Delta_K(t)$ of a knot K is determined by the coloured Jones function of K. In particular, $\Delta_K(e^h) = 1/J_K(h)$.*

These conjectures have been verified for torus knots by the second author [9]. Recent results of Steve Sawin [13] suggest the possibility of a complete proof of Conjecture 1 using the multiplicative structure of Vassiliev invariants.

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Note added in proof. Both conjectures have now been proved. Path integral arguments were first given by L. Rozansky (*A contribution of the trivial connection to the Jones polynomial and Witten's invariant of 3d manifolds*, 1993 preprint), and a mathematical proof has been given by D. Bar-Natan and S. Garoufalidis (*On the Melvin-Morton-Rozansky conjecture*, 1994 preprint).

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