Holonomy of pseudo-Riemannian cones

joint with:
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(M,g) pseudo-Riem. mf.
$\mathfrak{h} \subset so(V)$, $V = T_pM$,
its holonomy algebra

Def. $\mathfrak{h}$ decomposable
$:\Longleftrightarrow$ V contains a proper non-deg. $\mathfrak{h}$-inv. subspace.

(otherwise: indecomposable)

Wu - de Rham :

$\mathfrak{h}$ decomposable $\Longleftrightarrow$

M is loc. ps.-Riem. product

Def. $\mathfrak{h}$ reducible $:\Longleftrightarrow$

$\mathfrak{h}$ preserves a (possibly degenerate) proper subspace of V.

(otherwise : irreducible)

irred. $\Longrightarrow$ indecomp.

$\not\Longleftarrow$

$$\Big(\hat{M} = \underset{r}{\mathbb{R}^+} \times M,$$

$$\hat{g} = dr^2 + r^2 g\Big)$$

metric cone over M

$\hat{\mathfrak{h}}$ its holonomy alg.

Aim: relate

properties of $\hat{\mathfrak{h}}$ $\longleftrightarrow$

geometry of M

Ex. $\hat{\mathfrak{h}} = 0 \iff \begin{cases} \sec_M = 1 \\ \text{or } \dim M \le 1 \end{cases}$

$(\iff \hat{M}$ flat$)$

Thm. (Gallot '79)

(M,g) complete Riem.

Then $\hat{g}$ is irred. or $=0$.

completeness and positivity of g are necessary:

I. Counterexamples

Example 1 (completeness is needed even if $g>0$)

(M_1, g_1), (M_2, g_2) Riem. dim ≥ 2 and not both of const. curv. 1. Then

$$\Big(M = (0, \tfrac{\pi}{2}) \times M_1 \times M_2 ,$$
$$\qquad\quad s$$

$$g = ds^2 + \cos^2(s)\, g_1 + \sin^2(s)\, g_2 \Big)$$

is incomplete Riem. and

$\hat{g} \not\equiv 0$ is decomposable.

$$\left(\hat{M} \cong \hat{M}_1 \times \hat{M}_2 \right)$$

$$\begin{cases} r_1 = r\cos(s) \\ r_2 = r\sin(s) \end{cases}$$

<u>Example 2</u> (positivity is needed even for complete metrics)

(N, g_N) complete, $\dim \geqslant 2$ and not of const. curv. 1

Then

$$\left(M = \underset{s}{\mathbb{R}} \times N,\ g = -ds^2 + ch^2(s) g_N\right)$$

is complete ps.-Riem. and

$\hat{\mathfrak{h}} \neq 0$ is decomposable.

(The time-like v.f.

$X = r\,sh(s)\,\partial_r - ch(s)\,\partial_s$ generates

a parallel line.)

II. Local structure of (M, g) with decomp. $\hat{\mathfrak{h}}$

Thm. 1 $\hat{\mathfrak{h}}$ decomp. $\Longrightarrow$

(i) $\mathfrak{h} = so(p,q)$, where (p,q) = signature of g

(ii) $\exists$ dense open $M' \subset M$ s.t. any point $\in M'$ has neighborhood isometric to $\underset{s}{(a,b)} \times N_1 \times N_2$ with

$$g = \begin{cases} ds^2 + \cos^2(s)\, g_1 + \sin^2(s)\, g_2 \\ \text{or} \\ -ds^2 + ch^2(s)\, g_1 + sh^2(s)\, g_2 \end{cases}$$

Sketch of pf.

Assume $\hat{M}$ decomposable :

$$T\hat{M} = V_1 \overset{\perp}{\oplus} V_2$$ parallel decomposition

$$\Longrightarrow \begin{cases} \partial_r = X_1 + X_2 , & X_i \in V_i , \\ X_1 = \alpha \partial_r + X , & X \in TM , \\ X_2 = (1-\alpha)\partial_r - X \end{cases}$$

Lemma 1 The open subset

$$\mathcal{U} = \{ p \in \hat{M} \mid \alpha(p) \notin \{0,1\} \} \subset \hat{M}$$

is dense and the vector fields X_1, X_2 and X are nowhere isotropic on $\mathcal{U}$.

Curv. of $\widehat{M}$:

$$\begin{cases} \widehat{R}(\partial_r, \cdot) = 0 \\ \widehat{R}(Y,Z)W = R(Y,Z)W - \widehat{R}_1(Y,Z)W, \end{cases}$$

where $Y, Z, W \in TM$ and

$$\widehat{R}_1(Y,Z)W := \widehat{g}(Z,W)Y - \widehat{g}(Y,W)Z$$

$$\Rightarrow \underset{\widehat{V}_1}{\underset{\Uparrow}{\widehat{R}(X,Y_1)}} = \widehat{R}((1-\alpha)\partial_r - X_2, Y_1) = 0$$

$$\underset{\widehat{V}_2}{\underset{\Uparrow}{\widehat{R}(X,Y_2)}} = \widehat{R}(X_1 - \alpha\partial_r, Y_2) = 0$$

$$\Rightarrow \widehat{R}(X, \cdot) = 0 \Rightarrow R(X,Y) = \overline{R}_1(X,Y)\Big|_{TM}$$

$\Rightarrow \mathfrak{h} = so(p,q)$, since $\widehat{g}(X,X) \neq 0$ on $\mathcal{U} \neq \emptyset$. This proves (i).

Lemma 2 • $\partial_r \alpha = 0$, i.e. α is the pull back of a fct. on M

• $\mathcal{U} = \mathbb{R}^+ \times \mathcal{U}_1$

• $X = \frac{1}{r}\tilde{X}$, $\tilde{X} = \frac{1}{2}\operatorname{grad}(\alpha) \in \mathfrak{X}(M)$

• on $\mathcal{U}_1 \subset M$ we have

$$TM = L \overset{\perp}{\oplus} E_1 \overset{\perp}{\oplus} E_2$$

$$\nabla\tilde{X} : 1-2\alpha \, , 1-\alpha \, , -\alpha \, ,$$

where $L = \mathbb{R}\tilde{X}$, $E_i := TM \cap V_i$.

Lemma 3

(i) $E_1, E_2, E := E_1 \oplus E_2 \subset TM$ are involutive and $E_i \oplus L \subset TM$ are parallel on $\mathcal{U}_1$.

(ii) $\underbrace{N \subset \mathcal{U}_1 \text{ integral submf of } E}_{\text{through } x \in \mathcal{U}_1} \Rightarrow$ $E_i|_N \subset TN = E|_N$ parallel

Using the flow of $\tilde{X} = \frac{1}{2} \operatorname{grad}(\alpha)$ we obtain a diffeo.

$W \cong (a,b) \times N$, where

$W \subset U_1$ nbhd. of $x \in U_1$

$N \subset U_1$ level set of α.

$$g = \pm ds^2 + g_N(s) \qquad \Big(\pm = \operatorname{sign} \underbrace{(\alpha - \alpha^2)}_{= g(\tilde{X}, \tilde{X})}\Big)$$

By L.3 (ii), $N \underset{\text{loc.}}{\cong} N_1 \times N_2$,

$$g_N(s) = h_1(s) + h_2(s).$$

Using eigenspace decomp. of $\nabla \tilde{X}$, one proves

$$h_1 = \alpha g_1, \quad h_2 = \underbrace{\pm(1-\alpha)}_{>0} g_2$$

Solving the ODE $\tilde{X}(\alpha) = 2(\alpha - \alpha^2)$ yields $\alpha = \begin{cases} \cos^2(s) \\ \operatorname{ch}^2(s) \end{cases}$. $\blacksquare$

III Results for complete (M, g) with decomp. $\hat{\mathfrak{h}}$

Thm 2 (M, g) complete, $\hat{\mathfrak{h}}$ decomp. $\Longrightarrow$

$\exists$ dense open $M' \subset M$ s.t. any pt. $\in M'$ has neighborhood isometric to

(1) ps-Riem. mf. M_1 of const. curv. 1 or

(2) $M_2 = \mathbb{R}^+ \times N_1 \times N_2$

$$-ds^2 + ch^2(s)\, g_1 + sh^2(s)\, g_2,$$

where $\sec(g_2) = -1$ or $\dim N_2 \leq 1$.

$\left(\hat{M}_2 \cong \{r_1 > r_2\} \subset \hat{N}_1 \times \underbrace{\left(\mathbb{R}^+ \times N_2, -dr_2^2 + r_2^2 g_2 \right)}_{\text{time-like cone over } N_2} \right)$

Thm 3 (M, g) complete, compact, $\dim M \geq 2$, with decomp. hol. gp. of $\widehat{M}$ $\Longrightarrow$ (M, g) has const. curv. 1 and $\widehat{M}$ is flat.

Cor. Let (M, g) be a cp. s.c. complete indef. ps.-Riem. mf. Then $\widehat{g}$ is indecomp.

The pf. of Thms. 2-3 relies on:

Lemma (M, g) complete, $(\hat{M}, \hat{g})$ decomposable and W a component of $\mathcal{U}_1 = \{\alpha \neq 0, 1\} \subset M$ s.t. $0 < \alpha < 1$ on W. Then $\hat{W}$ is flat.

To prove Thm. 3 one shows that on a cp. mf. M one always has $0 \leq \alpha \leq 1$

IV Cones with indecomp. reducible holonomy

Now $\hat{\mathfrak{h}}$ is indecomp. but preserves a deg. subspace $V \subset T_p\hat{M}$.

We consider two cases:

IV.1) $\hat{\mathfrak{h}}$ preserves a non-triv. decomposition

$$T_p\hat{M} = V \oplus W$$

IV.2) $(\hat{M}, \hat{g})$ is Lorentzian

We show that IV.1) $\Rightarrow$

$(\hat{M}, \hat{g})$ has (loc.) a

para-Kähler structure, i.e.

parallel skew-symm.

$\hat{J} \in \Gamma(\operatorname{End} T\hat{M})$ s.t.

$$\hat{J}^2 = id_{T\hat{M}}$$

($\Rightarrow$ $\hat{g}$ has neutral signature
and dim M odd)

<u>Thm. 4</u> $\exists$ 1-1 correspondence between para-Kähler structures $(\hat{g}, \hat{J})$ on $(\hat{M}, \hat{g})$ and para-Sasakian structures (g, T) on (M, g).

The correspondence is given by

$$T \longmapsto \hat{J} = \hat{\nabla} T$$

Def. A <u>para-Sasakian str.</u> on (M, g) of signature $(n+1, n) = (-,\dots,-,+,\dots,+)$ is a time-like geodesic unit Killing v.f. T s.t.

$$J := \nabla T \big|_{T^\perp} \text{ defines}$$

an integrable para-cx. str. on $E = T^\perp$.

- We have a similar corresp. between para-h.K. cones $\widehat{M}$ and para-3-Sasakian mfs. M.

IV.2 Lorentzian cones

Thm 5 Let $\hat{M}$ be Lorentzian with indecomp. reducible hol. gp.

1) M s.c. $\Rightarrow$ $\hat{M}$ has parallel light-like v.f.

2) If $\hat{M}$ has signature $(+,-,\dots,-)$ and has a parallel light-like v.f., then $\forall p \in M$ $\exists$ nbhd.

$W \cong (a,b) \times N$, $(a,b) \subset \mathbb{R}$,

$g|_W = -ds^2 + e^{2s} g_N$, $g_N < 0$,

$\mathfrak{hol}(\hat{W}) \cong \mathfrak{hol}(N) \ltimes \mathbb{R}^{\dim N}$

(g complete $\Leftrightarrow$ g_N complete and $(a,b) = \mathbb{R}$)

3) If $\hat{M}$ has signature $(+,\dots,+,-)$ and has a parallel light-like v.f., then $\exists$ dense open $M' \subset M$: $\forall p \in M'$ $\exists$ nbhd. $W \cong (a,b) \times N$:

$$g|_W = -ds^2 + e^{2s} g_N , \quad g_N > 0,$$

$$\mathrm{Hol}(\hat{W}) = \mathrm{Hol}(N) \ltimes \mathbb{R}^{\dim N}.$$

(g complete $\Rightarrow$ $(a,b) = \mathbb{R}$)