String Theory – A Postgraduate Course for Physicists and Mathematicians

Thomas Mohaupt

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Classical relativistic point particles

In this chapter we revise classical relativistic point particles and set out our conventions and notation. Readers familiar with this material can skip through the chapter and use it as a reference when needed.

2.1 Minkowski Space

According to Einstein's theory of special relativity, space and time are combined into spacetime, which is modelled by Minkowski space $\mathbb{M}$.[1] The elements $P, Q, \ldots \in \mathbb{M}$ are *events*, which combine a moment of time with a position. We leave the dimension D of spacetime unspecified. D-dimensional Minkowski space $\mathbb{M} = \mathbb{M}^D$ is the D-dimensional affine space associated with the D-dimensional vector space $\mathbb{R}^D$. The difference between an affine space (or point space) and a vector space is that an affine space has no distinguished origin. In contrast in the vector space $\mathbb{R}^D$ we have the distinguished zero vector $\vec{0} \in \mathbb{R}^D$, which is the neutral element of vector addition. Vectors $x \in \mathbb{R}^D$ do not naturally correspond to points $P \in \mathbb{M}^D$, but to displacements relating a point P to another point Q

$$x = \overrightarrow{PQ} \, .$$

[1] For brevity's sake we will use 'Minkowksi space' instead of 'Minkowski spacetime.'

But we can use the vector space $\mathbb{R}^D$ to introduce linear coordinates on the point space $\mathbb{M}^D$ by chosing a point $O \in \mathbb{M}^D$ and declaring it the origin of our coordinate system. Then points are in one-to-one correspondence with position vectors:

$$x_P = \overrightarrow{OP} \;,$$

and displacements correspond to differences of position vectors:

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} \;.$$

While position vectors depend on a choice of origin, displacements are independent of this choice. This reflects that Minkowski space has translation symmetry. Having fixed an origin O, we can refer to points in Minkowski space in terms of their position vectors. The components x^μ, $\mu = 0, 1, \ldots, D = 1$ of vectors $x \in \mathbb{R}^D$ provide linear coordinates on $\mathbb{M}$. The coordinate x^0 is related to the time t, measured by an 'inertial', that is force-free, or freely falling observer, by $x^0 = ct$. We set the speed of light c to unity, $c = 1$, so that $x^0 = t$. The remaining coordinates x^i, $i = 1, \ldots, D = 1$ can be combined into a $(D-1)$-component vector $\vec{x} = (x^i)$. The components x^i parametrize space, as seen by the inertial observer.

In special relativity, the vector space $\mathbb{R}^D$ is endowed with the indefinite scalar product

$$x \cdot y = \eta_{\mu\nu} x^\mu y^\nu \;,$$

with Gram matrix

$$\eta = (\eta_{\mu\nu}) = \left(\begin{array}{cc} -1 & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_{D-1} \end{array} \right) \;. \qquad (2.1)$$

Inertial observes are required to use linear coordinates which are orthonormal with respect to this scalar product, that is, inertial frames are η-orthonormal frames.

Observe that we have chosen the 'mostly plus' convention, while part of the literature uses the 'mostly minus' convention, where the Gram matrix is multiplied by -1. We are using the index notation common in the physics literature, including Einstein's summation convention. It is understood that we have identified the vector space $V = \mathbb{R}^D$ with its dual V^* using the Minkowski metric. Thus a vector x has contravariant coordinates x^μ and covariant coordinates x_μ which are related by 'raising and lowering indices' $x_\mu = \eta_{\mu\nu} x^\nu$, and $x^\mu = \eta^{\mu\nu} x_\mu$.

For reference, we also list the corresponding line element.

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = -dt^2 + d\,\vec{x}^2 \ . \tag{2.2}$$ Mink_Line_Element

One can, and in many applications does, make use of other, curvilinear coordinate systems, such as spherical or cylindrical coordinates. But η-orthonormal coordinates are distinguished by the above standard form of the metric. The most general class of transformations which preserve this form are the Poincaré transformations

$$x^\mu \to \Lambda^\mu{}_\nu x^\nu + a^\mu \ ,$$

where $\Lambda = (\Lambda^\mu{}_\nu)$ is an invertible $D \times D$ matrix satisfying

$$\Lambda^T \eta \Lambda = \eta \ ,$$

and where $a = (a^\mu) \in \mathbb{R}^D$. The matrices Λ describe Lorentz transformations, which are the most general linear transformations preserving the metric. They contain rotations together with 'Lorentz boosts', which relate inertial frames travelling a constant velocity relative to each other. The Lorentz transformations form a six-dimensional Lie group, called the Lorentz group $O(1, D-1)$. Elements $\Lambda \in O(1, D-1)$ have determinant $\det \Lambda = \pm 1$, and satisfy $|Lambda^0{}_0| \geq 1$. The matrices with $\det \Lambda = 1$ form a subgroup $SO(1, D-1)$. This subgroup still has two connected component, since $\Lambda^0{}_0 \geq 1$ or $\Lambda^0{}_0 \leq -1$. The component containing the unit matrix $\mathbf{1} \in O(1, D-1)$ is connected and denoted $SO_0(1, D-1)$. This is the Lie group obtained by exponentiating the corresponding Lie algebra $\mathfrak{so}(1, D-1)$.

The Lorentz group and translation group combine into the Poincaré group, or inhomogeneous Lorentz group, $IO(1, D-1)$, which is a ten-dimensional Lie group. Note that since Lorentz transformations and translations do not commute, the Poincaré group is not a direct product. The composition law

$$(\Lambda, a) \circ (\Lambda', a') = (\Lambda\Lambda', a + \Lambda a')$$

shows that the Lorentz group operates on the translation group. More precisely, the translation group $\mathbb{R}^D$ happens to be a vector space, on which $O(1, D-1)$ acts by its fundamental representation. The Poincaré group is the semi-direct product of the Lorentz and translation group,

$$IO(1, D-1) = O(1, D-1) \ltimes \mathbb{R}^D \ .$$

Since the Minkowski metric (Mink_Line_Element 2.2 is defined by an indefinite scalar product, the distance-squared between events can be positive, zero or negative. This carries information about the causal structure of spacetime. Vectors are classified as time-like, light-like (or null), or space-like according to their norm-squared:

$$\begin{aligned} x \text{ time-like} &\Leftrightarrow \quad x \cdot x < 0 \, , \\ x \text{ light-like} &\Leftrightarrow \quad x \cdot x = 0 \, , \\ x \text{ space-like} &\Leftrightarrow \quad x \cdot x > 0 \, . \end{aligned}$$

Then, if $x = \overrightarrow{PQ}$ is the displacement between between two events, then these events are called time-like, light-like or space-like relative to each other, depending on x. The zero-component of x carries the information whether P is later than Q ($x^0 > 0$), or simultanous with Q ($x^0 = 0$) or earlier than Q ($x^0 < 0$), relative to a given Lorentz frame. If we fix the orientation of space and the direction of space, Lorentz frames are related by proper orthochronous Poincaré transformations, $(\Lambda, a) \in SO_0(1, D-1) \ltimes \mathbb{R}^D$. Under these transformations the temporal order of events is invariant, if and only if the events are time-like or light-like relative to each other. In contrast, the events that are relatively space-like can be put in any relative order by applying these transformations. This reflects that signals propagating with superluminal speed are not consistent with ‘causality’, that is the ideat that the past can influence the future but not the other way round: events which are time-like, light-like and space-like relative to one another are connected by straight lines corresponding to signals propagating with subluminal, luminal and superluminal speed, respectively.

2.2 Particles

The particle concept assumes that there are situations where matter can be modelled (possibly only approximately) as a system of extensionless mathematical points. The motion of such a point particle, or particle for short, is described by a parametrized curve, called the world-line. If we restrict ourselves to inertial frames, it is natural to choose the coordinate time t as the curve parameter. Then the worldline of the particle is a parametrized curve

$$C \, : I \to \mathbb{M} \, : t \mapsto x(t) = (x^\mu(t)) = (t, \vec{x}(t)) \, , \tag{2.3}$$

where $I \subset \mathbb{R}$ is the interval in time for which the particle is observed ($I = \mathbb{R}$ is included as a limiting case).

The velocity of the particle relative to the given inertial frame is

$$\vec{v} = \frac{d\vec{x}}{dt} \ . \tag{2.4}$$

Particles are equipped with an inertial mass m, which characterizes their resistance against a change of velocity. It is a standard result of special relativity that for particles with $m > 0$ the velocity is bound to be $v = \sqrt{\vec{v} \cdot \vec{v}} < 1$ (remember that we set $c = 1$). In special relativity it is consistent to admit particles with $m = 0$, which then must have speed $v = 1$. Particles propagating with $v > 1$ would have $m^2 < 0$ and are excluded by the causality postulate. Geometrically, massive and massless particles have time-like-like and light-like world lines, respectively. Here we call a curve time-like, light-like or space-like, if its tangent vector field is time-like, light-like or space-like, respectively.

Under Lorentz transformations, a description of particle motion in terms of t and $\vec{v}$ is not covariant, as both quantites are not Lorentz tensors. For massive particles one can use the proper time τ instead of the coordinate time t as a natural curve parameter. The proper time is the time measured on a clock attached to the particle. It is assumed that this clock has at any given moment the same rate as a non-accelerated clock in the inertial frame where the particle is momenarily at rest. Infinitesimally the relation between proper time and coordinate time is

$$[-dt^2 + d\vec{x}^2]_{|C} = -d\tau^2 \Rightarrow d\tau = dt\sqrt{1 - \vec{v}^2} \ ,$$

where as indicated the line element has been evaluated on the worldline C, and we have chosen proper time and coordinate time to have the same direction. The relation between finite intervals of proper time and coordinate time is found by integration:

$$\tau_2 - \tau_1 = \int_{t_1}^{t_2} dt\sqrt{1 - \vec{v}^2} \ . \tag{2.5}$$

If we synchronize proper time and coordinate time at $\tau = t = 0$, this implies

$$\tau(t) = \int_0^t dt'\sqrt{1 - \vec{v}^2} \ .$$

The proper time is by construction Lorentz invariant, as might also be checked by explicit computation. Geometrically, it corresponds to the ‘length’ of the world line C, as we will discuss below. For massless particles, which move with the speed of light, we cannot define a proper time, because there is no inertial frame relative to which a massless particle can be momentarily at rest. We will come back to massless particles later when we discuss the use of arbitrary curve parameters.

A manifestly covariant formulation of relativistic mechanics is obtained by starting with the Lorentz vector x^μ and the Lorentz scalar τ. Further Lorentz tensors arise by differentiation. The relativistic velocity is defined by

$$\dot{x}^\mu = \frac{dx^\mu}{d\tau} = \left(\frac{dt}{d\tau}, \frac{d\vec{x}}{dt}\frac{dt}{d\tau} \right) = \frac{1}{\sqrt{1-\vec{v}^2}} (1, \vec{v}) \; . \tag{2.6}$$

Note that relativistic velocity is normalised by construction:

$$\dot{x}^\mu \dot{x}_\mu = -1 \; . \tag{2.7}$$

Geometrically, $\dot{x} = (\dot{x}^\mu)$ is the tangent vector to the world line C. Since its norm-squared is negative, the worldline of a massive particle is time-like. This corresponds to the particle propagating with a speed smaller than the speed of light. Since the norm-squared of the tangent vector is constant along the curve, the proper time is what is called an affine curve parameter. Such curve parameters are unique up to affine transformations, $\tau \mapsto a\tau + b$, $a, b \in \mathbb{R}$, $a \neq 0$. The proper time τ is further distinguished among affine curve parameters, since the tangent vector is normalized specifically to minus unity.

By further differentiation we obtain the relativistic acceleration,

$$a^\mu = \ddot{x}^\mu \; .$$

One central postulate of mechanics is that force-free particles are unaccelerated relative to inertial frames.

The relativistic momentum of a particle is

$$p^\mu = m\dot{x}^\mu = (p^0, \vec{p}) = \left(\frac{m}{\sqrt{1-\vec{v}^2}}, \frac{m\vec{v}}{\sqrt{1-\vec{v}^2}} \right) \; . \tag{2.8}$$

The component $p^0 = E$ is the total energy of the particle. The norm-squared of p^μ is minus its mass squared

$$p^\mu p_\mu = -m^2 = -E^2 + \vec{p}^2 \; . \tag{2.9}$$ mass-shell

Solving for E (choosing E to be postive), we find

$$E = \sqrt{m^2 + \vec{p}^2} = m + \frac{1}{2m}\vec{p}^2 + \cdots$$

where we expanded for small momenta $\vec{p}$. In the rest frame, where $\vec{p} = \vec{0}$, we have $E = m(= mc^2)$, which tells us that the mass m is the total energy measured in the rest frame.

Force-free particles propagate with constant velocity, which means that their worldlines are straight lines. The relativistic version of Newton's 'second axiom' postulates that motion under a force is determined by the equation

$$\frac{dp^\mu}{d\tau} = f^\mu \;, \tag{2.10}$$

Newton2

where the Lorentz vector f^μ is the relativistic force. For constant mass this becomes

$$m\frac{d^2x^\mu}{d\tau^2} = f^\mu \;.$$

Thus if we know to describe the force acting on a particle by specifying a force vector field f^μ, we need to solve a system of linear ordinary second order differential equations to determine the particle's motion. Moreover, a force manifests itself as acceleration, $a^\mu \neq 0$, which results in a curved worldline.

We remark that there are situations where the mass m is not constant, for example for a rocket which accelerates by burning and emitting fuel. However, in fundamental theories we will be modelling elementary particles with a fixed mass.

2.3 A non-covariant action principle for relativistic particles

The equations of motion of all fundamental physical theories can be obtained from variational principles. In this approach a theory is defined by specifying its action functional, which is a functional on the configuration space. The equations of motion are the Euler-Lagrange equations obtained by imposing that the action is invariant under infinitesimal variations of the path between a given initial and final configuration, which are kept fixed.

For a point particle the configuration space is parametrized by its position $\vec{x}$ and velocity $\vec{v}$. The action functional takes the form

$$S[\vec{x}] = \int dt\, L(\vec{x}(t), \vec{v}(t)), t) \;.$$

The function L is called the Lagrangian and can depend on the position and velocity of the particle. In principle it can also also have an explicit dependence of time, as indicated, but this would correspond to situtations where an external, time-dependent force acts on the particle. In fundamental theories, invariance of the field equations under time-translations is a natural assumption, and forbids an explicit time dependence of L.

The action for a free, massive, relativistic particle is proportional to the proper time along the worldline, and given by minus the product of its mass and the proper time:

$$S = -m \int dt \sqrt{1 - \vec{v}^2}\ . \tag{2.11}$$

ActionI

The minus sign has been introduced so that L has the conventional form $L = T - V$, where T is the part quadratic in time derivatives, that is, the kinetic energy. The remaining part V is the potential energy. We work in units where the speed of light and the reduced Planck constant have been set to unity, $c = 1$, $\hbar = 1$. In such 'natural units' the action S is dimensionless. To verify that the action principle reproduces the equation of motion (Newton2 2.10), we consider the motion $\vec{x}(t)$ of a particle between the initial postion $\vec{x}_1 = \vec{x}(t_1)$ and the final position $\vec{x}_2 = \vec{x}(t_2)$. Then we compute the first order variation of the action under infinitesimal variations $\vec{x} \to \vec{x} + \delta\vec{x}$, which are arbitrary, except that they vanish and at the end points of the path, $\delta\vec{x}(t_i) = 0$, $i = 1, 2$. To compare the initial and deformed path we Taylor expand in $\delta\vec{x}(t)$:

$$S[\vec{x}(t) + \delta\vec{x}(t)] = S[\vec{x}(t), \vec{v}(t)] + \delta S[\vec{x}(t), \vec{v}(t)] + \cdots$$

where the omitted terms are of quadratic and higher order in $\delta\vec{x}(t)$. The equations of motion are then found by imposing that the first variation δS vanishes.

Practical manipulations are most easily performed using the following observations

1. The variation δ acts 'like a derivative.' For example for a function $f(\vec{x})$,

   $$\delta f = \partial_i f \delta x^i\ ,$$

   as is easily verified by Taylor expanding $f(\vec{x} + \delta\vec{x})$. Similarly, the 'obvious analogues' of the sum, constant factor, product and quotient rule apply, e.g. $\delta(fg) = \delta f g + f \delta g$.

2. $\vec{v} = \frac{d\vec{x}}{dt}$ is not an independent quantity, and therefore not varied independently. Consequently

   $$\delta\vec{v} = \delta\frac{d\vec{x}}{dt} = \frac{d}{dt}\delta\vec{x}\ .$$

3. To find δS we need to collect all terms proportional to $\delta\vec{x}$. Derivatives acting on $\delta\vec{x}$ have to be removed through integration by parts, which creates boundary terms.

Exercise: Verify that the variation of (2.11) takes the form

$$\delta S = -\int_{t_1}^{t_2} \left(\frac{d}{dt} \frac{m v_i}{\sqrt{1-\vec{v}^2}} \right) \delta x^i dt + \frac{m v_i}{\sqrt{1-\vec{v}^2}} \delta x^i \Big|_{t_1}^{t_2} \; . \tag{2.12}$$

In the variation the end points of the worldline are kept fixed, and therefore the second term, which is a boundary term resulting from integration by parts, vanishes. Since the variation $\delta\vec{x}$ is otherwise arbitrary, vanishing of the first term requires

$$\frac{d}{dt} \frac{m\vec{v}}{\sqrt{1-\vec{v}^2}} = \frac{d}{dt}\vec{p} = \vec{0} \; . \tag{2.13}$$ `Newton2.2`

These are the Euler Langrange equations of our variational problem.

Exercise: Verify that (2.13) is equivalent to (2.10) with $f^\mu = 0$. *Hint:* Rewrite $p^\mu = m\frac{dx^\mu}{d\tau}$ in terms of derivatives with respect to coordinate time t using the chain rule, and remember that the component p^0 is not independent, but related to the $\vec{p}$ by the 'mass shell condition' (2.9).

Remark: When performing the variation without specifying the Langrangian L, one obtains the Euler-Lagrange equations

$$\frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial v^i} = 0 \; . \tag{2.14}$$ `EL`

For $L = -m\sqrt{1-\vec{v}^2}$ this is easily seen to give (2.13). In my opinion it is more natural, convenient and insightful to obtain the equations of motion for a given concrete theory by varying the corresponding action, as done above, instead of plugging the Lagrangian into the Euler Lagrange equations. For instance, this procedure reminds one that there are in principle boundary terms that one has to worry about, as we will see when replacing particles by strings.

Remark: Variational derivatives. The condition that the action functional is stationary under infinitesimal variations of the path $x^i(t)$ can be expressed as the vanishing of the 'first variational derivative':

$$\frac{\delta S}{\delta x^i(t)} = 0 \; .$$

In physics such variational derivatives are ususally treated formally, by regarding $S[x(t)]$ as a 'function of infinitely many variables', **see textbooks on QFT,**

for example Becher/Böhm/Joos for an outline of the formal variational calculus. In order that variational derivatives are mathematically well defined, one must assume that the space of functions on which the functional is defined is sufficiently well behaved, for example a Banach space (that is, a vector space with a norm, which is complete with respect to the norm). In such cases one can define derivatives on infinite dimensional spaces, for example the Fréchet derivative. **Reference: Zeidler or other** In this book we will treat variational derivatives formally.

Remark: Physical Units. Readers who are confused about statements such as 'the action is dimensionless when using natural units' are referred to Appendix A for a quick review of physical units and dimensional analysis.

2.4 Canonical momenta and Hamiltonian

We now turn to the Hamiltonian description of the relativistic particle. In the Lagrangian formalism we use the configuration space variables $(\vec{x}, \vec{v}) = (x^i, v^i)$. In the Hamiltonian formalism, the velocity $\vec{v}$ is replaced by the canonical momentum

$$\pi^i := \frac{\partial L}{\partial v_i} \ . \tag{2.15}$$

For the Lagrangian $L = -m\sqrt{1-\vec{v}^2}$, the canonical momentum agrees with the kinetic momentum, $\vec{\pi} = \vec{p} = (1-\vec{v}^2)^{-1/2} m\vec{v}$. However, conceptually canoncial and kinetic momentum are different quantities. A standard example where the two quantities are not equal is a charged particle in a magnetic field.

The Hamiltonian $H(\vec{x}, \vec{\pi})$ is obtained from the Lagrangian $L(\vec{x}, \vec{v})$ by a Legrendre transformation:

$$H(\vec{x}, \vec{\pi}) = \vec{\pi} \cdot \vec{v} - L(\vec{x}, \vec{v}(\vec{x}, \vec{\pi})) \ . \tag{2.16}$$

For $L = -m\sqrt{1-\vec{v}^2}$ the Hamiltonian is equal to the total energy:

$$H = \vec{\pi} \cdot \vec{v} - L = \vec{p} \cdot \vec{v} - L = \frac{m}{\sqrt{1-\vec{v}^2}} = p^0 = E \ . \tag{2.17}$$

As we will see below, this is not true in general.

Describing relativistic particles using (2.11) has the following disadvantages.

- We can only describe massive particles, but photons, gluons and the hypothetical gravitons underlying gravity are believed to be massless. How can we describe massless particles?

- The independent variables $\vec{x}, \vec{v}$ are not Lorentz vectors. Therefore our formalism lacks manifest Lorentz covariance. How can we formulate an action principle that is Lorentz covariant?
- We have picked a particular curve parameter to parametrize the worldline, namely the inertial time with respect to a reference Lorentz frame. While this is a distinguished choice, 'physics', that is observational data, cannot depend on how we label points on the worldline. But how can we formulate an action principle that is manifestly covariant with respect to reparametrizations of the worldline?

We will answer these issues in reverse order.

2.5 Length, proper time and reparametrizations

A (parametrized) curve is a map from a parameter interval $I \subset \mathbb{R}$ into a space, which we take to be Minkowski space $\mathbb{M}$:

$$C \ : I \ni \sigma \longrightarrow x^{\mu}(\sigma) \in \mathbb{M} \ . \tag{2.18}$$

In this setting σ is an arbitrary curve parameter. We will assume in the following that this map is smooth, or that at least that the derivatives we compute exist and are continuous. We can 'reparametrize' the curve by introducting a new curve parameter $\tilde{\sigma} \in \tilde{I}$ which is related to σ by an invertible map

$$\sigma \to \tilde{\sigma}(\sigma) \ , \quad \text{where} \quad \frac{d\tilde{\sigma}}{d\sigma} \neq 0 \ . \tag{2.19}$$

While this defines a different map (a different parametrized curve), we are interested in the image of this map in $\mathbb{M}$, and therefore regard $\tilde{C} : \tilde{I} \to \mathbb{M}$ as a different description (parametrization) of the same curve. The quantity $d\tilde{\sigma}/d\sigma$ is the Jacobian of this reparametrization.

Often one imposes the stronger condition

$$\frac{d\tilde{\sigma}}{d\sigma} > 0 \ , \tag{2.20}$$

which means that the orientation (direction) of the curve is preserved.

The tangent vector field of a curve is obtained by differentiating with respect to the curve parameter:

$$x'^{\,\mu} := \frac{dx^{\mu}}{d\sigma} \ . \tag{2.21}$$

Recall that a curve $C\ : I \rightarrow \mathbb{M}$ is called space-like, light-like or space-like if its tangent vector field is space-like, light-like or space-like, respectively, for all $\sigma \in I$. Note that this property is reparametrization invariant.

For a space-like curve $I = [\sigma_1, \sigma_2] \rightarrow \mathbb{M}$, the length is defined as

$$L = \int_{\sigma_1}^{\sigma_2} \sqrt{\eta_{\mu\nu} \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\sigma}} d\sigma \ . \tag{2.22}$$

Exercise: Verify that the length of a space-like curve is reparametrization invariant.

For a time-like curve we can define the analogous quantity

$$\tau(\sigma_1, \sigma_2) = \int_{\sigma_1}^{\sigma_2} d\sigma \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\sigma}} \ . \tag{2.23}$$

This quantity is the proper time of special relativity, and the associated tangent vector field can be interpreted as the velocity field along the worldline of a massive relativistic particle.

Let us verify this statement. We can use τ as a curve parameter by defining:

$$\tau(\sigma) = \int_{\sigma_1}^{\sigma} d\sigma' \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\sigma'} \frac{dx^\nu}{d\sigma'}} \ , \tag{2.24}$$

where $\sigma_1 \leq \sigma \leq \sigma_2$. By differentiating this function we obtain the Jacobian

$$\frac{d\tau}{d\sigma} = \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\sigma}} \ . \tag{2.25}$$

If we choose σ to be the time t measured in an inertial frame, wo obtain the standard special-relativistic relation between the differentials of proper time and of inertial time. Therefore we should find that the tangent vector field $dx^\mu/d\tau$ is normalized, and has constant length-squared -1. To verify this we compare the tangent vector fields of the two parametrizations:

$$\frac{dx^\mu}{d\tau} = \frac{dx^\mu}{d\sigma} \frac{d\sigma}{d\tau} = \frac{\frac{dx^\mu}{d\sigma}}{\sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\sigma}}} \ . \tag{2.26}$$

This implies that

$$\eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = -1 \ . \tag{2.27}$$

Expressing the proper time integral using τ itself as curve parameter, we obtain

$$\int_{\sigma_1}^{\sigma_2} d\sigma \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\sigma}} = \int_{\tau_1}^{\tau_2} d\tau = \tau_2 - \tau_1 \ . \tag{2.28}$$

This completes verifying the statement.

We note the following characterization of the proper time τ as a distinguished curve parameter: parametrizing a time-like curve by its proper time is equivalent to normalizing the tangent vector field to length-squared -1. Similarly, parametrizing a space-like curve by its length is equivalent to normalizing the tangent vector field to length-squared $+1$. Note that for light-like curve there is no quantity analogous to length or proper time. We will come back to this when we introduce an action for massless particles.

2.6 A covariant action for massive relativistic particles

Using the concepts of the previous section, we introduce the following action:

$$S[x,x'] = -m \int d\sigma \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\sigma}} \; . \tag{2.29}$$

ActionII

Up to the constant factor $-m$, the action is given by the 'length', more precisely by the proper time of the world line. The dimensionful factor m makes the action dimensionless (with respect to natural units where $c = 1$, $\hbar = 1$). We now use an arbitrary curve parameter σ, and configuration space variables $(x, x') = (x^\mu, x'^\mu)$ which transform covariantly under Lorentz transformations. The action (2.29) is covariant in the following sense:

- The action is invariant under reparametrisations $\sigma \to \tilde{\sigma}(\sigma)$ of the world-line.
- The action is manifestly invariant under Poincaré transformations,

$$x^\mu \to \Lambda^\mu{}_\nu x^\nu + a^\mu \; , \tag{2.30}$$

  where

$$(\Lambda^\mu{}_\nu) \in O(1, D-1) \quad \text{and} \quad (a^\mu) \in \mathbb{M} \tag{2.31}$$

  are constant.

To verify that the new action (2.29) leads to the same field equations as (2.11), we perform the variation $x^\mu \to x^\mu + \delta x^\mu$ and obtain:

$$\frac{\delta S}{\delta x^\mu} = 0 \Leftrightarrow \frac{d}{d\sigma} \left(\frac{m \, x'^{\,\mu}}{\sqrt{-x' \cdot x'}} \right) = 0 \; . \tag{2.32}$$

To get the physical interpretation, we choose the curve parameter σ to be proper time τ:

$$\frac{d}{d\tau}\left(m\frac{dx^\mu}{d\tau}\right) = m\ddot{x}^\mu = 0 \;, \tag{2.33}$$

where 'dot' denotes the derivative with respect to proper time. This is indeed (2.10) with $f^\mu = 0$.

The general solution of this equation, which describes the motion of a free massive particle in Minkowski space is the straight world line

$$x^\mu(\tau) = x^\mu(0) + \dot{x}^\mu(0)\tau \;. \tag{2.34}$$

Remark: Reparametrizations vs Diffeomorphisms. In part of the literature reparametrization invariance is referred to as diffeomorphism invariance. We use the term reparametrization, rather than diffemorphism, to emphasize that we interpret the map $\sigma \mapsto \tilde{\sigma}$ *passively*, that is as changing the label which refers to a given point. In contrast, an active transformation would map a given point to another point. When writing such a diffeomorphism in terms of local coordinates, expressions agree, up to a $(-)$–sign **This will be illustrated by an examle in Part 2. Add crossreference here.**. In physics it is a standard assumption that the passive and active interpretation of symmetries are equivalent. **Refer to Kiefer's book for discussion in context of gravity.**

Remark: Local vs global in mathematical and physical terminology. In mathematics 'local' refers to statements which hold on open neighbourhoods around each point, whereas 'global' refers to statements holding for the whole space under consideration. In contrast, physicists call symmetries 'global' or 'rigid' if the transformation parameters are independent of spacetime, and 'local' if the transformation parameters are functions on spacetime. In the case of the point particle action above, Poincaré transformations are global symmetries, while reparametrizations are local. I will try to reduce the risk of confusion by saying 'rigid symmetry' rather than 'global symmetry', but when a symmetry is referred to as local, it is meant in the physicist's sense. Also, it is common for physicists to talk about statements which are true locally (in the mathematician's sense), but not necessarily true globally, using 'global terminology'. I will follow this linguistic convention and will often leave it to the careful reader to clarify whether a statement formulated in 'global terminology' is actually a global result in the strict sense.[2]

[2]**An explicit example will be given later when we discuss the actions of the**

2.7 Particle Interactions

So far we have considered free particles. Interactions can be introduced by adding terms which couple the particle to fields. The most important examples are the following:

- If the force f^μ has a potential, $f_\mu = \partial_\mu V(x)$, then the equation of motion (2.10) follows from the action
$$S = -m \int \sqrt{-\dot{x}^2} d\tau - \int V(x(\tau)) d\tau \ . \tag{2.35}$$
For simplicity, we took the curve parameter to be proper time. In the second term, the potential V is evaluated along the worldline of the particle.

- If f^μ is the Lorentz force acting on a particle with charge q, that is $f^\mu = F^{\mu\nu}\dot{x}_\nu$, then the action is
$$S = -m \int \sqrt{-\dot{x}^2} d\tau - q \int A_\mu dx^\mu \ . \tag{2.36}$$
In the second term, the (relativistic) vector potential A_μ is integrated along the world line of the particle
$$\int A_\mu dx^\mu = \int A_\mu(x(\tau)) \frac{dx^\mu}{d\tau} d\tau \ . \tag{2.37}$$
The resulting equation of motion is
$$\frac{d}{d\tau}\left(m \frac{dx^\mu}{d\tau} \right) = q F^{\mu\nu} \dot{x}_\nu \ , \tag{2.38}$$
where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength tensor. Equation (2.38) is the manifestly covariant version of the Lorentz force
$$\frac{d\vec{p}}{dt} = q \left(\vec{E} + \vec{v} \times \vec{B} \right) \ . \tag{2.39}$$

- The coupling to gravity can be obtained by replacing the Minkowski metric $\eta_{\mu\nu}$ by a general (pseudo-)Riemannian metric $g_{\mu\nu}(x)$:
$$S = -m \int d\tau \sqrt{-g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu} \ . \tag{2.40}$$
The resulting equation of motion is the geodesic equation
$$\ddot{x}^\mu + \Gamma^\mu{}_{\nu\rho} \dot{x}^\nu \dot{x}^\rho = 0 \ , \tag{2.41}$$
(with affine curve parameter τ.)

---

conformal Lie Algebra and of the conformal group on the string worldsheet.

Exercise: Verify that the variation of the above actions gives rise to the corresponding field equations.

Remark: In all three examples the interaction is introduced by coupling the particle to a background field.[3] A different way to describe interactions is to allow that the wordlines of particles split or join. In a quantum theory both descriptions are in fact related, since particles and fields can be viewed as different types of excitations of 'quantum fields.' Particles correspond to eigenstates of the particle number operator, whereas fields correspond to coherent states which are eigenstates of the particle creation operator. **Add reference to quantum mechanics book illustrating this with the harmonic oscillator.** The description of quantum particles in terms of splitting and joining worldlines is called the 'worldline formalism,' which we will not cover in this book. **Add references for worldline formalism.** The worldsheet formulation of string theory that we will be developing is the analogue of the wordline formalism for particles.

2.8 Canonical momenta and Hamiltonian for the covariant action

From the action (ActionII 2.29)

$$S = \int L d\sigma = -m \int d\sigma \sqrt{- x'^{\,2}} \;, \tag{2.42}$$

we obtain the following canonical momentum vector:

$$\pi^\mu = \frac{\partial L}{\partial\, x'_{\;\mu}} = m \frac{x'^{\,\mu}}{\sqrt{- x'^{\,2}}} = m \dot{x}^\mu \;. \tag{2.43}$$

A new feature compared to action (ActionI 2.11) is that the components of the canonical momentum are not independent, but subject to the constraint

$$\pi^\mu \pi_\mu = -m^2 \;. \tag{2.44}$$

Since canonical and kinetic momenta agree,

$$\pi^\mu = p^\mu \;, \tag{2.45}$$

[3]Background field means that the field is 'given', rather than to be found by solving field equations. The full coupled theory of particles and fields would require to specify the field equations for the field. In electrodynamics the Maxwell equations would be specified in addition to the Lorentz force.

and we can interprete the constraint as the mass shell condition $p^2 = -m^2$. The Hamiltonian associated to (2.29) is

$$H = \pi^\mu \dot{x}_\mu - L = 0 \; . \tag{2.46}$$

Thus the Hamiltonian is not equal to the total energy, but rather vanishes. To be precise, since $H \propto p^2 + m^2$, the Hamiltonian does not vanish identically, but only for field configurations which satisfy the mass shell condition. Thus $H = 0$ is a constraint which needs to be imposed on top of the dynamical field equations. This is sometimes denoted $H \simeq 0$, and one says that the Hamiltonian is weakly zero. This type of 'Hamiltonian constraint' arises when mechanical or field theoretical systems are formulated in a manifestly Lorentz covariant or manifestly reparametrization invariant way. The study of 'constrained dynamics' is a research subject in its own right. **Add references for constrained dynamics, including Dirac, Sundermeyer, Henneux/Teitelboim.** We will not need to develop this systematically, because all the constraints we will encounter will be of 'first class.' This means that they are consistent with the dynamical evolution, and can therefore be imposed as initial conditions. We will demonstrate this explicitly for strings later **Add cross reference**.

Remark: Hamiltonian and time evolution. For those readers who are familiar with the formulation of mechanics using Poisson briacketes, we add that while the Hamiltonian is weakly zero, it still generates the infinitesimal time evolution of physical quantities. Similarly, in the quantum version of the theory, the infinitesimal time evolution of an operator O in the Heisenberg picture is still given by its commutator with the Hamiltonian. Moreover 'consistency with the time evolution' means to commute with the Hamiltonian. For the relativistic particle this is clear because vanishing of the Hamiltonian is the only constraint.

By accepting that constraints are the prize to pay for a covariant formalism, we can describe relativistic massive particles in a Lorentz covariant and reparametrization invariant way. But we still need to find a way to include massless particles.

2.9 A covariant action for massless and massive particles

To include massless particles, we use a trick which works by introducing an auxiliary field $e(\sigma)$ on the wordline. We require that $e\,d\sigma$ is a reparamentrization invariant line element on the world line:

$$e\,d\sigma = \tilde{e}\,d\tilde{\sigma}\ .$$

This implies that e transforms inversely to a coordinate differential:

$$\tilde{e}(\tilde{\sigma}) = e(\sigma)\frac{d\sigma}{d\tilde{\sigma}}\ ,$$

Note that since e defines a one-dimensional volume element, e does not have zeros. We take $e > 0$ for definiteness, and this condition is preserved under reparametrizations which preserve orientation.

Using the invariant line element, we write down the following action:

$$S[x,e] = \frac{1}{2}\int e d\sigma \left(\frac{1}{e^2}\left(\frac{dx^\mu}{d\sigma}\right)^2 - K \right)\ , \tag{2.47}$$

ActionIII

where K is a real constant.

The action (2.47) has the following symmetries

- $S[x,e]$ is invariant under reparametrisations $\sigma \to \tilde{\sigma}$.
- $S[x,e]$ is invariant under Poincaré transformations $x^\mu \to \Lambda^\mu{}_\nu x^\nu + a^\mu$.

The action depends on the fields $x = (x^\mu)$ and e. Performing the variations $x \to x + \delta x$ and $\to e + \delta e$, respectively, we obtain the following equations of motion.

$$\frac{d}{d\sigma}\left(\frac{x'^{\,\mu}}{e}\right) = 0\ , \tag{2.48}$$

$$x'^{\,2} + e^2 K = 0\ . \tag{2.49}$$

Exercise: Derive the above equations of motion by variation of the action (2.47).

The equation of motion for e is algebraic, and tells us that for $K > 0$ the solution is a time-like curve, while for $K = 0$ it is light-like and for $K < 0$ space-like. To show that the time-like case brings us back to (2.29), we set $K = m^2$

and solve for the auxiliary field e:

$$e = \frac{\sqrt{-\,x'^{\,2}}}{m}\ , \tag{2.50}$$

where we have used that $e > 0$. Substituting the solution for e into (ActionIII 2.47) we recover the action (ActionII 2.29) for a massive particle of mass m.

The advantage of (ActionIII 2.47) is that it includes the case of massless particles as well. Let us consider $K = m^2 \geq 0$. Instead of solving for e we now fix it by 'imposing a gauge', which in this case means picking a particular parametrization.

- For $m^2 > 0$, we impose the gauge

  $$e = \frac{1}{m}\ . \tag{2.51}$$

  The equations of motion become[4]:

  $$\ddot{x}^\mu = 0\ , \tag{2.52}$$

  $$\dot{x}^2 = -1\ . \tag{2.53}$$

  The second equation tells us that this gauge is equivalent to choosing the proper time τ as curve parameter.

- For $m^2 = 0$, we impose the gauge

  $$e = 1\ . \tag{2.54}$$

  The equations of motion become:

  $$\ddot{x}^\mu = 0\ , \tag{2.55}$$

  $$\dot{x}^2 = 0\ . \tag{2.56}$$

  The second equation tells us that the worldline is light-like, as expected for a massless particle. In this case there is no proper time, but choosing $e = 1$ still corresponds to choosing a distinguished curve parameter. Observe that the first, dynamical equation of motion only simplifies to $\ddot{x}^\mu = 0$ if we choose e to be constant. Conversely, the equation $\ddot{x}^\mu = 0$, is only invariant under affine reperamatrizations $\sigma \mapsto a\sigma + b$, $a \neq 0$ of the worldline. Imposing $e = 1$ (or any other constant value) corresponds to

[4] We use a 'dot' to denote the differentiation with respect to the curve parameter selected by our gauge condition.

choosing an affine curve parameter. Since for light-like curves the concept of length or proper time does not exist, choosing an affine curve parameter serves as a subsitute. We can in fact fix the normalization of the affine parameter by imposing that $p^\mu = \dot{x}^\mu$, where p^μ is the momentum of the massless particle. While a relativistic velocity cannot be defined for massless particles, the relativistic momentum vector is defined, and related to measurable quantities.

Remark: Readers who are familiar with field theory will observe that the first term of action (2.47) looks like the action for a one-dimensional free massless scalar field. Readers familiar with general relativity will recognize that the one-dimensional invariant volume element $e\,d\sigma$ is analogous to the D-dimensional invariant volume element $\sqrt{|g|}d^Dx$ appearing in the Einstein-Hilbert action, and in actions describing the coupling of matter to gravity. Since $g = \det(g_{\mu\nu})$ is the determinant of the metric, by analogy we can interpret e as the square root of the determinant of an intrinsic one-dimensional metric on the worldline. A one-dimensional metric has only one component, and a moments thought shows that this component is given by $e^2(\sigma)$. Therefore e^{-2} is the single component of the inverse metric. Taking this into account the first term of (2.47) is the action for a one-dimensional free massless scalar field coupled to one-dimensional gravity. The second term is analogous to a cosmological constant.

Exercise: Work out the case where $m^2 < 0$. What is the geometrical interpretation of the resulting equations?

Remark: The action (2.47) is not only useful in physics, but also in geometry, because it allows to formulate a variational principle which treats null (light-like) curves on the same footing as non-null (time-like and space-like) curves. Actions of the type (2.47) and their higher-dimensional generalizations are known as 'sigma models' in physics. Geometrically (2.47) is an 'energy functional' (sometimes called Dirichlet energy functional) for maps between two semi-Riemannian spaces,[5] the worldline and Minkowski space. Maps which extremize the energy functional are called harmonic maps, and in the specific case of a one-dimensional action functional, geodesic curves. And indeed, we have seen above that the solutions for the Minkowski space are stratight lines in affine

[5]By semi-Riemannian we refer to a manifold equipped with a symmetric and non-degenerate rank 2 tensor field. While we call such tensor fields a metric, we do not require it to be positive, and admit metrics of indefinite signature, such as the Minkowski metric.

parametrization, and therefore geodesics.

Chapter 3

Classical relativistic strings

In this chaper we introduce classical relativistic strings as generalizations of classical relativistic particles. Our treatment mostly follows Chapter 2 of **Green-Schwarz+Witten**, from which we have taken our conventions and notations.

3.1 The Nambu-Goto action

We now generalize relativistic particles to relativistic strings. The worldline of a particle is replaced by the worldsheet Σ, which is a surface embedded into Minkowksi space:

$$X : \Sigma \ni P \longrightarrow X(P) \in \mathbb{M} \,. \tag{3.1}$$

On spacetime we choose linear coordinates associated to a Lorentz frame, denoted $X = (X^\mu)$, where $\mu = 0, 1, \ldots, D-1$. On the worldsheet we choose local coordinates $\sigma = (\sigma^0, \sigma^1) = (\sigma^\alpha)$. Depending on the global structure of the worldsheet, it might not be possible to cover Σ with a single coordinate system. While we can work using local coordinates, we will have to verify that our equations are covariant with respect to reparametrizations. For the time being we do not consider interactions where strings can split or join, so that Σ has the topology of a strip (open strings) or of a cylinder (closed strings). At each point of Σ we can choose a time-like tangent vector ('the direction towards the future') and a space-like tangent vector ('the direction along the string'). We adopt the convention that the coordinate σ^0 is time-like (that is, the corresponding tangent vector ∂_0 is a time-like vector), while the coordinate σ^1 is space-like[1]:

$$\dot{X}^2 \leq 0 \;, \quad (X')^2 > 0 \;. \tag{3.2}$$

Here we use the following notation for tangent vectors:

$$\begin{aligned} \dot{X} &= (\partial_0 X^\mu) = \left(\frac{\partial X^\mu}{\partial \sigma^0} \right) \;, \\ X' &= (\partial_1 X^\mu) = \left(\frac{\partial X^\mu}{\partial \sigma^1} \right) \;. \end{aligned} \tag{3.3}$$

We also make conventional choices for the range of the worldsheet coordinates. The space-like coordinate takes values

$$\sigma^1 \in [0, \pi] \;,$$

where as the time-like coordinate takes values

$$\sigma^0 \in [\sigma^0_{(1)}, \sigma^0_{(2)}] \subset \mathbb{R} \;.$$

The limiting case $\sigma^0 \in \mathbb{R}$ is allowed and describes the asymptotic time evolution of a string from the infinite past to the infinite future.[2]

The Nambu-Goto action is the direct generalization of (ActionII 2.29), and thus proportional to the area of the worldsheet Σ, measured with the metric on Σ induced by the Minkowski metric:

$$S_{\text{NG}}[X] = -TA(\Sigma) = -T \int_\Sigma d^2 A \;. \tag{3.4}$$

The constant T must have the dimension $(\text{length})^{-2}$ or energy/length, in units where $\hbar = 1$, $c = 1$. It is therefore called the string tension. This quantity generalizes the mass of a point particle and is the only dimensionful parameter entering into the definition of the theory.[3]

From differential geometry we know that the Minkowski metric $\eta_{\mu\nu}$ induces a metric $g_{\alpha\beta}$ on Σ by 'pull back':

$$g_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} \;. \tag{3.5}$$

[1] Exceptionally, $\dot{X}$ might become light-like at special points. In particular, we will see that the endpoints of an open string have to move with the speed of light.

[2] While σ^0 is 'worldsheet time', and in general different from Minkowksi time $X^0 = t$, we can use it as time-like coordinate on spacetime.

[3] Note that T is a constant, while the energy and length of a string at a given time depends on its state and motion. We will come back to this point later. **Add cross reference**

From this we can form the invariant area element

$$d^2A = d^2\sigma\sqrt{|\det(g_{\alpha\beta})|}\ , \tag{3.6}$$

with which we measure areas on Σ. Note that $\det(g_{\alpha\beta}) < 0$ since Σ has one time-like and one space-like direction, which is why we have to take the absolut value. Combining the above formulae, the Nambu Goto action is

$$S = -T\int d^2\sigma\sqrt{|\det\left(\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}\right)|}\ . \tag{3.7}$$ NG1

The area, and, hence, the Nambu Goto action, is invariant under reparametrizations of Σ,

$$\sigma^\alpha \to \tilde{\sigma}^\alpha(\sigma^0, \sigma^1)\ , \quad \text{where} \quad \det\left(\frac{\partial\tilde{\sigma}^\alpha}{\partial\sigma^\beta}\right) \neq 0\ . \tag{3.8}$$

Exercise: Verify that the Nambu-Goto action is invariant under reparametrizations.

The action is also invariant under Poincaré transformations of $\mathbb{M}$,

$$X^\mu \to \Lambda^\mu{}_\nu X^\nu + a^\mu\ .$$

This symmetry is manifest since the Lorentz indices are contracted in the correct way to obtain a Lorentz scalar.

To perform calculations, it is useful to write out the action more explicitly:

$$S_{\text{NG}} = \int d^2\sigma\mathcal{L} = -T\int d^2\sigma\sqrt{(\dot{X}\cdot X')^2 - \dot{X}^2(X')^2}\ . \tag{3.9}$$ NG

Here $\mathcal{L}$ is the Lagrangian density, or Langrangian for short.

The worldsheet momentum densities are defined as

$$P^\alpha_\mu = \frac{\partial\mathcal{L}}{\partial\partial_\alpha X^\mu}\ .$$

Evaluating the components explicitly, we find:

$$\begin{aligned}
\Pi_\mu := P^0_\mu := \frac{\partial\mathcal{L}}{\partial\dot{X}^\mu} &= -T\frac{(X')^2\dot{X}_\mu - (\dot{X}\cdot X')X'_\mu}{\sqrt{(\dot{X}\cdot X')^2 - \dot{X}^2(X')^2}}\ , \\
P^1_\mu := \frac{\partial\mathcal{L}}{\partial X'_\mu} &= T\frac{\dot{X}^2 X'_\mu - (\dot{X}\cdot X')\dot{X}_\mu}{\sqrt{(\dot{X}\cdot X')^2 - \dot{X}^2(X')^2}}\ .
\end{aligned} \tag{3.10}$$

Note that $P^0_\mu = \Pi_\mu$ is the canonical momentum (density). While these expressions look complicated, a more intuitive form is obtained if we assume that we can choose coordinates such that $X' \cdot \dot{X} = 0$, $\dot{X}^2 = -1$, $(X')^2 = 1$, which we can always do at least at a given point:[4]

$$(P^\alpha_\mu) = T(\dot{X}_\mu \,, -X'_\mu) = T(-\partial^0 X_\mu \,, -\partial^1 X_\mu) \,. \tag{3.11}$$

This can be viewed as generalizing the 'momentum flow' $p^\mu = m\dot{x}^\mu$ along the worldline of a relativistic particle.

The equations of motion are found by imposing that the action (3.9) is invariant under variations $X \to X + \delta X$, subject to the condition that the initial and final positions of the string are kept fixed: $\delta X(\sigma^0 = \sigma^0_{(1)}) = 0$, $\delta X(\sigma^0 = \sigma^0_{(2)}) = 0$. When carrying out the variation we can simplify our work by using the expressions (3.10):

$$\delta S = \int d^2\sigma \left(P^0_\mu \delta \dot{X}^\mu + P^1_\mu \delta X'^\mu \right) \,.$$

We need to perform an integration by parts which creates two boundary terms:

$$\delta S = \int_0^\pi d\sigma^1 \left[P^0_\mu \delta X^\mu \right]^{\sigma^0_{(2)}}_{\sigma^0_{(1)}} + \int_{\sigma^0_{(1)}}^{\sigma^0_{(2)}} d\sigma^0 \left[P^1_\mu \delta X^\mu \right]^{\sigma^1=\pi}_{\sigma^1=0} - \int d^2\sigma \partial_\alpha P^\alpha_\mu \delta X^\mu \,.$$

The first boundary vanishes because the variational principle imposes to keep the initial and final position fixed. However the second boundary term does not vanish automatically, and to have a consistent variational principle we have to impose that

$$\delta S = \int d\sigma^0 \left[P^1_\mu \delta X^\mu \right]^{\sigma^1=\pi}_{\sigma^1=0} \stackrel{!}{=} 0 \,. \tag{3.12}$$

These conditions are satisfied by imposing boundary conditions, for which we have the following options:

1. Periodic boundary conditions:

$$X(\sigma^1) = X(\sigma^1 + \pi) \,. \tag{3.13}$$

   This corresponds to closed strings, where the worldsheet can only have space-like boudaries corresponding to the initial and final configuration.

2. Neumann boundary conditions:

$$P^1_\mu \big|_{\sigma^1=0,\pi} = \left. \frac{\partial \mathcal{L}}{\partial X'_\mu} \right|_{\sigma^1=0,\pi} = 0 \,. \tag{3.14}$$

[4]See Zwiebach's book for a detailed account of how to construct coordinates systems. **Add reference**

Since P^1_μ, evaluated at $\sigma^1 = 0, \pi$ is the component of the worldsheet momentum density normal to the boundary, Neumann boundary conditions imply momentum conservation at the ends of the string. They describe open strings whose ends can move freely. As we will see in a later exercise, the ends of an open string always move with the speed of light.

3. Dirichlet boundary conditions. For space-like directions $i = 1, \ldots, D-1$ we can impose Dirichlet boundary conditions,

$$P^0_i\big|_{\sigma^1=0,\pi} = \frac{\partial \mathcal{L}}{\partial \dot{X}_i}\bigg|_{\sigma^1=0,\pi} = 0 \; .$$

Since this implies that the tangential component of the worldsheet momentum vanishes at the boundary, it correponds to keeping the ends of the string fixed in the i-th direction:

$$X^i(\sigma^1 = 0) = x_0 \; , \quad X^i(\sigma^1 = \pi) = x_1 \; . \tag{3.15}$$

In this case momentum is not conserved at the ends of the string, as expected since the boundary conditions break translation invariance. To restore momentum conservation one must couple open strings which have Dirichlet boundary conditions to new types of dynamical objects, called D-branes.[5]

Remark: D-branes. Since we can impose Neumann boundary conditions along some directions and Dirichlet boundary conditions in others, there are D-branes of various dimensions. If Neumann boundary conditions are imposed in p space-like directions and as well in time, and Dirichlet boundary conditions in the remaining $D-p-1$ space-like directions, then the resulting D-branes is referred to as a D$-p-$brane= Dirichlet p-brane. 'Space-filling' D-branes with $p = D-1$ correspond to imposing Neumann boundary conditions in all directions, while D-0-branes are 'D-particles' and D-1-branes are D-strings, etc. D-branes are interpreted as 'collective excitations' or 'solitons' of string theory. Imposing Dirichlet condition in time makes sense if we Wick rotate to imaginary time and consider strings in a Euclidean 'spacetime.' In this context D-branes are interpreted a instantons, that is as configurations with stationary and finite Euclidean action, which give rise to 'non-perturbative' contributions to physical quantities. In Euclidean space we can in particular impose Dirichlet boundary

[5] Which raises the question whether string theory is a theory of strings, only.

conditions in all directions and thus obtain $D-(-1)$ branes which are called D-instantons.

By imposing any of the above boundary conditions we can cancel the boundary term in the variation of the action. The vanishing of the remaining, non-boundary terms implies the following equations of motion:

$$\partial_\alpha P^\alpha_\mu = 0 \; . \tag{3.16}$$

While, given (3.10), this looks very complicated, it becomes the two-dimensional wave equation upon choosing coordinates where $\dot{X}\cdot X' = 0$, $\dot{X}^2 = -1$, $X'^2 = 1$. Instead of showing how such a coordinate system can be constructed **Add reference to Zwiebach's book or Scherk's review**, we will derive this result in a different way using the Polyakov action later **Add cross reference**.

One thing to note is that the canonical momenta are not independent. We find two constraints:

$$\begin{aligned} \Pi^\mu X'_\mu &= 0 \; , \\ \Pi^2 + T^2 (X')^2 &= 0 \; . \end{aligned} \tag{3.17}$$

The canonical Hamiltonian (density) is obtained from the Lagrangian by a Legendre transformation:

$$\mathcal{H}_{\text{can}} = \dot{X}\Pi - \mathcal{L} = 0 \; . \tag{3.18}$$

As for the relativistic particle, the Hamiltonian is not equal to the energy (or here energy density), and is weakly zero.

Problem: Verify that that the canonical momenta are subject to the two constraints given above. Show that the Hamiltonian constraint follows from the other two constraints.

3.2 The Polyakov action

3.2.1 Action, symmetries, equations of motion

The Polyakov action is related to the Nambu-Goto action in the same way as the point particle action (2.47) is related to (2.29). That is, we replace the area by the corresponding energy functional, or, in physical terms, sigma model.

This requires to introduce an intrinsic metric $h_{\alpha\beta}(\sigma)$ on the worldsheet Σ. The Polyakov action is

$$S_{\mathrm{P}}[X,h] = -\frac{T}{2}\int d^2\sigma\sqrt{h}h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}\ , \tag{3.19}$$

PolAct

where $h = -\det(h_{\alpha\beta}) = |\det(h_{\alpha\beta})|$, generalizes the auxiliary field e in (ActionIII 2.47). The intrinsic metric $h_{\alpha\beta}$ is a priori unrelated to induced metric $g_{\alpha\beta}$, but we require that $h_{\alpha\beta}$ has the same signature $(-+)$, because we interprete Σ as the worldsheet of a relativistic string. The embedding $X\ :(\Sigma, h_{\alpha\beta}) \to (\mathbb{M}, \eta_{\mu\mu})$ is now a map between two semi-Riemannian manifolds.

The Polyakov action has the following local symmetries with respect to the worldhsheet Σ:

1. Reparametrizations $\sigma \to \tilde{\sigma}(\sigma)$, which act by

$$\begin{aligned} \tilde{X}^\mu(\tilde{\sigma}) &= X^\mu(\sigma)\ , \\ \tilde{h}_{\alpha\beta}(\tilde{\sigma}) &= \frac{\partial\sigma^\gamma}{\partial\tilde{\sigma}^\alpha}\frac{\partial\sigma^\delta}{\partial\tilde{\sigma}^\beta}h_{\gamma\delta}(\sigma)\ . \end{aligned} \tag{3.20}$$

2. Weyl transformations:

$$h_{\alpha\beta}(\sigma) \to e^{2\Lambda(\sigma)}h_{\alpha\beta}(\sigma)\ . \tag{3.21}$$

Remarks:

1. Note that Weyl transformations do not act on the coordinates and are therefore different from reparametrizations. Mathematicians usually call them conformal transformations, because they change the metric but preserve the conformal structure of $(\Sigma, h_{\alpha\beta})$.

2. The invariance of the action under Weyl transformation is special for strings. One can write down Polyakov-type actions for particles, membranes and higher-dimensional p-branes, but they are not Weyl invariant.

3. Combining Weyl with reparametrization invariance, one has three local transformations which can be used to gauge-fix the metric $h_{\alpha\beta}$ completely. Thus $h_{\alpha\beta}$ does not introduce new local degrees of freedom: it is an auxiliary field.

Apart from these local symmetries we also have global symmetries, namely the invariance of the Polyakov action under Poincaré transformations on spacetime. This symmetry is manifest, since X^μ are coordinates adapted to an inertial frame, and all Lorentz indices are properly contracted.

The equations of motion are obtained by imposing stationarity with respect to the variations $X^\mu \to X^\mu + \delta X^\mu$ and $h_{\alpha\beta} \to h_{\alpha\beta} + \delta h_{\alpha\beta}$.

Exercise: Show that resulting equations of motion are:

$$\frac{1}{\sqrt{h}}\partial_\alpha\left(\sqrt{h}h^{\alpha\beta}\partial_\beta X^\mu\right) = 0 \, , \tag{3.22}$$

$$\partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2}h_{\alpha\beta}h^{\gamma\delta}\partial_\gamma X^\mu \partial_\delta X_\mu = 0 \, . \tag{3.23}$$

As for the Nambu-Goto action, we have to make sure that boundary terms vanish, which gives us the choice between periodic, Neumann and Dirichlet boundary conditions.

The X-equation (3.22) is the two-dimensional wave equation on the semi-Riemannian manifold $(\Sigma, h_{\alpha\beta})$. This can be rewritten in various ways:

$$\Box X^\mu = 0 \Leftrightarrow \nabla_\alpha \nabla^\alpha X^\mu = 0 \Leftrightarrow \nabla_\alpha \partial^\alpha X^\mu = 0 \, , \tag{3.24}$$

where ∇_μ is the covariant derivative with respect to the worldsheet metric $h_{\alpha\beta}$. The h-equation (3.23) is algebraic and can be used to eliminate $h_{\alpha\beta}$ in terms of the induced metric $g_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}$. More precisely:

Exercise: Show that (3.23) implies

$$\det(g_{\alpha\beta}) = \frac{1}{4}\det(h_{\alpha\beta})(h^{\gamma\delta}g_{\gamma\delta})^2 \, . \tag{3.25}$$

Use this to show that upon imposing (3.23) the intrinsic and induced metric are conformally equivalent, that is, they differ only by a Weyl transformation. By substituting back into the Polyakov action you will obtain the Nambu-Goto action.

3.2.2 Interpretation as a two-dimension field theory

The advantage of the Polyakov action is that it does not involve a square root and takes the form of a standard two-dimensional field theory action for free massless scalar fields. This allows us to take an alternative point of view and to interpret Σ as a two-dimensional 'spacetime', populated by D scalar fields $X = (X^\mu)$, which take values in the 'target space' $\mathbb{M}^D$. We can now use methods, results and intution from field theory. We will call this point of view the 'worldsheet perspective' in contrast to the 'spacetime perspective' where ActionIII (2.47) is interpreted in terms of a string in Minkowksi space.

When studying a field theory on a semi-Riemannian manifold, one defines the energy momentum tensor of an action by its variation with respect to the metric.[6]

The energy-momentum tensor of the Polyakov action is:

$$T_{\alpha\beta} := -\frac{1}{T}\frac{1}{\sqrt{h}}\frac{\delta S_P}{\delta h^{\alpha\beta}} = \frac{1}{2}\partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{4} h_{\alpha\beta} h^{\gamma\delta} \partial_\gamma X^\mu \partial_\delta X_\mu \ . \tag{3.26}$$

Expressing the h-equation of motion in terms of $T_{\alpha\beta}$ gives

$$T_{\alpha\beta} = 0 \ . \tag{3.27}$$

Two comments are in order. Firstly this is a constraint, not a dynamical equation, and we will show later that it can be imposed as an initial condition. Secondly, this equation resembles the Einstein equation, which in four dimensions takes the form $R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa^2 T_{\mu\nu}$, where $g_{\mu\nu}$ is the metric, $R_{\mu\nu}$ the Ricci tensor, R the Ricci scalar and κ the gravitational coupling constant. The Einstein Hilbert action which generates the l.h.s. of this equation is $S_{EH} = \frac{1}{\kappa^2}\int d^4x \sqrt{|g|} R$. Could we modify the Polyakov action by adding a two-dimensional Einstein-Hilbert term? The answer is that adding an Einstein-Hilbert term does not change the theory, at least as far as the local dynamics is concerned: variation of the two-dimensional Einstein-Hilbert action results in a total derivative, and therefore does not contribute to the equations of motion. In other words, $T_{\alpha\beta} = 0$ is already the two-dimensional Einstein equation.

If we consider the worldsheet Σ globally, something more interesting happens. While the Einstein-Hilbert action does not change under small variations of the metric, it need not be zero, and turns out to be a topological invariant of the worldsheet, the so-called Euler number. We will come back to this later. **Add crorss reference.**

[6] This can also be applied to field theories on a flat spacetime, by introducing a background metric that can then be varied. An alternative defintion on a flat spacetime is through the Noether theorem. The resulting energy momentum tensor is in general not symmetric, and may differ from the one obtained by variation of the metric by a total derivative. However, both types of energy momentum tensors are physically equivalent, i.p. one obtains the same conserved quantities by integration over space **Refer to literature and/or later discussion.**

Let us now investigate the properties of the energy momentum tensor. On a flat worldsheet $T_{\alpha\beta}$ is conserved (has a vanishing divergence). By the equivalence principle[7] we expect it therefore to be covariantly conserved on $(\Sigma, h_{\alpha\beta})$.

Exercise: Show that $T_{\alpha\beta}$ is covariantly conserved:

$$\nabla^\alpha T_{\alpha\beta} = 0 \, , \tag{3.28}$$ NablaT

on shell, that is, modulo the equations of motion.

Exercise: Show that $T_{\alpha\beta}$ is traceless:

$$h^{\alpha\beta} T_{\alpha\beta} = 0 \, . \tag{3.29}$$

This relation holds off shell, that is, without imposing the equations of motion. Since $T_{\alpha\beta}$ is symmetric and traceless, it only has two independent components.

Remark: Note that the trace of a tensor is always defined by contraction with the metric. This is in general different from taking the trace in the matrix sense, which would not be reparametrization invariant.

3.2.3 The conformal gauge

One way to use symmetries is to impose 'gauge conditions' which bring expressions to a standard form. The Polyakov action has three local symmetries: the reparametrizations of two coordinates and Weyl transformations. Since the metric $h_{\alpha\beta}$ has three independent components, counting suggests that we can fix its form completely, in particular that we can impose the so-called conformal gauge where it takes the form of the standard two-dimensional Minkowski metric:

$$h_{\alpha\beta} \stackrel{!}{=} \eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \, . \tag{3.30}$$ ConfGauge

This is indeed true locally, because any two-dimensional semi-Riemannian metric is conformally flat and can be written locally as the product of the standard flat metric with a conformal factor. Since we can apply Weyl transformations

[7] Here we refer to Einstein's equivalance principle, which states that if we go to a freely falling reference frame (corresponding to Riemannian normal coordinates), physics on a curved spacetime takes the same form as in special relativity, at the given point (the point relative to which we have introduced Riemannian normal coordinates = the origin of the freely falling frame).

in addition to reparametrizations, the conformal factor can be removed, leaving us with the standard flat metric:

$$h_{\alpha\beta} \to e^{2\Omega(\sigma)}\eta_{\alpha\beta} \to \eta_{\alpha\beta} \ .$$

Globally the story is more complicated, since the reparametrizations needed for the first step do not need to exist. **Add cross or reference to literature.**

We now work out various useful formulas for the Polyakov string in the conformal gauge. Some care is required when imposing gauge conditions on the action itself rather than the equations of motion. In the present case, substituting the gauge condition (3.30) into the Polyakov action gives.

ConfGauge

$$S_P \quad = \quad -\frac{T}{2}\int d^2\sigma \eta^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu \ . \tag{3.31}$$

Variation of this action with respect to X gives indeed the correct equation of motion, namely the gauge fixed version of the previous X-equation:

$$\Box X^\mu = -(\partial_0^2 - \partial_1^2)X^\mu = 0 \tag{3.32}$$

This is the standard 'flat' two-dimensional wave equation, which is known to have the general solution:

$$X^\mu(\sigma) = X^\mu_L(\sigma^0 + \sigma^1) + X^\mu_R(\sigma^0 - \sigma^1) \ , \tag{3.33}$$

2dWaveGeneral

describing decoupled left- and right-moving waves.

However, not all solutions of the two-dimensional wave equations are solutions of string theory. First of all, we have to impose boundary conditions. In the conformal gauge, the consistent boundary conditions take the following form:

$$\begin{aligned} X^\mu(\sigma^1 + \pi) = X^\mu(\sigma) \ , &\qquad \text{periodic} \ , \\ X'_\mu\Big|_{\sigma^1 = 0,\pi} = 0 \ , &\qquad \text{Neumann} \ , \\ \dot{X}_\mu\Big|_{\sigma^1 = 0,\pi} = 0 \ , &\qquad \text{Dirichlet} \ . \end{aligned} \tag{3.34}$$

Morever, the equations coming from the h-variation of the Polyakov action must now be added by hand:

$$T_{\alpha\beta} = 0 \ . \tag{3.35}$$

The energy momentum tensor is traceless

$$\text{Trace}(T) = T^\alpha{}_\alpha = \eta^{\alpha\beta}T_{\alpha\beta} = -T_{00} + T_{11} = 0 \ , \tag{3.36}$$

and since this holds off-shell we only have two non-trivial constraints:

$$\begin{aligned} T_{01} = T_{10} &= \frac{1}{2}\dot{X}X' = 0 \;, \\ T_{00} = T_{11} &= \frac{1}{4}(\dot{X}^2 + X'^2) = 0 \;. \end{aligned} \tag{3.37}$$

In the Hamiltonian formulation of the Nambu-Goto action, constraints arose from relations between the canonical momenta. For the Polyakov action, the canonical momenta are:

$$\Pi^\mu = \frac{\partial \mathcal{L}_P}{\partial \dot{X}_\mu} = T\dot{X}^\mu \;, \tag{3.38}$$

and the canonical Hamiltonian is:

$$H_{\rm can} = \int_0^\pi d\sigma^1 \left(\dot{X}\Pi - \mathcal{L}_P \right) = \frac{T}{2}\int_0^\pi d\sigma^1 \left(\dot{X}^2 + X'^2 \right) \;. \tag{3.39}$$

Thus $T_{00} = T_{11} = 0$ implies that the Hamiltonian vanishes on shell.

Exercise: Compute the worldsheet momentum densities $P^\alpha_\mu = \partial\mathcal{L}/\partial(\partial_\alpha X^\mu)$ of the Polyakov action in the conformal gauge. Using that $\Pi_\mu = P^0_\mu$ is the canonical momentum, show that the constraints (3.37) are equivalent to the constraints (3.17).

Remark: By comparing how the constraints arise for the Nambu-Goto and Polyakov action, we see that in the first case they follow directly from the definition of the canonical momenta, while in the second case they arise through the equation of motion of an auxiliary field. This is a specal case of the distinction between primary and secondary constraints **Add Reference Dirac or other constrained dynamics text**. This distinction is independent of the one between first and second class constraints. Our constraints are first class in both cases.

In field theory on a flat spacetime, invariance under translations implies energy-momentum conservation. When applying this to worldsheet field theory, the conserved current associated with shifts in σ^β is $T^{\alpha\beta}$. [8]

Exercise: Verify that

$$\partial_\alpha T^{\alpha\beta} = 0 \;, \tag{3.40}$$

holds on-shell.

[8] Since our world sheet is not global Minkowski space but, for the propagation of a single string, an infinite strip or cylinder, this discussion will be modified later to include boundary conditons.

Remark: This is the gauge-fixed version of the covariant conservation equation (3.28)

We have thus verified that as long as we impose the gauge fixed h-equation by hand, we can obtain the gauge-fixed versions of all other equations by variation of the gauge-fixed action. In general one needs to be careful when imposing conditions directly on the action instead of the field equations, as such conditions need not be consistent with the variational principle. In the above example we just had to add one field equation, which is a non-dynamical constraint, by hand. In other cases, for example when performing dimensional reductions of actions, it can happen that the truncation of the action is not consistent, meaning that solutions to the Euler-Langange equations of the truncated actions are not solutions to the Euler-Lagrange equations of the original action.

3.2.4 Lightcone coordinates

Equation (3.33) suggests to introduce lightcone coordinates (also called null coordinates):

$$\sigma^{\pm} := \sigma^0 \pm \sigma^1 \; . \tag{3.41}$$

We adopt a convention where we write σ^a, with $a = +, -$ for lightcone coordinates and σ^α, with $\alpha = 0, 1$ for non-null coordinates.

To relate quantities in both types of coordinate systems, we compute the Jacobian of the coordinate transformation and its inverse:

$$(J_\alpha{}^a) = \frac{D(\sigma^+, \sigma^-)}{D(\sigma^0, \sigma^1)} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \; , \quad (J_a{}^\alpha) = \frac{D(\sigma^0, \sigma^1)}{D(\sigma^+, \sigma^-)} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \; . \tag{3.42}$$

Tensor indices are converted using these Jacobians. In particular, the lightcone expressions for coordinate differentials and derivatives are:

$$d\sigma^{\pm} = d\sigma^0 \pm d\sigma^2 \; , \quad \partial_{\pm} = \frac{1}{2} \left(\partial_0 \pm \partial_1 \right) \; . \tag{3.43}$$

For reference, we note that the standard Minkowski metric $\eta_{\alpha\beta}$ takes following form in lightcone coordinates:

$$(\eta_{ab}) = -\frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \; , \quad (\eta^{ab}) = -2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \; . \tag{3.44}$$

The lightcone components of the energy-momentum tensor are:

$$T_{++} = \frac{1}{2}(T_{00} + T_{01}) \; , \quad T_{--} = \frac{1}{2}(T_{00} - T_{01}) \; , \quad T_{+-} = 0 = T_{-+} \; . \tag{3.45}$$

Note that the trace, evaluated in lightcone coordinates, is:

$$\mathrm{Trace}(T) = \eta^{ab}T_{ab} = 2\eta^{+-}T_{+-} = -4T_{+-} \ . \tag{3.46}$$

Thus ‘tracelessness’ means $T_{+-} = 0$, and the two independent components are T_{++} and T_{--}.

We can write the gauge-fixed action in lightcone coordinates:

$$\begin{aligned} S_P &= -\frac{T}{2}\int d^2\sigma \eta^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu \\ &= \frac{T}{2}\int d^2\sigma = (\dot{X}^2 - X'^2) = 2T\int d^2\sigma \partial_+ X^\mu \partial_- X_\mu \ . \end{aligned} \tag{3.47}$$

For reference, the equations of motion take the form

$$\Box X^\mu = -(\partial_0^2 - \partial_1^2)X^\mu = -4\partial_+\partial_- X^\mu = 0 \ . \tag{3.48}$$

Thus, in lightcone coordinates it is obvious that the general solution decomposes into independent left- and right-moving waves with arbitrary profile:

$$X^\mu(\sigma) = X_L^\mu(\sigma^0 + \sigma^1) + X_R^\mu(\sigma^0 - \sigma^1) \ . \tag{3.49}$$

We also give the constraints in lightcone coordinates:

$$\begin{aligned} T_{++} = 0 \quad &\Leftrightarrow \quad \partial_+ X^\mu \partial_+ X_\mu = 0 \Leftrightarrow \dot{X}_L^2 = 0 \ , \\ T_{--} = 0 \quad &\Leftrightarrow \quad \partial_- X^\mu \partial_- X_\mu = 0 \Leftrightarrow \dot{X}_R^2 = 0 \ . \end{aligned} \tag{3.50}$$

We did not list $T_{+-} = 0$, because this condition holds off shell.

`conservation_laws`

3.2.5 From symmetries to conservation laws

One of the most central insights in physics is that symmetries, to be precise, rigid symmetries, lead to conservation laws. The formal statement of this relation is the famous first Noether theorem, which shows that given an action with a rigid symmetry, we can construct a ‘conserved current,’ from which in turn a ‘conserved charge’ is obtained by integration of the current over a space-like hypersurface. We will not explain this theorem in generality, but refer the interested reader to the literature **add references**. Instead we will go through two instructive and relevant examples. Before starting, let us emphasize that Noether’s theorem can be used for both spacetime symmetries (related to group actions on spacetime) and internal symmetries (related to group actions on vector or other bundles over spacetime). In our situation this terminology is potentially ambigous, because symmetries that are spacetime symmetries from the spacetime point of view are internal symmetries from the worldsheet point of view. In the following it will usually be clear from the context what we mean.

Conformal transformations on Σ

When imposing the conformal gauge $h_{\alpha\beta} \stackrel{!}{=} \eta_{\alpha\beta}$ we made use of both reparametrizations and Weyl transformations. However, we have not completely gauge-fixed these symmetries, because some reparametrizations only change the metric by a conformal factor, and can therefore be undone by a compensating Weyl transformation:

$$\eta_{\alpha\beta} \to e^{2\Omega(\sigma)}\eta_{\alpha\beta} \to \eta_{\alpha\beta} \ .$$

We will refer to reparametrizations which have this restricted form as 'conformal transformations,'[9] and to the gauge-fixed theory as 'conformally invariant.' In a later exercise **Add cross reference or move exercise forward to this position.** we will show that conformal transformations take the following form in light cone coordinates:

$$\sigma^+ \to \tilde{\sigma}^+(\sigma^+) \ , \quad \sigma^- \to \tilde{\sigma}^-(\sigma^-) \ .$$

That is, conformal transformations are precisely those reparametrizations which do not mix the lightcone coordinates.

We have already verified that the energy momentum tensor is conserved, $\partial^\alpha T_{\alpha\beta} = 0$. In lightcone coordinates, this statement reads

$$\partial_- T_{++} = 0 \ , \quad \partial_+ T_{--} = 0 \ ,$$

which implies that each independent component only depends on one of the lightcone coordinates,

$$T_{++} = T_{++}(\sigma^+) \ , \quad T_{--} = T_{--}(\sigma^-) \ .$$

Therefore our conserved current $T_{\alpha\beta}$ decomposes into two 'chiral conserved currents.' The decoupling of left- and right-moving quantities, which we have

[9] This terminology is common in physics while in mathematics what we call Weyl transformations are often called conformal transformations.

already seen in other expressions before, is characteristic for massless two-dimensional theories, and for strings. Given this chiral decomposition, it is sufficient to look at one chiral current, say T_{++}.

Since we are interested in string theory rather than in two-dimensional scalar field theory on global Minkowski space, we need to impose boundary conditions. Let us choose periodic boundary conditions for definiteness. This implies that T_{++} is periodic, $T_{++}(\sigma^+ + \pi) = T_{++}(\sigma^+)$. While integration of T_{++} over σ^1 gives indeed a conserved charge, we might note at this point that we can create further conserved currents by multiplying T_{++} by an arbitrary (smooth) periodic function $f(\sigma^+)$ since

$$\partial_- \left(f(\sigma^+) T_{++} \right) = 0 \ .$$

We claim that the corresponding conserved charge is:

$$L_f = T \int_0^\pi d\sigma^1 f(\sigma^+) T_{++} \ .$$

Exercise: Show that L_f is a conserved charge, i.e. verify that

$$\frac{d}{d\sigma^0} L_f = 0 \ .$$

Since $f(\sigma^+)$ is periodic, we can expand it in a Fourier series. The Fourier basis $\{e^{2im\sigma^+} | m \in \mathbb{Z}\}$ then provides us with a basis $\{\tilde{L}_m | m \in \mathbb{Z}\}$ for the conserved charges, where

$$\tilde{L}_m = T \int_0^\pi d\sigma^1 e^{2im\sigma^1} T_{++} \ .$$

Exercise: Why can we write $e^{2im\sigma^1}$ instead of $e^{2im\sigma^+}$ in the above formula?

By repeating the same steps for T_{--} we obtain a second infinite set of conserved charges,

$$L_m = T \int_0^\pi d\sigma^1 e^{-2im\sigma^1} T_{--} \ .$$

Using the conserved charges, we can rewrite the constraints $T_{++} = 0 = T_{--}$ as

$$L_m = 0 = \tilde{L}_m \ .$$

These equations hold on-shell. Since we have shown that $L_m, \tilde{L}_m$ are conserved we have now shown that we can impose the constraints as initial conditions: if they hold at one (worldsheet) time σ^0, they hold for all times. This verifies our previous claims that the constraints are 'first class', that is consistent with time evolution and therefore easier to deal with as other types of constraints.

Exercise: Repeat this analysis for open strings, and show that there is only one infinite set of conserved charges:

$$L_m = 2T \int_0^\pi d\sigma^1 \left(e^{im\sigma^1} T_{++} + e^{-im\sigma^1} T_{--} \right) \ .$$

Hint: with open string boundary conditions, left- and right-moving waves are no longer independent, but related at the boundary. This explains why there is only one set of charges. The derivation can be simplified by formally combining left- and right moving quantities into a single quantity which is periodic on the doubled interval $\sigma^1 \in [-\pi, \pi]$.

Poincaré Transformations on $\mathbb{M}$. Momentum and angular momentum of the string

The Polyakov action is invariant under global Poincaré transformations of M:

$$X^\mu \to \Lambda^\mu{}_\nu X^\nu + a^\mu \ . \tag{3.51}$$

Instead of invoking the Noether theorem, we use a short-cut, dubbed the Noether trick to identify the corresponding conserved currents. For simplicity, we focus on translations $X^\mu \to X^\mu + a^\mu$. This is a rigid symmetry transformation, but consider what happens when we 'gauge it', that is, when we promote the transformation paramater a^μ to ta function $a^\mu(\sigma)$ on the worldsheet. To avoid confusion, let us stress that we have to apply the Noether trick from the world-sheet perspective, because we are given an action on Σ. From this viewpoint, translations in $\mathbb{M}$ are internal symmetries. An infinitesimal 'gauged' translation acts by $\delta X^\mu = \delta a^\mu(\sigma)$. This is no longer a symmetry of the action, but we know that the action becomes invariant if we restrict to rigid translations. Therefore the infinitesimal variation of the action must take the form

$$\delta S = \int d^2\sigma \partial_\alpha a^\mu P^\alpha_\mu \ . \tag{3.52}$$

Integration by parts gives

$$\delta S = -\int d^2\sigma a^\mu \partial_\alpha P^\alpha_\mu \ . \tag{3.53}$$

More over, once we impose the equations of motion, the action is stationary with respect to all variations, including gauged translations. Hence the current P^α_μ must be conserved on shell:

$$\partial_\alpha P^\alpha_\mu = 0 \ . \tag{3.54}$$

P^α_μ, with μ fixed, is the conserved current on Σ associated with translations in the μ-direction of $\mathbb{M}$, in other words, the momentum density along the μ-direction.

Remark: While above we assumed the conformal gauge, the method works without gauge fixing. We quote the result for the conserved current:

$$P^\alpha_\mu = -T\sqrt{h}h^{\alpha\beta}\partial_\beta X_\mu \stackrel{c.g.}{=} -T\partial^\alpha X_\mu \; , \tag{3.55}$$

In the last step we imposed the conformal gauge to check consistency with the derivation above.

To find the angular momentum density, we have to follow the same procedure for Lorentz transformations in $\mathbb{M}$. The result for the conserved current is

$$J^\alpha_{\mu\nu} = X_\mu P^\alpha_\nu - X_\nu P^\alpha_\mu \tag{3.56}$$

The associated conserved charges are obtained by integration of the time-like component of the current along any space-like hypersurface $\sigma^0 = \text{const.}$ of Σ:

$$\begin{aligned} P_\mu &= \int_0^\pi d\sigma^1 P^0_\mu = T\int_0^\pi d\sigma^1 \dot{X}_\mu \\ J_{\mu\nu} &= \int_0^\pi d\sigma^1 J^0_{\mu\nu} = T\int_0^\pi d\sigma^1 \left(X_\mu \dot{X}_\nu - X_\nu \dot{X}_\mu \right) \end{aligned} \tag{3.57}$$

Remark: These charges are conserved on shell by construction. You can verify this using the general Stokes theorem, which converts a volume integral into a surface integral. You'll have to assume that the currents vanish at space-like infinity (if you take Σ to be global Minkowski space), or use the boundary conditions. Alternatively you can check directly that $dP_\mu/d\sigma^0 = 0$, $dJ_{\mu\nu}/d\sigma^0 = 0$.

The quantities P_μ and $J_{\mu\nu}$ are the total relativistic momentum of the string. This interpretation is already justified by the fact they are the conserved charges associated to translations and Lorentz transformations.

3.2.6 Explicit solutions

So far we have extracted information without solving the equations of motion explicitly. We now turn to this remaining problem. We need to select those solutions of the wave equation which satisfy the constraints and one of the possible boundary conditions. Let us start with the boundary conditions, and consider periodic boundary conditions for definiteness. Since the solution is periodic in σ^1, it is clear that it must take the form

$$X^\mu(\sigma) = a^\mu + b^\mu\sigma^0 + \sum_{n\neq 0} c^\mu_n e^{-2in\sigma^-} + \sum_{n\neq 0} d^\mu_n e^{-2in\sigma^+} \ ,$$

where $a^\mu, b^\mu \in \mathbb{R}$ and $(c^\mu_n)^* = c^\mu_{-n}$ and $(d^\mu_n)^* = d^\mu_{-n}$, since X^μ is real.

Note that the term linear in σ^0 is allowed by the boundary conditons and solves the wave equation. To see explicity that X^μ splits into left- and right moving parts, note that $p^\mu\sigma^0 = \frac{1}{2}p^\mu(\sigma^+ + \sigma^-)$.[10]

The conventional parametrization is string theory looks somewhat different:

$$X^\mu(\sigma) = x^\mu + \frac{1}{\pi T}p^\mu\sigma^0 + \frac{i}{2}\sqrt{\frac{1}{\pi T}}\sum_{n\neq 0}\frac{1}{n}\alpha^\mu_n e^{-2in\sigma^-} + \frac{i}{2}\sqrt{\frac{1}{\pi T}}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}^\mu_n e^{-2in\sigma^+} \ , \tag{3.58}$$

X-solution

where $x^\mu, p^\mu \in \mathbb{R}$ and $(\alpha^\mu_n)^* = \alpha^\mu_{-n}$ and $(\tilde{\alpha}^\mu_n)^* = \tilde{\alpha}^\mu_{-n}$. We now explain why this slightly complicated looking parametrization is convenient.

Let us first compute the total momentum:

$$P^\mu = T\int_0^\pi d\sigma^1 \dot{X}^\mu = p^\mu \ .$$

Thus the coefficient p^μ of the term proportional to σ^1 is the total momentum. Next, compute the motion of the center of mass

$$x^\mu_{CM} = \frac{1}{\pi}\int_0^\pi d\sigma^1 X^\mu(\sigma) = x^\mu + p^\mu\sigma^0 \ .$$

For time-like p^μ we can match this with the worldline of a massive relativistic particle,

$$x^\mu(\tau) = x^\mu(0) + \frac{dx^\mu}{d\tau}(0)\tau \ ,$$

and conclude that the center of mass of the string behaves like a relativistic particle and moves on a straight line on Minkowski space. Since $p^\mu = m\dot{x}^\mu$, worldsheet time σ^0 and proper time τ are related by $\sigma^0 = m^{-1}\tau$. The mass of the string is give by $P^\mu P_\mu = p^\mu p_\mu = -m^2$, and we will work out explicit expressions later.

Exercise: Work out the modifications needed to decribe the center of mass motion of massless string states. **Must reemember to solve this myself...**

[10]While this is less canonical, the constant part a^μ is usually split symmetrically.

We have thus seen that the motion of a string decomposes into two parts. The zero mode part, which describes the motion of its center of mass behaves like a relativistic particle. The remaining terms, which describe the various possible excitations correspond, for periodic boundary conditions, to left- and rightmoving waves.

X-solution

The string tension T appears explicitly in (3.58) and many other equations. It is convenient to use so-called string units where one sets

$$T = \frac{1}{\pi} \quad \text{in addition to} \quad c = 1 \;, \hbar = 1 \;.$$

This is an example of a system of units where all quantities are measured in multiples of fundamental constants. Another example are Planck units where $G_N = 1$, $\hbar = 1$, $c = 1$. We will discuss the relation between both systems later. **Add cross reference.**

Using our explicit solution, we can now evaluate the expressions found previously for the conserved charges associated with conformal transformations. Observe that $L_m, \tilde{L}_m$ are the Fourier components of $T_{\pm\pm}$, which we can evaluate at any worldsheet time σ^0, because they are conserved. Evaluating the Fourier components of

$$T_{\pm\pm} = \frac{1}{2}(\partial_\pm X)^2 \tag{3.59}$$

at $\sigma^0 = 0$ we find:

$$\begin{aligned} L_m &:= T\int_0^\pi d\sigma^1 \; e^{-2im\sigma^1} T_{--} = \frac{\pi T}{2} \sum_{n=-\infty}^{\infty} \alpha_{m-n} \cdot \alpha_n \;, \\ \tilde{L}_m &:= T\int_0^\pi d\sigma^1 \; e^{2im\sigma^1} T_{++} = \frac{\pi T}{2} \sum_{n=-\infty}^{\infty} \tilde{\alpha}_{m-n} \cdot \tilde{\alpha}_n \;. \end{aligned} \tag{3.60}$$

While x^μ does not appear in $T_{\pm\pm}$, we have included the momentum p^μ in the sum by defining

$$\alpha_0 = \tilde{\alpha}_0 = \frac{1}{\sqrt{4\pi T}} p \overset{\pi T = 1}{=} \frac{1}{2} p \;. \tag{3.61}$$

The constraints $T_{\pm\pm} = 0$ imply

$$L_m = \tilde{L}_m = 0 \;. \tag{3.62}$$

By evaluating the canonical Hamiltonian,

$$H = \int_0^\pi d\sigma^1 \; \left(\dot{X}\Pi - \mathcal{L}\right) = \frac{T}{2}\int_0^\pi (\dot{X}^2 + (X')^2)^2 = L_0 + \tilde{L}_0 \;, \tag{3.63}$$

we see that it coincides with the so-called worldsheet Hamiltonian $L_0 + \tilde{L}_0$.[11]

The 'Hamiltonian constraint' $H = 0$ allows us to express the mass of a state in terms of its Fourier coefficients:

$$\begin{aligned} H &= L_0 + \tilde{L}_0 = \frac{\pi T}{2} \sum_{n=-\infty}^{\infty} (\alpha_{-n} \cdot \alpha_n + \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n) \\ &= \frac{p^2}{4} + \pi T (N + \tilde{N}) = 0 \,, \end{aligned} \tag{3.64}$$

where we defined the total occupation numbers

$$N = \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n \;, \quad \tilde{N} = \sum_{n=1}^{\infty} \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n \;.$$

This provides us with the 'stringy mass shell condition':

$$M^2 = -p^2 = 4\pi T (N + \tilde{N}) \;.$$

Since $L_0 = \tilde{L}_0$ we have the additional constraint

$$N = \tilde{N} \;.$$

This is called level matching, because it implies that left- and right-moving modes contribute equally to the mass. The physical interpretation of the other constraints $L_m = 0 = \tilde{L}_m$, $m \neq 0$ will be discussed later.

For references, we also give the results for open string with Neumann boundary conditions, and recommend filling out the intermediate steps as an exercise. The solution is

$$X^\mu(\sigma) = x^\mu + p^\mu \sigma^0 + \frac{i}{\sqrt{\pi T}} \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\sigma^0} \cos(n\sigma^1) \;, \tag{3.65}$$ FourierOpen

This can still be decomposed into left- and right-mvoing parts $X = X_L(\sigma^+) + X_R(\sigma^-)$:

$$X^\mu_{L/R}(\sigma^\pm) = \frac{1}{2} x^\mu + \frac{1}{\pi T} p^\mu_{L/R} \sigma^\pm + \frac{i}{2\sqrt{\pi T}} \sum_{n \neq 0} \frac{1}{n} \alpha^\mu_{n\;(\mathrm{L/R})} e^{-in\sigma^\pm} \;, \tag{3.66}$$

but the boundary conditions $X'^\mu(\sigma^1 = 0, \pi) = 0$ imply

$$p_L = p_R = \frac{1}{2} p \;, \quad \alpha_{n(\mathrm{L})} = \alpha_{n(\mathrm{R})} \;. \tag{3.67}$$

[11] We will see later that $L_0 + \tilde{L}_0$ generates translations of σ^0, which explains this terminology. **Add cross reference.**

We see that due to the boundary conditions left- and rightmoving waves are reflected at the boundaries and combine into standing waves. As a consequence, there are only half as many independent oscillations as for closed strings.

Exercise: Show that the ends of an open string must move with the speed of light. **Hint:** You need to use constraints as well as the boundary conditions.

We mentioned before that with Neumann boundary conditions there is only one set of conserved charges L_m. Here is their explicit form in terms of Fourier coefficients:

$$\begin{aligned} L_m &= 2T \int_0^\pi d\sigma^1 \left(e^{im\sigma^1} T_{++} + e^{-im\sigma^1} T_{--} \right) = \frac{T}{4} \int_{-\pi}^{\pi} e^{im\sigma^1} \left(\dot{X} + X' \right)^2 \\ &= \frac{1}{2} \pi T \sum_n \alpha_{m-n} \alpha_n \end{aligned} \tag{3.68}$$

where we defined $\alpha_0 = p$. The canonical Hamiltonian is $H = L_0$.

As for closed strings, the Hamiltonian constraint $H = L_0 = 0$ is the mass shell condition:

$$M^2 = -p^2 = 2\pi T N \ .$$

3.2.7 Open strings with Dirichlet boundary conditions

Exercise: Work out the details for Dirichlet boundary conditions and compare to the literature.

Hints: Dirichlet boundary conditions allow a term linear term in σ^1, but no tmer linear in worldsheet time σ^0. For the osciallators, cos is replaced by sin. Thus $\dot{X} = 0$ at the ends (ends don't move), but $X' \neq 0$ at the ends, corresponding to exchange of momentum with a D-brane. Consider what happens when you force the ends of an open strings two live on to spatially separated D-branes.

3.2.8 Non-oriented strings

to be added

Chapter 4

Quantized relativistic particles and strings

In this chapter we introduce the covariant quantization of relativistic particles and strings, and explain their relation to the quantization of (scalar) fields. The detailed analysis of the resulting Hilbert space of states for relativistic strings will be performed in Part 3, after we have developed various useful tools from two-dimensional conformal field theory in Part 2.

4.1 Quantized relativistic particles

The usual heuristic approach to 'quantization' is to identify the canonical coordinates and canonical momenta of a classical theory, and then to promote them to self-adjoint operators acting on a separable Hilbert space $\mathcal{H}$, which satisfy canoncial commutation relations. The canonical commutation relations can be motivated by replacing the Poisson brackets of the classical theory by quantum commutators, through the formal substitution rule $\{\cdot,\cdot\} \to -i[\cdot,\cdot]$. In the following we will not assume that the reader is familiar with Poisson brackets, and simply postulate the canonical commutation relations.

In the case of a free non-relativistic particle with (Cartesian) coordinates x^i and canonical (= kinetic) momentum p^j, the canonical commutation relations are

$$[x^i, p^j] = i\delta^{ij} \ , \qquad (4.1)$$ xp-non-rel

where we have set $\hbar = 1$, and where the unit operator on the Hilbert sapce $\mathcal{H}$ is understood on the right hand side. In the following we will procede formally and ignore the technical complications caused by the fact that x^i, p^j are unbounded operators on an infinite dimensional Hilbert spaces. In particular we will not specify the domains of these operators, and we will assume throughout that any Hermitian operator we encounter has a complete set of eigenstates.

For a relativistic particle the natural generalization of (4.1) is

$$[x^\mu, p^\nu] = i\eta^{\mu\nu} \ . \tag{4.2}$$

Qxp

However, we know that the components of the relativistic momentum are subject to the mass shell condition $p^2 + m^2 = 0$. One option is to solve this constraint in the classical theory, and then to quantize the theory using only gauge-inequivalent quantities. A specific version of this procedure is the so-called light cone quantization, which will be discussed later **add cross reference**. Any such scheme has the disadvantage that Lorentz invariance is no longer manifest, because one has to solve for one component of the relativistic momentum in terms of others. Here we will follow the complementary, covariant approach, where canonical commutation relations are imposed on Lorentz covariant quantities. The constraint $p^2 + m^2$ will be imposed as a condition which selects physical states later.

We therefore start by constructing a representation space $\mathcal{F}$ for the relations (4.2), which we call the Fock space. This is done by postulating the existence of a distinguished state, the vacuum $|0\rangle$, which is translation invariant:

$$p^\nu |0\rangle = 0 \ .$$

The Fock space $\mathcal{F}$ is then generated by applying operators build out of x^μ. Since we assume that p^ν has a complete set of eigenstates, this amounts to constructing momentum eigenstates.

Exercise: Show that the operator $e^{ik\cdot x}$, where $k = (k^\mu) \in \mathbb{R}^D$, creates eigenstates of p^ν with eigenvalue k^ν:

$$|k\rangle := e^{ik\cdot x}|0\rangle \Rightarrow p^\nu |k\rangle = |k^\nu\rangle \ .$$

The scalar product between momentum eigenstates is defined using the Dirac delta function:

$$\langle k|k'\rangle = \delta^D(k - k') \ .$$

Since momentum eigenstates are not normalizable, they are not physical states, though they are extremely useful in handling them. To obtain normalizable states we form superpositions, 'wave packages,' of the form

$$|\Phi\rangle = \int d^D k \; \tilde{\Phi}(k)|k\rangle$$

and require them to be square-integrable:

$$|\langle\Phi|\Phi\rangle| < \infty \; .$$

Since the scalar product between wave packages is

$$\langle\Phi|\Phi'\rangle = \int d^D k d^D k' \overline{\tilde{\Phi}(k)} \tilde{\Phi}(k') \langle k|k'\rangle = \int d^D k \overline{\tilde{\Phi}(k)} \tilde{\Phi}(k) \; ,$$

and the resulting Hilbert space $\mathcal{H}$ is isomorphic to $L^2(\mathbb{R}^D)$, the space of square-integrable functions in D variables.

This is, however, not the Hilbert space of physical states, because we still have to impose the constraint $p^2 + m^2 = 0$. Therefore, we define the physical subspaces $\mathcal{F}_{\text{phys}} \subset \mathcal{F}$ and $\mathcal{H}_{\text{phys}} \subset \mathcal{H}$ as

$$\mathcal{F}_{\text{phys}} = \{|\Phi\rangle \in \mathcal{F} | (p^2 + m^2)|\Phi\rangle = 0\}$$

and

$$\mathcal{H}_{\text{phys}} := \{|\Phi\rangle \in \mathcal{H} | (p^2 + m^2)|\Phi\rangle = 0\} \subset \mathcal{F}_{\text{phys}} \subset \mathcal{F}$$

States satisfying the mass shell constraint can be parametrized as

$$|\Phi\rangle = \int d^D k \delta(k^2 + m^2) \tilde{\phi}(k) |k\rangle \; ,$$

where the δ-function forces the momenta entering into the wave package to live on the mass shell $k^2 + m^2 = 0$. The mass shell condition has two solutions for the energy k^0:

$$k^0 = \pm\sqrt{\vec{k}^2 + m^2} \; .$$

Restricting to positive energies $k^0 > 0$, and using that (see (delta-composed B.1)

$$\theta(k^0)\delta(k^2 + m^2) = \frac{1}{2|k^0|} \delta\left(k^0 - \sqrt{\vec{k}^2 + m^2}\right)$$

we obtain

$$|\Phi\rangle = \int \frac{d^{D-1}\vec{k}}{2\omega} \tilde{\phi}(k) |k\rangle \; ,$$

where $\omega = \sqrt{\vec{k}^2 + m^2}$, and where k is restricted to values on the hyperboloid $k^2 + m^2 = 0$, $k^0 > 0$.

The scalar product between two normalizable states satisfying the mass shell condition $|\Phi_i\rangle \in \mathcal{H}_{\text{phys}}$, $i = 1, 2$ is

$$\langle \Phi_1 | \Phi_2 \rangle = \int \frac{d^{D-1}\vec{k}}{2\omega} \tilde{\phi}_1^*(k) \tilde{\phi}_2(k) \; .$$

The resulting Hilbert space $\mathcal{H}_{\text{phys}}$ is isomorphic to $L^2(\mathbb{R}^{D-1}, d\mu)$, where

$$d\mu = \frac{d^{D-1}\vec{k}}{2\omega}$$

is the Lorentz invariant measure on the mass hyperboloid $k^2 + m^2 = 0$, $k^0 > 0$.

As an extension, one can also consider negative $k^0 < 0$. The corresponding modes are not interpreted as describing negative energy states, but as positive energy modes of an 'antiparticle.' The particle-antiparticle Hilbert space consists of two orthogonal copies of $L^2(\mathbb{R}^{D-1}, d\mu)$. We remark that particles and antiparticles are in general distinct, and carry charges of opposite sign under all global symmetries. Thus identifying particle and antiparticle and working with just one component of the mass hyperboloid is only possible for particles which are real, or neutral, in the sense that they do not carry charge under any global symmetry. Examples for neutral particles are photons and gravitons.

Exercise: Consider the x-representation[1] of physical states $|\Phi\rangle \in \mathcal{H}_{\text{phys}}$ in terms of 'position space wave functions'

$$\Phi(x) = \langle x | \Phi \rangle \; ,$$

and show that $\Phi(x)$ satisfies the Klein-Gordon equation:

$$(-\Box + m^2)\Phi(x) = 0 \; ,$$

where $\Box = \partial_\mu \partial^\mu = -\partial_0^2 + \Delta$ is the wave operator.

Remark: For solutions of the Klein-Gordon equation one can define a positive definite and time-independent scalar product using the conserved current

$$j^\mu = i(\Phi^* \partial^\mu \Phi - \partial^\mu \Phi^* \Phi) \; ,$$

which has the interpretation of a charge density. One can show that this scalar product agrees with the one which we have introduced above using momentum eigenstates (the k-representation), and momentum space wave functions.

bra-ket

[1]See Appendix B.3 for a short summary of the bra-ket notation we are using here

Note that when taking $\Phi(x)$ to be complex, the conserved current cannot be interpreted as a probability density, and that the naive x-representation scalar product

$$\begin{aligned} \langle \Phi_1 | \Phi_2 \rangle &= i \int_{x^0=\text{const}} d^{D-1}\vec{x} j^0 \\ &= i \int_{x^0=\text{const}} d^{D-1}\vec{x} \Phi_1^*(x) \partial^0 \Phi_2(x) - (\partial^0 \Phi_1(x)) \Phi_2^*(x) \end{aligned}$$

is indefinite. This can be corrected by interpreting Fourier modes of $\Phi(x)$ with negative k^0 as momentum modes of an antiparticle carrying opposite charge to the particle with $k^0 > 0$. The conserved current is then interpreted as a charge rather than probability density. Since states with positive and negative k^0 are orthogonal with respect to the above scalar product, and since its restriction to states with positive (negative) k^0 is positive (negative) definite, one can make the scalar product positive definite by introducing a relative sign between the two complementary subspaces. We will see this from the perspective of momentum space in the next section, although we will give the function $\Phi(x)$ a different physical interpretation.

4.2 Field quantization and Quantum Field Theory

As the above exercise has shown, the Klein-Gordon equation can be viewed as the implementation of the constraint $p^2 + m^2 = 0$ on physical states, when states are realized in the x-representation, that is as position space wave functions $\Phi(x) = \langle x | \Phi \rangle$. With this interpretation the Klein-Gordon equation is analogous to the Schrödinger equation.

There is an alternative interpretation of the Klein Gordon field as a classical field, analogous to the electromagnetic field. While the interpretation is different, the mathematics of solving the Klein-Gordon equation and representing the solution as a Fourier integral remains the same. We will now discuss what happens if we quantize a classical complex Klein Gordon field and compare the result to the quantization of a relativistic particle discussed above. We start by noting that the Klein-Gordon equation

$$(-\Box + m^2)\Phi(x) = 0$$

follows from the action principle

$$S[\Phi] = \int d^D x \left(-\partial_\mu \Phi \partial^\mu \Phi^* - m^2 \Phi \Phi^* \right) ,$$

and the canonically conjugated momentum is

$$\Pi(x) = \frac{\partial L}{\partial \partial_0 \Phi} = \partial_0 \Phi^* .$$

The Fourier representation of the general solution can be parametrized in the following form:

$$\Phi(x) = \int d^D k \left(\theta(k^0) \delta(k^2 + m^2) \tilde{\phi}_+(k) + \theta(-k^0) \delta(k^2 + m^2) \tilde{\phi}_-^*(k) \right) .$$

Here we used the step function

$$\theta(y) = \begin{cases} 1 , & \text{for } y \geq 0 , \\ 0 , & \text{for } y < 0 . \end{cases}$$

to separate the two components of the hyperboloid $k^2 + m^2 = 0$. Carrying out the k^0-integration using the δ-function we obtain

$$\Phi(x) = \frac{1}{(2\pi)^{(D-1)/2}} \int \frac{d^{D-1}\vec{k}}{2\omega_{\vec{k}}} \left(\phi_+(k) e^{ikx} + \phi_-^*(k) e^{-ikx} \right)_{k^0 = \omega_{\vec{k}}} .$$

Here $\phi_\pm(k)$ are suitably rescaled versions of $\tilde{\phi}_\pm(k)$. We now quantize the complex Klein-Gordon field by declaring $\Phi(x)$ to be an operator satisfying the canonical commutation relation

$$[\Phi(x), \Pi(y)]_{x^0 = y^0} = i\delta^{D-1}(\vec{x} - \vec{y}) .$$

Since $\Phi(x)$ depends on time x^0, we are in the Heisenberg picture of quantum mechanics, where operators depend on time while states are time-independent. We only need to specify the commutator at equal times, because the commutator at other times is fixed by time evolution. Comparing back to the relativistic particle, we see that the spatial coordinate is treated as a continuous index labelling degrees of freedom located at different points of space. This is the original idea of 'field quantization' **Add reference Dreimännerarbeit**. The operator $\Phi(x)$ can be represented as a Fourier integral, with the Fourier coefficients $\phi_\pm(k)$ promoted to operators. Complex conjugated quantities are now interpreted has Hermitian conjugate operators, and denoted $\Phi^\dagger(x)$, $\phi_\pm^\dagger(k)$.

Exercise: Substitute the Fourier decomposition of $\Phi(x)$ into the canonical commutation relations, and show that the quantized Fourier modes satisfy the relations:

$$[\phi_\pm(\vec{k}), \phi^\dagger_\pm(\vec{k}')] = 2\omega_{\vec{k}} \delta^{D-1}(\vec{k} - \vec{k}') \ , \quad [\phi_\pm(\vec{k}), \phi^\dagger_\mp(\vec{k}')] = 0 \ . \tag{4.3}$$ CRphipm

Now we can construct a Fock space based on a ground state $|0\rangle$ defined by the properties

$$\phi_\pm(\vec{k})|0\rangle = 0 \ , \quad \langle 0|0\rangle = 1 \ .$$

The states

$$|\vec{k}\rangle_\pm = \phi^\dagger_\pm(\vec{k})|0\rangle$$

are interpreted as momentum eigenstates for two particles, which are related by $\Phi \to \Phi^\dagger$, and are interpreted as a particle and its antiparticle. The mutual scalar products between such states are

$${}_\pm\langle \vec{k}|\vec{k}'\rangle_\pm = \langle 0|\phi_\pm(\vec{k})\phi^\dagger_\pm(\vec{k}')|0\rangle = \langle 0|[\phi_\pm(\vec{k})\phi^\dagger_\pm(\vec{k}')]|0\rangle = 2\omega_{\vec{k}}\delta^{D-1}(\vec{k} - \vec{k}')$$

and

$${}_\mp\langle \vec{k}|\vec{k}'\rangle_\pm = 0 \ .$$

Thus particle and antiparticle states are orthogonal. By taking square-integrable superpositions we obtain two orthogonal copies of the Hilbert space $L^2(\mathbb{R}^{D-1}, d\mu)$. This is the same Hilbert space that we obtained above by quantizing the relativistic particle, provided that we include both sheets of the mass hyperboloid. The advantage of field quantization is that it gives us more, because multiple application of creation operators $\phi^\dagger_\pm(\vec{k})$ allows us to obtain multiparticle momentum eigenstates

$$\phi^\dagger_+(\vec{k}_1)\phi^\dagger_+(\vec{k}_2)\cdots\phi^\dagger_-(\vec{k}'_1)\phi^\dagger_-(\vec{k}'_2)\cdots|0\rangle \ .$$

Since creation operators commute among themselves, the multiparticle Hilbert space obtained by taking square integrable superpositions of momentum eigenstates has the form

$$\mathcal{H}^{\text{multiparticle}}_{\text{phys}} = \mathbb{C} \oplus \mathcal{H}^+_{\text{phys}} \oplus \bigvee^2 \mathcal{H}^+_{\text{phys}} \oplus \cdots \oplus \mathcal{H}^-_{\text{phys}} \oplus \bigvee^2 \mathcal{H}^-_{\text{phys}} \oplus \cdots \ ,$$

where $\mathbb{C}$ is the zero-particle sector spanned by $|0\rangle$, where $\mathcal{H}^\pm_{\text{phys}} \simeq L^2(\mathbb{R}^{D-1}, d\mu)$ are the one-particle Hilbert spaces for particle and antiparticle, and where $\bigvee^k \mathcal{H}^\pm_{\text{phys}}$ denotes the k-th symmetrized tensor power.

The commutation relations (CRphipm 4.3) resemble those of harmonic oscillators labled by the momentum $\vec{k}$. In the quantum field theory literature it is common to use rescaled operators

$$a(\vec{k}) = (2\omega_{\vec{k}})^{-1/2}\phi_+(k) \ , \quad b(\vec{k}) = (2\omega_{\vec{k}})^{-1/2}\phi_-(k) \ ,$$

and their Hermitian conjugates

$$a^\dagger(\vec{k}) = (2\omega_{\vec{k}})^{-1/2}\phi_+^\dagger(k) \ , \quad b^\dagger(\vec{k}) = (2\omega_{\vec{k}})^{-1/2}\phi_-^\dagger(k) \ .$$

These satisfy standard harmonic oscillator 'with a continuous index $\vec{k}$.'

$$[a(k), a^\dagger(\vec{k})] = \delta^{D-1}(\vec{k} - \vec{k}') \ , \quad [b(k), b^\dagger(\vec{k})] = \delta^{D-1}(\vec{k} - \vec{k}') \ ,$$

and the mode expansion of the field operator takes the form

$$\Phi(x) = \frac{1}{(2\pi)^{(D-1)/2}} \int \frac{d^{D-1}\vec{k}}{\sqrt{2\omega_{\vec{k}}}} \left(a(k)e^{ikx} + b^\dagger(k)e^{-ikx} \right)_{k^0=\omega_{\vec{k}}} \ .$$

While one can use either parametrization, we remark that while

$$d\mu = \frac{d^{D-1}}{2\omega_{\vec{k}}} \quad \text{and} \quad 2\omega_{\vec{k}}\delta^{D-1}(\vec{k} - \vec{k}')$$

are a Lorentz invariant measure and a Lorentz invariant δ-function on the mass shell $k^2 + m^2 = 0$, the expressions

$$\frac{d^{D-1}}{\sqrt{2\omega_{\vec{k}}}} \quad \text{and} \quad \delta^{D-1}(\vec{k} - \vec{k}')$$

are not Lorentz invariant. This also implies that states of the form $\Phi^\dagger(\vec{k})|0\rangle = \sqrt{2\omega_{\vec{k}}}|0\rangle$ are Lorentz invariant, while states of the form $a^\dagger(\vec{k})|0\rangle$ are not.

The Lorentz invariance of $d\mu$ is clear because we obtained it by localizing the volume element d^Dk on the mass shell using $\delta(k^2 + m^2)$. Similarly since

$$\delta^D(k - \vec{k}') = \delta(k^0 - k'^0)\delta^{D-1}(\vec{k} - \vec{k}')$$

and, using (delta-composed B.1)

$$\delta(k \cdot k - k' \cdot k') = \frac{1}{2|k^0|}\delta(k^0 - k'^0) \ ,$$

we have

$$\delta^D(k - \vec{k}') = \delta(k \cdot k - k' \cdot k')2|k^0|\delta^{D-1}(\vec{k} - \vec{k}') \ ,$$

which implies that $2|k^0|\delta^{D-1}(\vec{k} - \vec{k}')$ is Lorentz invariant.

Instead of a complex classical field, we could have considered a real classical field. This case is recovered from the above formulae by setting $\Phi(x) = \Phi^{\dagger}(x)$, which implies $\phi_{+}(k) = \phi_{-}(k)$ and $a(\vec{k}) = b(\vec{k})$. Thus particle and antiparticle are identified. Note that this identification requires that there is no symmetry or interaction which acts on the phase of the scalar field Φ. For example, if we couple a scalar field to an electromagnetic field, then particle and antiparticle automatically carry opposite charges and cannot be identified. More technically, the minimal coupling of a scalar field to an electromagnetic field requires a symmetry of $\Phi(x)$ under local $U(1)$ phase transformations, and therefore $\Phi(x)$ must be complex.

We finish this section with some additional remarks. Firstly, the appearance of δ-functions in the commutation relations for the operators $\Phi(x)$ and $\phi_{\pm}(k)$ reflects an issue that is similar to the fact that momentum eigenstates are not normalizable and therefore do not lie in the Hilbert space. The 'operators' $\Phi(x)$ and $\phi_{\pm}(k)$ are not proper operators but operator valued distributions. While momentum eigenstates must be combined into 'wave packets' to obtain states in the Hilbert space, operator valued distributions become operators on the Hilbert space once they are applied to suitable testfunctions. **Add reference, for example Haag.** However, in practice it is more convenient to work with momentum eigenstates, δ-functions and field operators $\Phi(x)$ than with the actual Hilbert space and operators defined thereon.

Secondly, we have seen that the advantage of field quantization ('second quantization') over particle quantization ('first quantization') is that it directly leads to a multiparticle theory, which is what one needs in a relativistic setting where particles can be created and annihilated. Therefore quantum field theory is the standard formulation of quantized relativistic systems based on point particles, although methods based on the 'first quantized' approach (somtimes called worldline formalism) have some uses. Unfortunately, the analogous 'second quantized' approach to string theory, or string field theory, is very complicated and not as much developed as quantum field theory. However the first quantized approach to the quantum theory of relativistic string is workable and by now highly developed.

4.3 Quantized relativistic strings

From the classical solutions we found in the previous chapter we know that a free relativistic string in Minkowski space is, as far as its degrees of freedom are concerned, the combination of a relativistic particle, corresponding to its center of mass motion, and an infinite set of harmonic oscillators corresponding to left- and right-moving waves (possibly coupled by the boundary conditions). Therefore we should expect that the Hilber space is the product of the one a relativistic particle with infinitely many harmonic oscillators. But instead of postulating canonical relations piecemeal for x^μ, p^ν and the Fourier coefficients, we can use the canonical coordinates $X^\mu(\sigma^0, \sigma^1)$ and the canonical momenta $\Pi^\nu(\sigma^0, \sigma^1)$. We work with the Polyakov action in the conformal gauge, and can interprete X^μ either as embedding coordinates for a string in spacetime, or as a set of scalar fields on the worldsheet Σ. In the conformal gauge $\Pi^\mu = T\dot{X}^\mu$. Both X^μ and Π^ν are time-dependent operators, and we are thus in the Heisenberg picture of quantum mechanics. Canonical commutators are imposed at equal worldsheet time, and σ^1 is treated as a continuous index, similar as μ is a discrete index. We consider periodic boundary conditions for definiteness. Then the canonical commutation relations are:

$$[X^\mu(\sigma^0, \sigma^1)\ , \Pi^\nu(\sigma'^0, \sigma'^1]_{\sigma^0=\sigma'^0} = i\eta^{\mu\nu}\delta_\pi(\sigma^1 - \sigma'^1)\ , \tag{4.4}$$

QXP

where

$$\delta_\pi(\sigma^1) = \frac{1}{\pi}\sum_{k=-\infty}^{\infty} e^{-2ik\sigma^1} = \delta_\pi(\sigma^1 + \pi)$$

is the periodic δ-function with period π, see Appendix B.2.

Exercise: Use the canonical commutation relations (4.4) to derive the following commutation relations among the Fourier mode operators:

$$[x^\mu\ , p^\nu] = i\eta^{\mu\nu}\ , \quad [\alpha^\mu_m\ , \alpha^\nu_n] = m\delta_{m+n,0}\ , \quad [\tilde{\alpha}^\mu_m\ , \tilde{\alpha}^\nu_n] = m\delta_{m+n,0}\ .$$

Since X^μ is Hermitian, it also follows that

$$(x^\mu)^\dagger = x^m u\ , \quad (p^\mu)^\dagger = p^\mu\ , \quad (\alpha^\mu_m)^\dagger = \alpha^\mu_{-m}\ , \quad (\tilde{\alpha}^\mu_m)^\dagger = \tilde{\alpha}^\mu_{-m}\ .$$

Comparing the relations for α^μ_m to the standard relations between creation and annihilation operators of a harmonic oscillator

$$[a, a^\dagger] = 1\ ,$$

we see that we indeed get an infinite set of harmonic oscillators. More precisely, setting

$$\alpha_m^\mu = \begin{cases} \sqrt{m} a_m^\mu & \text{for } m > 0 \;, \\ \sqrt{-m}(a_{-m}^\mu)^\dagger & \text{for } m < 0 \;, \end{cases}$$

we obtain

$$[a_m^\mu \;, (a_n^\nu)^\dagger] = \eta^{\mu\nu}\delta_{m,n} \;.$$

While these are standard harmonic oscillator relations for $\mu, \nu \neq 0$, we obtain an additional minus sign in the relations for $\mu = \nu = 0$.

To explore this further, let us build a Fock space $\mathcal{F}_{\rm osc}$ by starting with a ground state $|0\rangle$, defined by

$$\alpha_n^\nu |0\rangle \;, \quad n > 0 \;.$$

If we include the zero mode part corresponding to x^μ, p^ν we should of course add the condition $p^\nu|0\rangle = 0$. But we already know how to deal with this 'zero mode part' of the Fock space and concentrate on the 'oscillator part' $\mathcal{F}_{\rm osc}$ for the time being.

Oscillator eigenstates are generated by applying creation operators α_{-m}^μ, $m > 0$ to the ground state. We take the ground state to be normalized as $\langle 0|0\rangle = 1$. Now consider scalar products of the form

$$\left(\alpha_{-m}^\mu|0\rangle \;, \alpha_{-n}^\nu|0\rangle\right) = \langle 0|\alpha_m^\mu \alpha_{-n}^\nu|0\rangle = \langle 0|[\alpha_m^\mu \;, \alpha_{-n}^\nu]|0\rangle = \eta^{\mu\nu}\delta_{m,n} \;.$$

Thus the natural, relativistically covariant scalar product on $\mathcal{F}_{\rm osc}$ is indefinite, while the space of states in a quantum theory must have a positive definite scalar product. However, we still need to impose the constraints $L_m = 0$ (and $\tilde{L}_m = 0$). Since solving them before quantisation obscures relativistic covariance, we impose them on states of the quantum theory, and define the subspace of physical states $\mathcal{F}_{\rm phys} \subset \mathcal{F}$ by requiring that the matrix elements of $L_m, \tilde{L}_m$ vanish between physical states $|\Phi\rangle, |\Phi'\rangle$:

$$\langle \Phi|L_m|\Phi'\rangle = 0 \;, \quad \langle \Phi|\tilde{L}_m|\Phi'\rangle = 0 \;.$$

As we will see later, only the constraints with $m \neq 0$ takes this form, while there is a modification for $m = 0$.

While one might have hoped that $\mathcal{F}_{\rm phys}$ is positive definite, the situation is more complicated. First of all, the best one can achieve is that $\mathcal{F}_{\rm phys}$ is positive semi-definite:

$$\langle \Phi|\Phi\rangle \geq 0 \;.$$

One can show that there are always 'null states,' which are physical and different from the zero state, but have zero norm squared. The presence of these states is related to the residual residual symmetry under conformal transformations, which has not been fixed by imposing the conformal gauge. Physical states which differ by a null states are related by conformal transformations, and should thus been identified. Writing $|\Phi\rangle \sim |\Phi'\rangle$ when to physical states differ by a null state, the candidate for the physical Hilbert space is

$$\mathcal{H} = \mathcal{F}_{\text{phys}} / \sim \ .$$

In Part 2 of this book we will develop various tools from conformal field theory which will help us to better understand the structure of the Hilbert space. The spacetime interpretation of the resulting physical states will then be a main subject of Part 3.

Continuing our preview of things to come we remark that further conditions need to be imposed to guarantee that $\mathcal{F}_{\text{phys}}$ is positive semi-definite. The so-called 'no-ghost theorem' shows the absence of negative norm states for $D \leq 26$, while for $D > 26$ negative norm states always exist.[2] This implies that strings in Minkowksi space can only be quantised consistently if $D \leq 26$. Moreover, once string interactions are included, negative norm states, which naively have been projected out by imposing the constraints can re-occur as intermediate states in loop diagrams. Since for consistency they then have to be allowed as asymptotic states as well, the essential 'unitarity' or positive definiteness of the quantum theory is lost. The only case where negative norm states can be decoupled consistently for strings in a Minkowski background is in $D = 26$ dimensions. Since, at least at length scales accessible to current experiments, we live in a spacetime of dimension 4, this raises the question how to account for the the extra dimensions. We will come back to this question in Part 4 of the book.

[2]By 'negative norm states' we refer to states $|\Phi\rangle$ where $\langle\Phi\ ,\Phi\rangle < 0$. 'Negative norm-squared states' would be more accurate, but somewhat tedious.

Appendix A

Units

Physical Units

For the benefits of readers without a strong physics background, we add a few remarks on physical units. Physical quantites are measured by comparing to an agreed standard unit of measure. In the SI system one uses the units meter, second, kilogram, ampere, and degree Kelvin for the quanitites length, time, mass, current and temperature. All further units are products of these basic units. The basic quantities are referred to as ‘dimensions,’ but note that while physical quantities can be multiplied, they can only be added if they ‘have the same dimension’, that is are measured in the same unit. Thus the units or dimensions do not form a vector space **reference**. For our purposes only those quanties which derived from length, time and mass will be relevant. As a slightly non-trivial example consider the action functional (ActionI 2.11). When using SI units this takes the form

$$S = -mc^2 \int dt \sqrt{1 - \frac{\vec{v}^2}{c^2}}\ . \tag{A.1}$$

Action-dimensionful

In mechanics, the action of a force on a particle is the integral of work done over time, and therefore has dimension energy $\times$ time. The actions used in formulating actions principles are generalizations of this quantity and have the same dimension.

Like most of the theoretical and mathematical literature, we will not use SI units, but ‘natural units.’ Such units are based on using constants of nature instead of conventionally chosen units. One such constant is speed of light c, which we set to unity, $c = 1$. In the resulting system of units, time and space are measured in the same units (light seconds, light years, etc), velocities are dimensionless and measured in multiples of the speed of light, while mass and energy have the same dimension. In high energy physics it is common to specify the masses of elementary particles in energy units (electron volts) rather than mass units. Setting $c = 1$ in (`Action-dimensionful` A.1), we recover (`ActionI` 2.11), which at this point has dimension Energy $\times$ time = Mass $\times$ time. As a further simplification one also sets to unity the reduced Planck quantum of action, $\hbar = \frac{h}{2\pi} = 1$. In the resulting 'natural' system of units, mass/energy and time/distance have inverse units to one other, and action is dimensionless.

Remark: Dimensional analysis is useful to constrain the structure and sometimes the qualitative behaviour and order of magnitude of physical quantities. **Ref: Zeidler Vol 1, other articles on metrology.** Moreover it is often invoked in 'naturalness arguments,' which go along the line that once a physical quantity is expressed as a pure number by setting constants of nature to unity, this number should be 'of order one' or 'not very large or very small' unless there is a 'good reason.' For example, the cosmological constant, when expressed in natural units is considered 'unnaturally small.' **Add numerical values? Add tables for units or refer to textbook?** As indicated, naturalness arguments are subject to interpretation (what is 'natural') and thus to debate. Their value lies in probably more in instigating discussion, rather than providing answers.

Appendix B

Dirac δ-functions, Fourier analysis and bra-ket notation

B.1 Dirac δ-functions

The Dirac δ-function is the generalized function (or distribution) associated with the functional which evaluates a function at a given point:

$$\delta_{x_0}[f] = f(x_0) \ .$$

Using the δ-function, this is represented as

$$\int \delta(x - x_0) f(x) = f(x_0) \ ,$$

where $\delta(x - x_0)$ is a 'generalized function supported at the point $x = x_0$'. While a function with the above properties does not exist, one can define a generalized function provided that the functions f with which it is paired are sufficiently well behaved. For example, if f is a Schwartz function, which means that f is smooth and that f and all its derivatives decay faster than any power at infinity, then the dual space (with respect to a suitable norm) of tempered distributions contains the δ-function. Properties of distributions are established by integrating them against test functions, that is functions in the dual space.

For example, using integration by parts one shows

$$\int dx \delta'(x - x_0) f(x) = -f'(x_0) \ .$$

Similarly, given a function $g(x)$ which has a single zero $g(x_0) = 0$ such that $g'(x_0) \neq 0$, one can use the substitution $y = g(x)$ to show that

$$\delta(g(x) - g(x_0)) = \frac{1}{|g'(x_0)|} \delta(x - x_0) \ .$$

For functions $g(x)$ with multiple zeros $x_1, x_2 \ldots, x_N$, with $g'(x_i) \neq 0$ this generalizes to

$$\delta(g(x) - g(x_0)) = \sum_{i=1}^{N} \frac{1}{|g'(x_i)|} \delta(x - x_i) \ . \tag{B.1}$$

delta-composed

Fourier

B.2 Fourier sums and Fourier integrals

Periodic functions with period 2π, $f(x + 2\pi) = f(x)$ can be represented by a Fourier series

$$f(x) = \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} c_k e^{ikx} \ .$$

The Fourier coefficients are determined by the inverse transformation

$$c_k = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} dx\, f(x) e^{-ikx} \ .$$

A sufficient condition for the Fourier series to exist is that that the sequence c_n is square-summable, $\sum_{k=-\infty}^{\infty} |c_k| < \infty$.

For a general period P, that is for functions where $f(x + P) = f(x)$, the formulae are modified as follows:

$$f(x) = \frac{1}{\sqrt{P}} \sum_{k=-\infty}^{\infty} c_k e^{(2\pi i x)/P} \ ,$$

and

$$c_k = \frac{1}{\sqrt{P}} \int_{x_0}^{x_0+P} f(x) e^{(-2\pi i k x)/P} \ .$$

Fourier sums can be extended to generalized functions. One example is the periodic Dirac function or 'Dirac comb,' $\delta_P(x) = \delta_P(x + P)$. By applying the inverse Fourier formula formally, we immediately obtain its representation as a Fourier sum:

$$\delta_P(x) = \frac{1}{P} \sum_{k=-\infty}^{\infty} e^{(2\pi i x k)/P} \ .$$

Non-periodic functions which decay sufficiently fast at infinity admit a representation as Fourier integrals, which can be viewed as 'expansions in plane waves.'

The D-dimensional Fourier transformation

$$f(x) = \frac{1}{(2\pi)^{D/2}} \int d^D k \tilde{f}(k) e^{ikx}$$

has the inverse

$$\tilde{f}(k) = \frac{1}{(2\pi)^{D/2}} \int d^D x f(x) e^{-ikx} .$$

Sometimes it is convenient to distribute the powers of $\sqrt{2\pi}$ asymmetrically.

The D-dimensional Dirac δ-function can be represented as the Fourier transform of the constant function $(2\pi)^{-D/2}$:

$$\delta^D(x) = \frac{1}{(2\pi)^D} \int d^D k e^{ikx} .$$

This follow immediately by formally applying the inverse Fourier transformation.

B.3 Bra-ket formalism

bra-ket

Dirac's bra-ket formalism allows to formally manipulate states and operators by using the analogy with finite dimensional vectors and matrices. Consider a non-relativistic particle in one dimension. We denote the position and momentum operator by Q, P and the corresponding eigenstates $|x\rangle, |k\rangle$:

$$Q|x\rangle = x|x\rangle , \quad P|k\rangle = k|k\rangle .$$

We normalize eigenstates of Q according to:

$$\langle x|x'\rangle = \delta(x - x') .$$

The operator Q has the spectral representation

$$Q = \int dx \; x|x\rangle\langle x|$$

and matrix elements:

$$\langle x|Q|x'\rangle = x\delta(x - x') .$$

Since we assume that Q has a complete spectrum of eigenstats, we can write general states as superpositions of position eigenstates:

$$|\psi\rangle = \int dx \psi(x) |x\rangle \ .$$

The scalar product between such states is

$$\langle \psi_1 | \psi_2 \rangle = \int dx \psi_1^*(x) \psi_2(x) \ .$$

Normalizable states satisfy

$$\langle \psi | \psi \rangle = \int dx |\psi(x)|^2 < \infty \ .$$

The resulting Hilbert space is isomorphic to $L^2(\mathbb{R})$.

The expansion coefficients $\psi(x)$ can be projected out using $\langle x|$:

$$\psi(x) = \langle x | \psi \rangle \ .$$

They can be identified with the usual position space wave functions of quantum mechanics. Therefore $\psi(x)$ is called the 'x-representation' or position space representation of the 'abstract state' $|\phi\rangle$.

Momentum eigenstates in the x-representation satisfy

$$P\psi(x) := -i \frac{d}{dx} \psi(x) = k \psi(x) \ .$$

Solutions of this eigenvalue equation, that is momentum eigenstates in the x-representation. are plane waves

$$\psi(x) = c e^{ikx} \ .$$

We normalize momentum eigenstates such that

$$\langle k | k' \rangle = \delta(k - k') \ .$$

Since

$$\delta(k) = \frac{1}{2\pi} \int dx e^{ikx} \ ,$$

this implies

$$\langle x | k \rangle = \frac{1}{\sqrt{2\pi}} e^{ikx} \ , \quad \langle k | x \rangle = \frac{1}{\sqrt{2\pi}} e^{-ikx} \ .$$

In the main part of this book, the momentum operator is often denoted p and we use x for both the 'position operator'[1] and its eigenvalues.

[1] In a relativistic theory x^μ is not Hermitian on physical states and should not be interpreted

as being related to time or the observable position of a particle. On physical states one can define a Hermitian position operator, but its eigenstates are not arbitrarily sharply (δ-function like) localized. Moreover, it is not clear how to define a time operator. The Hamilton operator should be bounded from below which prevents one from defining a time operator by Fourier transformation.