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Notes on Topological STRING THEORY

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Some References (Reviews)

M. Vonk hep-th/0504147

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M. Marino hep-th/0410165

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B. Greene hep-th/9702155

I. Topological Field Theory

- The Theory has some physical operators

$$O_1(x_1), \ O_2(x_2) \cdots O_n(x_n)$$

- The Lagrangian $L(\varphi_i, \partial\varphi_i)$ depends on the fields and their derivatives

- The theory is defined on a manifold M. One can choose a metric on it (call it $h_{\alpha\beta}$)

- If the correlation function

$$\langle O_1(x_1) \cdots O_n(x_n) \rangle = \int \mathcal{D}\varphi \, O_1(x_1) \cdots O_n(x_n) \, e^{i\int d^D x \, L(\varphi, \partial\varphi)}$$

Does not depend on the metric $h_{\alpha\beta}$, then our theory is a topological Field Theory.

- Exemple $(D=3)$ Chern-Simons theory

$$L = \mathrm{Tr}\left(A \wedge dA - \tfrac{2}{3} A \wedge A \wedge A\right)$$

is invariant under $A' = gAg^{-1} - g\,dg^{-1}$

A is a matrix-valued function (say $SU(N)$) g is a $SU(N)$ matrix

- The Lagrangian is invariant up to a term

$$\mathrm{Tr}(g\,dg^{-1}\wedge dg\wedge dg^{-1})$$

But this term is a topological invariant

$$\frac{1}{24\pi^2}\int_M \mathrm{Tr}(g\,dg^{-1}\wedge dg\wedge dg^{-1}) = m \in \mathbb{Z}$$

So defining the action

$S = \frac{k}{4\pi}\int_M L \rightarrow$ changes by $2\pi k m$ (k is an integer)

* Define the partition function

$Z = \int DA\, e^{iS[A]} \rightarrow$ it contains <u>no</u> metric.

** <u>Comment:</u> Chern-Simons theory on a compact 3 manifold has no quantum anomalies.

II Cohomological Field Theory

* This theory in general uses a metric but correlation functions do not depend on it.

* It has a symmetry operator Q.

The variation under Q of any operator O_i is

$$\delta_\varepsilon O_i = i\varepsilon [Q, O_i]_\pm$$ ("+" if O_i is an odd operator, "−" if O_i is an even operator)

* $Q^2 = 0 \to Q$ is nilpotent

* Physical operators are invariant $[Q, O_i]_\pm = 0$

* The vacuum is symmetric $Q|0\rangle = 0$ and hence we have equivalence relation

$$O_i \sim O_i + [Q, \Lambda]_\pm$$

Since $\langle 0| O_i \cdots O_{i+k} \cdots O_{i_n}|0\rangle$ does not change under the redefinition given above $\langle 0|O_i \cdots [Q, \Lambda_{i+k}]_\pm \cdots O_{i_n}|0\rangle = 0$

* Energy-momentum tensor is Q-exact

$$T_{\alpha\beta} = \frac{\delta S}{\delta h^{\alpha\beta}} = \{Q, G_{\alpha\beta}\}_+$$ For some $G_{\alpha\beta}$

Therefore: (Below we assume a graded commutator everywhere)

$$\frac{\delta}{\delta h^{\alpha\beta}}\langle O_{i}\cdots O_{in}\rangle=\frac{\delta}{\delta h^{\alpha\beta}}\left(\int \mathcal{D}\varphi\, O_{i_1}\cdots O_{i_n}\, e^{iS[\varphi]}\right)=$$

$$=i\int \mathcal{D}\varphi\, O_{i}\cdots O_{in}\frac{\delta S}{\delta h^{\alpha\beta}}e^{iS}=i\langle O_{i}\cdots O_{in}[Q,G_{\alpha\beta}]\rangle=0$$

* One way to ensure these properties is to have a Lagrangian which is Q-exact, i.e.,

$$L=[Q,V]$$

As a result for $e^{\frac{i}{\hbar}S}=e^{\frac{i}{\hbar}[Q,\int_M V]}$ and $\frac{d}{d\hbar}\langle O_{i}\cdots O_{in}\rangle=0$

$\hbar\to$ Planck constant

We can calculate correlators in the classical limit.

Nonlocal OPERATORS

* For the momentum P_α we have:

$$P_\alpha=[Q,G_\alpha]$$

Consider $O_\alpha^{(1)}=i[G_\alpha,O^{(0)}]$ where $[O^{(0)},Q]=0$

Then $\frac{d}{dx^\alpha}O^{(0)}=i[P_\alpha O^{(0)}]=[Q,O_\alpha^{(1)}]$

Define an one form operator:

$$O^{(1)}=O_\alpha^{(1)}dx^\alpha\Rightarrow dO^{(0)}=[Q,O^{(1)}]$$

* Integrate it over the closed curve, we get

$$[Q, \int_\gamma O^{(1)}] = 0$$

* Therefore we have constructed a class of non-local Physical operators from $O^{(0)}$

$$[Q; O^{(0)}] = 0 ; \quad [Q : O^{(1)}] = d O^{(0)} \dots \quad [Q : O^{(n)}] = d O^{(n-1)}$$

$$d O^{(n)} = 0$$

Some properties of 2 dimensional Cohomological Field Theories.

* Correlation functions.

$$\langle O_1 \dots O_n \rangle_\Sigma = \langle O_1 \dots O_i O_a \rangle_{\Sigma_1} \eta^{ab} \langle O_b O_{i+1} \dots O_n \rangle_{\Sigma_2}$$

Genus $g_\Sigma = g_{\Sigma_1} + g_{\Sigma_2}$

The metric η^{ab} is an inverse of the 2 point function $C_{ab} = \langle O_a, O_b \rangle$

The structure conctants C_{abc} in $O_a O_b = \sum_c C_{ab}^c O_c$ can be computed from $C_{abc} = \langle O_a, O_b, O_c \rangle_0 \longrightarrow$ genus of a sphere

* $\frac{d C_{abc}}{d t^d} = \langle O_a O_b O_c \int O_d^{(2)} \rangle$ when turning the deformation $t^a O_a^{(2)}$ into the Lagrangian

* From the metric and structure constants one can compute any correlation function.

III $N=(2,2)$ SUSY

Introduce a superspace $(z;\bar{z};\theta^{\pm};\bar{\theta}^{\pm})$

$$(\theta^{+})^{*}=\bar{\theta}^{-};\quad (\theta^{-})^{*}=\bar{\theta}^{+}$$

SUSY invariant measure

$$\int d^2z\, d^4\theta=\int dz\, d\bar{z}\, d\theta^{+} d\theta^{-}\, d\bar{\theta}^{+} d\bar{\theta}^{-}$$

SUSY invariant action

$$S=\int d^2z\, d^4\theta\; K(\Phi^{i}\bar{\Phi}^{j})$$

where K is a real function of Superfields $\Phi^{i}(z,\bar{z},\theta^{\pm},\bar{\theta}^{\pm})$

* Hamiltonian $H=-i(\partial_{+}-\partial_{-})$ $\quad \partial_{+}=\frac{\partial}{\partial z};\quad \partial_{-}=\frac{\partial}{\partial\bar{z}}$
* Momentum $P=-i(\partial_{+}+\partial_{-})$
* Lorentz generator $M=2z\partial_{+}-2\bar{z}\partial_{-}+\theta^{+}\frac{d}{d\theta^{+}}-\theta^{-}\frac{d}{d\theta^{-}}+\bar{\theta}^{+}\frac{d}{d\bar{\theta}^{+}}-\bar{\theta}^{-}\frac{d}{d\bar{\theta}^{-}}$

SUSY ALGEBRA

* Generators of SUSY: $Q_\pm = \frac{\partial}{\partial\theta^\pm} + i\bar\theta^\pm \partial_\pm$; $\bar Q_\pm = -\frac{\partial}{\partial\bar\theta^\pm} - i\theta^\pm \partial_\pm$

* SUSY covariant derivatives
$$D_\pm = \frac{\partial}{\partial\theta^\pm} - i\theta^+\partial_\pm \, ; \quad \bar D_\pm = -\frac{\partial}{\partial\bar\theta^\pm} + i\theta^\pm \partial_\pm$$
(D-s commute with Q-s)

$$\{Q_\pm ; \bar Q_\pm\} = P \pm H \qquad [M \, Q_\pm] = \mp Q_\pm \, ; \quad [M ; \bar Q_\pm] = \mp \bar Q_\pm$$

$$\{D_\pm ; \bar D_\pm\} = -P \mp H \qquad [M ; D_\pm] = \mp D_\pm \, ; \quad [M ; \bar D_\pm] = \mp \bar D_\pm$$

Let us restrict the superfield Φ in a SUSY invariant way (chiral superfield)
$$\bar D_\pm \Phi = 0 \Rightarrow \Phi = \varphi(z\bar z) + \psi_+(z'\bar z')\theta^+ + \psi_-(z', \bar z')\theta^- + F(z', \bar z')\theta^+\theta^-$$

where $z' = z - i\theta^+\bar\theta^+$

Performing integration over Grassman variables θ we get the Lagrangian (after eliminating auxiliary fields F^i)

$$L = -g_{i\bar j}(\partial^\alpha \varphi^i)(\partial_\alpha \bar\varphi^j) - 2i g_{i\bar j} \bar\psi_-^{\bar j} \Delta_+ \psi_-^i - 2i g_{i\bar j} \bar\psi_+^{\bar j} \Delta_- \psi_+^i - R_{i\bar j k \bar e} \psi_+^i \psi_-^k \bar\psi_+^{\bar j} \bar\psi_-^{\bar e}$$

where $g_{i\bar j} = \frac{\partial^2 K}{\partial\varphi^i \partial\bar\varphi^{\bar j}}$; $\Gamma^i_{jk} = g^{i\bar e} \partial_k g_{\bar e j}$; $\partial_k = \frac{\partial}{\partial\varphi^k}$

$$\Delta_\pm \psi^i = \partial_\pm \psi^i + \Gamma^i_{jk}(\partial_\pm \varphi^j)\psi^k \, ; \quad R_{i\bar j k \bar e} = g^{m\bar n}(\partial_{\bar e} g_{m\bar j})(\partial_k g_{\bar n i}) - \partial_k \partial_{\bar e} g_{i\bar j}$$

+

* We have also two R-symmetries R_V and R_A

$$R_V\ (\theta^+;\bar\theta^+) \to (e^{-i\alpha}\theta^+;\ e^{i\alpha}\bar\theta^+)\ ;\ (\theta^-;\bar\theta^-) \to (e^{-i\alpha}\theta^-;\ e^{i\alpha}\bar\theta^-)$$

$$R_A\ (\theta^+;\bar\theta^+) \to (e^{-i\alpha}\theta^+;\ e^{i\alpha}\bar\theta^+)\ ;\ (\theta^-;\bar\theta^-) \to (e^{i\alpha}\theta^-;\ e^{-i\alpha}\bar\theta^-)$$

They multiply operators D by a constant.

* Coresponding generators

$$F_V = -\theta^+\frac{d}{d\theta^+} - \theta^-\frac{d}{d\theta^-} + \bar\theta^+\frac{d}{d\bar\theta^+} + \bar\theta^-\frac{d}{d\bar\theta^-}$$

$$F_A = -\theta^+\frac{d}{d\theta^+} + \theta^-\frac{d}{d\theta^-} + \bar\theta^+\frac{d}{d\bar\theta^+} - \bar\theta^-\frac{d}{d\bar\theta^-}$$

* Algebra

$$[F_V, Q_\pm] = Q_\pm \qquad [F_A; Q_\pm] = \pm Q_\pm$$

$$[F_V, \bar Q_\pm] = -\bar Q_\pm \qquad [F_A, \bar Q_\pm] = \mp \bar Q_\pm$$

* Are these symmetries also the ones for quantum theory?

Important Points (zero modes)

* The Path integral

$$\int D\psi_- D\psi_+ \exp(\bar\psi_- \Delta_+ \psi_-) \text{ is zero if } \Delta_+\psi_- = 0 \text{ (zero modes)}$$

* $k_- \equiv \dim \operatorname{Ker} \Delta_+ - \dim \operatorname{Ker} \Delta_+^{+}$ is called the index of the operator Δ_+. (Obviously $k_+ = \dim \operatorname{Ker} \Delta_- - \dim \operatorname{Ker} \Delta_-^{T} = -k$

* Atiyah-Singer inxdex theorem

$$k_- = \int_{\varphi(\Sigma)} C_1(M)$$

Assume we have k zero modes for ψ_- and $\bar{\psi}_+$ and 0 zero modes for $\bar{\psi}_-$ and ψ_+.

* Absorbtion of zero modes:

Define a path integral

$$\int D\psi_+ D\psi_- D\bar{\psi}_+ D\bar{\psi}_- \{ g_{i_1\bar{j}_1} \psi_-^{i_1} \bar{\psi}_+^{\bar{j}_1} \cdots g_{i_k \bar{j}_k} \psi^{i_k} \bar{\psi}^{\bar{j}_k} \} e^{-S}$$

i.e. insert as many fermions as ~~many~~ zero modes we have

* To have the invariance ~~und~~ of the path integral under $U(1)_V$ and $U(1)_A$:

1) R_V is present in the quantum theory for any Kahler target space (~~the~~ fields ψ_- and $\bar{\psi}_+$ have opposite charges

2) R_A axial symmetry is present if we have Calabi-Yau manifolds as target spaces ($C_1(M) = 0$)

* Note: all this is true if we neglect a term with $R_{i\bar{j}k\bar{l}}$ in the action

IV Twisted Theory (A, B twists)

* We considered a case of a flat world-sheet ($\mathbb{C}$ or T^2). What happens when the world-sheet is not flat?

* We would like to write down a variation of the form

$$\delta \varphi^i = \varepsilon Q \varphi^i$$

ε was a constant, but for an arbitrary world-sheet metric there is no covariantly constant spinor

* We need to define "new" Lorentz generators under which some of "Q-s and ψ-s become "scalars".

$$M_A = M - F_V \ / \ M_B = M - F_A$$

$$[M_A, Q_+] = -2Q_+; \ [M_A, Q_-] = 0; \ [M_A, \bar{Q}_+] = 0; \ [M_A, \bar{Q}_-] = 2\bar{Q}_-$$

$$[M_B, Q_+] = -2Q_+; \ [M_B; Q_-] = 2Q_-, \ [M_B, \bar{Q}_+] = 0; \ [M_B; \bar{Q}_-] = 0$$

And define

$$Q_A = \bar{Q}_+ + Q_-; \quad Q_B = \bar{Q}_+ + \bar{Q}_-; \quad Q_A^2 = Q_B^2 = 0.$$

* $Q_A (Q_B)$ is a scalar for $M_A (M_B)$ and therefore it can be taken as a symmetry operator in a metric independent way.

two dimensional CFT

* $T_{\alpha\beta}$ - Energy momentum tensor $\partial_\alpha T^{\alpha\beta} = 0$

* $T_{z\bar{z}} = T_{\bar{z}z} = 0$; $T_{zz} = T(z)$ is holomorphic

* Expand $T(z)$: $T(z) = \sum L_m z^{-m-2}$

* For a closed string $T(z)$ and $\bar{T}(\bar{z})$ are independent

* Conserved $U(1)$ current $J(z) = \sum J_m z^{-m-1}$; again $J(z)$ and $\bar{J}(\bar{z})$ are independent.

* Algebra: (Left part of it)

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{n+m,0}$$

$$[L_m, J_n] = -n J_{m+n}; \quad [J_m, J_n] = \frac{c}{3} m \delta_{m+n}$$

c is a constant central charge.

* A twist $L_0^A = L_0 - \frac{1}{2} J_0$ $\quad \bar{L}_0^A = \bar{L}_0 + \frac{1}{2}\bar{J}_0$

* B twist $L_0^B = L_0 + \frac{1}{2} J_0$; $\bar{L}_0^B = \bar{L}_0 - \frac{1}{2}\bar{J}_0$

* New current $\tilde{T}(z) = T(z) + \frac{1}{2}\partial J(z)$

$$\tilde{L}_m = L_m - \frac{1}{2}(m+1) J_m$$

$[\tilde{L}_m, \tilde{L}_n] = (m-n)\tilde{L}_{m+n} \to$ no central change

$\underline{\overline{V}}$ A model, B model

* Fermion field content:

$$\psi_+^i \equiv \psi_z^i \in \Omega^{1,0} \otimes \phi^*(T^{(1,0)}M) \; ; \; \bar{\psi}_-^{i} \equiv \psi_{\bar{z}}^{\bar{i}} \in \Omega^{0,1} \otimes \phi^*(T^{(0,1)}M)$$

$$\psi_-^i \equiv \chi^i \in \phi^*(T^{(1,0)}M) \; ; \; \bar{\psi}_+^{i} \equiv \chi^{\bar{i}} \in \phi^*(T^{(0,1)}M)$$

For example ψ_+^i becomes world-sheet (1,0) form after twisting and its index i means that it takes values in the holomorphic tangent bundle on M.

* The Lagrangian

$$L = -2t\left(g_{i\bar{j}}(\partial_z\phi^i)(\partial_{\bar{z}}\bar{\phi}^{\bar{j}}) + g_{i\bar{j}}(\partial_{\bar{z}}\phi^i)(\partial_z\bar{\phi}^{\bar{j}}) + i g_{i\bar{j}}\psi_z^i \Delta_{\bar{z}}\chi^{\bar{j}} + \right.$$

$$\left. + i g_{i\bar{j}}\psi_{\bar{z}}^{\bar{j}}\Delta_z\chi_-^i + \frac{1}{2}R_{i\bar{j}k\bar{\ell}}\psi_z^i\psi_{\bar{z}}^{\bar{j}}\chi^k\chi^{\bar{\ell}}\right);$$

t is an inverse to the Planck constant; $\Delta_+ = \Delta_z$; $\Delta_- = \Delta_{\bar{z}}$

* The Lagrangian can be covariantized to any world sheet.

* One can write $L' = \{Q_A, V\}$ where

$$V = g_{i\bar{j}}(\psi_z^i\partial_{\bar{z}}\bar{\phi}^{\bar{j}} + (\partial_z\phi^i)\psi_{\bar{z}}^{\bar{j}})$$

· $L - L' = 2t\, g_{i\bar{j}}\left((\partial_z\phi^i)(\partial_{\bar{z}}\bar{\phi}^{\bar{j}}) - (\partial_{\bar{z}}\phi^i)(\partial_z\bar{\phi}^{\bar{j}})\right)$

* Therefore $S - S' = t\int_{\phi(\Sigma)}\omega$ with $\omega = 2i g_{i\bar{j}}\, dz^i \wedge d\bar{z}^{\bar{j}}$ → Kähler form

* The integral depends on a Homology class.

$$S - S' = t\,\omega\,\beta \quad \text{where} \quad \beta \in H_2(M)$$

* Conclusions: 1) The theory is topological with respect to the world-sheet metric.

2) $\frac{dS'}{dt}$ is Q_A exact $\to$ we can calculate the path integral in the classical limit.

* The theory is „half topological", it depends on a Kähler class.

* Note: When deriving the relation between L and L' we have used equations of motion with respect ψ^i_z and $\psi^{\bar{i}}_{\bar{z}}$. Below we shall not use the correlators which involve these fields. This problem however can be cured in general. One defines a new operator $\tilde{Q}_A$ whose action on fields ψ equals to the one of Q_A up to equations of motion, and its action on the other fields is the same as of Q_A.

* General local operator

$$O_B = B_{i_1 \dots i_p, \bar{j}_1 \dots \bar{j}_q}(\varphi)\,\chi^{i_1} \dots \chi^{i_p}\,\chi^{\bar{j}_1} \dots \chi^{\bar{j}_q}$$

No $\psi_z, \bar{\psi}_{\bar{z}}$ fields or $\partial_z, \partial_{\bar{z}}$ here since it would introduce a world-sheet metric.

* Identifying $\chi^i \to d\varphi^i$; $\bar{\chi}^{\bar{i}} \to d\bar{\varphi}^{\bar{i}}$ we get

$B \to p,q$ form, $Q_A \to$ de Rham cohomology operator

* Consider a Calabi-Yau target space with complex dimension m.

* The number of zero modes for $\Delta_{\bar{z}}$ is

$$K = m(1-g); \quad g \to \text{genus}$$

* Therefore to get a nonzero correlator we need to take either $g=1$, or $g=0$ and m insertions of χ^i and $\bar{\chi}^{\bar{i}}$. For $g>1$ we need to absorb $\psi_{\bar{z}}^{i}$ modes which is impossible in a metric independent way, and with local operators.

* Calculations in the classical limit mean

$$\partial_{\bar{z}} \phi^i = \partial_z \phi^{\bar{i}} = 0$$

i.e, ϕ is a holomorphic map from Σ to M

· Q_A cohomology classes $\{O_S\}$ a Poincare dual to the homology classes $\to \{S\}$ basis of cycles of homology on M.

$$\int_{S_P} A^{(p)} = \int_M A^{(p)} \wedge S^{(2m-p)} \quad \text{for any } A^{(p)}$$

* Consider a correlator

$$\langle O_{S_1}(P_1) \ldots O_{S_r}(P_r) \rangle - P_i \text{ are marked Points on the world-sheet}$$

* In order the correlator to be nonzero, the sum of degrees of the forms S_i must be $2m(1-g)$

$$\langle O_{S_1}(p^1) \cdots O_{S_r}(p^r)\rangle_\beta = e^{-t\omega\beta} \# \mathcal{M}_{g,r}(M,\beta,S^i)$$

$\mathcal{M}_{g,r}(M,\beta,S^i) \Rightarrow$ moduli space of holomorphic maps of a Riemann surface Σ of degree g with r labeled points P_i on it into a certain cohomology class $\beta \in H_2(M)$ in such a way that $\phi(P_i)$ is mapped into S^i

* Operators O correspond to differential forms on M. In the path integral we are integrating over a moduli space $\mathcal{M}_{g,r}$.

* One can always „pull back" differential forms on the manifold to differential forms on the moduli space using distinguished points P_i.

* In the topological string (different from the topological field theory) for fixed points P_i are introduced corresponding moduli.

* Gromov-Witten invariants (generalization for the topological stringΣ of the numbers $\# \mathcal{M}_{g,r}(M,\beta,S^i)$)

B model

* Fermionic field content

$$\psi_+^i \in \Omega^{1,0} \otimes \phi^*(T^{(1,0)}M); \quad \psi_-^i \in \Omega^{0,1} \otimes \phi^*(T^{(1,0)}M)$$

$$\bar{\psi}_+^{\bar{i}} \in \phi^*(T^{(0,1)}M); \quad \bar{\psi}_-^{\bar{i}} \in \phi^*(T^{(0,1)}M)$$

* Introduce

$$\eta^{\bar{i}} = \bar{\psi}_+^{\bar{i}} + \bar{\psi}_-^{\bar{i}}; \quad \theta_i = g_{i\bar{j}}(\bar{\psi}_+^{\bar{j}} - \bar{\psi}_-^{\bar{j}}); \quad \rho_z^i = \psi_+^i; \quad \rho_{\bar{z}}^i = \psi_-^i$$

* The Lagrangian

$$L = -t\Big(g_{i\bar{j}}\, \eta^{\alpha\beta}(\partial_\alpha \phi^i)(\partial_\beta \bar{\phi}^{\bar{j}}) + i g_{i\bar{j}}\, \eta^{\bar{j}}(\Delta_{\bar{z}} \rho_z^i + \Delta_z \rho_{\bar{z}}^i) + i\theta_i(\Delta_{\bar{z}} \rho_z^i - \Delta_z \rho_{\bar{z}}^i)$$
$$+ \frac{1}{2} R_{i\bar{j}k\bar{l}}\, \rho_z^i \rho_{\bar{z}}^k \eta^{\bar{j}} \theta^{\bar{l}}\Big)$$

* Again $L' = -it\{Q_B, V\}$ with $V = g_{i\bar{j}}(\rho_z^i \partial_{\bar{z}} \bar{\phi}^{\bar{j}} + \rho_{\bar{z}}^i \partial_z \bar{\phi}^{\bar{j}})$

* $L - L' = -t\Big(i\theta_i(\Delta_{\bar{z}} \rho_z^i - \Delta_z \rho_{\bar{z}}^i) + \frac{1}{2} R_{i\bar{j}k}{}^{l}\, \rho_z^i \rho_{\bar{z}}^k \eta^{\bar{j}} \theta_l\Big)$

  $L - L'$ is (1,1) form, integral over a two dimensional manifold of it is metric independent.

* Rescaling $\theta_i \to t\theta_i$, the path integral changes by an overal factor of t.

* The theory does not depend on Kahler moduli (The variation of the action with respect to ω is Q_B exact)

* Local operators:

$$O_B = B_{\bar{i}_1 \cdots i_p}{}^{j_1 \cdots j_q}(\phi\bar{\phi})\, \eta^{\bar{i}_1} \cdots \eta^{\bar{i}_p}\, \theta_{j_1} \cdots \theta_{j_q}$$

$\bar{\eta}^i \to d\bar{\phi}^i$, $\theta_i \to \frac{\partial}{\partial \phi^i}$; $O_B \to (0,p)$ form with values in $\Lambda^q T^{(1,0)} M$.

* Correlation functions are nonzero for

$$p = q = m(1-g)$$

* Equations of motion from L' (it contains also $g_{i\bar{j}}(\partial_{\bar{z}}\varphi^i)(\partial_z \varphi^{\bar{j}})$ term

$$\partial_z \varphi^i = \partial_{\bar{z}} \varphi^i = \partial_z \bar{\varphi}^i = \partial_{\bar{z}} \bar{\varphi}^i = 0$$

ϕ is a constant map. Space of constant maps $\to$ M manifold itself.

* $B_{\bar{i}_1 \cdots \bar{i}_m}{}^{j_1 \cdots j_m} \to B_{\bar{i}_1 \cdots \bar{i}_m}{}^{j_1 \cdots j_m}\, \Omega_{j_1 \cdots j_m}\, \Omega_{k_1 \cdots k_m}$

* Observables of the B model are integrals of wedge products of forms over the tangent space.

To summarize

Observables

A model on X $\longrightarrow$ Kahler moduli on X

B model on X $\longrightarrow$ Complex structure moduli on X

VI Topological STRING THEORY

* For a topological string theory correlators extra integration is required with respect to the moduli of the world sheet

* They are constructed in a manner similar to the bosonic string

* In the bosonic string a genus g Free energy is

$$\int_{M_g} \langle | \prod_{i=1}^{3g-3} b(\mu_i) |^2 \rangle$$

where μ_i are Beltrami differentials (anti-holomorphic 1 forms on Σ with values on holomorphic tangent bundle) and

$$b(\mu) = \int_\Sigma b_{zz} \mu_{\bar z}^{z} \quad ; \quad b \to \text{ghost}$$

* For a topological string

$$F_g = \int_{M_g} \langle | \prod_{i=1}^{3g-3} G^-(\mu_i))^2 \rangle \quad \text{(Recall } N=2 \text{ SCFT } (J, G^+, G^-, T))$$

The full free energy $F = \sum_{g=0}^{\infty} \lambda^{2-2g} F_g$ $\quad \lambda \to$ string coupling constant

* Partition function: $Z = \exp F$

* For a topological string theory on a target Calabi-Yau maifold of a complex dimension 3 one obtains nonvanishing 3-point function of a total degree (3, 3) at genus 0, a nonvanishing one point function of degree (1,1) at genus 1, and nonvanishing partition function at all genera $g>1$

* Further insertions of local operators is not possible but one can insert nonlocal operators

* Define a „new" action

$$S[t] = S_0 + t^a \int O_a^{(2)} \to \text{invariant under } U(1)_A \text{ and } U(1)_V$$

One can find nonzero correlators $\langle \int O_1^{(2)} .. \int O_n^{(2)} \rangle$

Holomorphic anomaly (Let us consider B-model)

* To calculate correlation functions of primary operators we take t-derivatives of the corresponding perturbed partition function $Z(t)$ and consequently set $t=0$

$$S[t] = S_0 + t^a \int_\Sigma O_a^{(2)} \to \bar{Q}_B \text{ exact term}$$

* The complex conjugate of the holomorphic deformation $\bar{t}^a \int O_a^{(2)}$ turns out to be O_B exact, but since the insertion G^- is not a BRST invariant we still have a nonzero dependence of $F(\bar{t})$ due to contributions of boundary terms in the moduli space.

* $$\frac{\partial F_g}{\partial \bar{t}^{\bar{a}}} = \frac{1}{2} C_{\bar{a}\bar{b}\bar{c}}\, e^{2K} G^{\bar{b}d} G^{\bar{c}e} \left(D_d D_e F_{g-1} + \sum_{r=1}^{g-1} D_d F_r D_e F_{g-r} \right)$$

G - certain Kahler metric (Zamolodchikov metric)

K - its Kahler potential, D_a - covariant derivatives on this space.

* From this equation one can inductively determine the $\bar{t}^a$ dependence of the partition function if the holomorphic dependence is known.