Fibonacci Sequences

Consider a typical Fibonacci sequence, f_n (n = 1,2,3,................):

1, 2, 3, 5, 8, 13, 21, 34, ...

where the first two terms, f_1 and f_2 , are given and the remaining terms are given by $f_{n+2} = f_{n+1} + f_n$. Note that the first two terms can be any numbers, positive or negative, but not 0.

Now consider the sequence consisting of the ratios of the successive terms in the Fibonacci sequence to the previous term, $r_n = f_{n+1}/f_n$, (n = 1,2,3,................):

2, 3/2, 5/3, 8/5, 13/8, 21/13, 34/21,..............................

The numbers in the sequence of ratios seem to be staying quite small and positive, unlike those in the original Fibonacci sequence which, due to the addition of terms and the fact that the first two terms can be positive or negative, become larger and larger and possibly negative.

So, here's a question:

Does the sequence of ratios converge (i.e. sooner or later, are successive terms the same (to some specified number of decimal places))? If so, to what number does it converge and how does this depend on the starting terms in the Fibonacci sequence?

Do some experiments with Excel, say to 6 decimal places, and you will see that it does converge (quite quickly to a small number of decimal places) and, amazingly, it seems to converge to the same number, regardless of the starting values in the original sequence (even when these are very large or small, positive or negative, or rational or irrational). In fact, the number to which it converges is a fundamental number called the Golden Ratio, which is $(1 + \sqrt{5})/2$. This number occurs in various fields of maths and in practical applications, such as architecture and paper sizing.

Now here is your job:

Assume that the sequence of ratios does converge, so that $r_{N+1} = r_N$, for some integer N. Now remember that the terms in the original Fibonacci sequence are given by $f_{n+2} = f_{n+1} + f_n$ and the terms in the sequence of ratios are given by $r_n = f_{n+1}/f_n$. Use these facts to show that $r_N = (1 + \sqrt{5})/2$.

Note that you will get an equation which has two solutions, one of which you can rule out because it is negative and the ratios clearly, sooner or later, are positive, regardless of whether the starting values are positive or negative.

As an exercise, if you have done complex numbers at school, you might like to think about what happens in the case of a Fibonacci sequence with complex terms, and why.