Liverpool Lectures on String Theory
Problem Set 1
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Problem 1 Relativistic force.

The relativistic version of Newton's second law is

$$\frac{d\vec{p}}{dt} = \frac{d}{dt}\left(\frac{m\vec{v}}{\sqrt{1-\vec{v}^2}}\right) = \vec{F}$$

Show that this can be written in manifestly covariant form as

$$\frac{dp^\mu}{d\tau} = f^\mu$$

and find the relation between $\vec{F}$ and the relativistic force vector f^μ. *Hint:* use the relation between the differentials dt and $d\tau$ of coordinate time and proper time and the mass shell condition $-(p^0)^2 + \vec{p}^2 = -m^2$.

Problem 2 Action principle for massive particle in a potential.

Show that the variation of the action

$$S = -m\int d\tau\sqrt{-\dot{x}^2} - \int V(x(\tau))d\tau \tag{1}$$

gives the equation of motion for a massive particle subject to a force of the form $f_\mu = \partial_\mu V$.

Problem 3 Lorentz force from action principle.

The manifestly covariant form of the Lorentz force

$$\vec{F} = q\left(\vec{E} + \vec{v}\times\vec{B}\right) \tag{2}$$

is

$$f^\mu = qF^{\mu\nu}\dot{x}_\nu \ , \tag{3}$$

where τ = proper time, q = charge, A_μ = (relativistic) vector potential, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ = field strength tensor. Show that the equation of motion $m\ddot{x}^\mu = f^\mu$ is obtained by variation of the action

$$S = -m\int d\tau\sqrt{-\dot{x}^2} - q\int A_\mu dx^\mu \tag{4}$$

where the second term is the integral of the vector potential along the world line.

Problem 4 Massive particle in curved space (=gravitational field).

This assumes that you have had a first course in GR already. Then you will know that the coupling to gravity is found by replacing the Minkowski metric $\eta_{\mu\nu}$ by (pseudo-)Riemannian metric $g_{\mu\nu}(x)$. The resulting action for a massive particle is

$$S = -m \int d\tau \sqrt{-g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu} \ . \tag{5}$$

Show that by variation of x^μ one obtains the following equations of motion

$$\ddot{x}^\mu + \Gamma^\mu_{\nu\rho}\dot{x}^\nu \dot{x}^\rho = 0 \ , \tag{6}$$

where $\Gamma^\mu_{\nu\rho}$ are the Christoffel symbols (of second type) of the metric $g_{\mu\nu}$. What is the geometrical meaning of this equation?

Problem 5 An action for massless and massive particles.

Consider the action

$$S[x,e] = \frac{1}{2}\int e d\sigma \left(\frac{1}{e^2}\left(\frac{dx^\mu}{d\sigma}\right)^2 - m^2 \right) \ , \tag{7}$$

where σ is an arbitrary curve parameter and $ed\sigma$ is the invariant line element.

1. Show that the action is invariant under reparametrisations of the wordline and under Poincaré transformation of Minkowski space.

2. Perform variations of x^μ and of e to obtain the equations of motion for x^μ and e respectively.

3. For $m^2 > 0$, eliminate e by its equation of motion, and substitute the result back into (7). Show that you obtain the action for a free massive particle.

4. Instead of eliminating e, you can set it to a constant value by a reparametrisation. This is possible for both $m^2 > 0$ and $m^2 = 0$. Go to such a parametrisation and find the equations of motion in this gauge. What is the interpretation of the resulting equations? In the case $m^2 > 0$, what is the interpretation of the curve parameter?

If you have a good background in general relativity or differential geometry, you can do this problem for a particle propagating in a general space-time with metric $g_{\mu\nu}(x)$, instead of Minkowski space.