Using mathematics to understand coral reef dynamics

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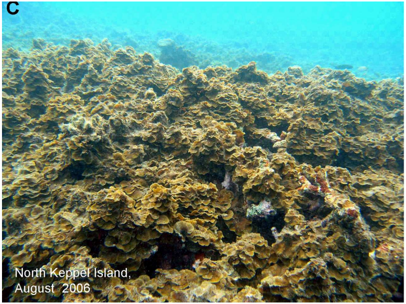
Coral reef states

Coral-dominated

Algal-dominated

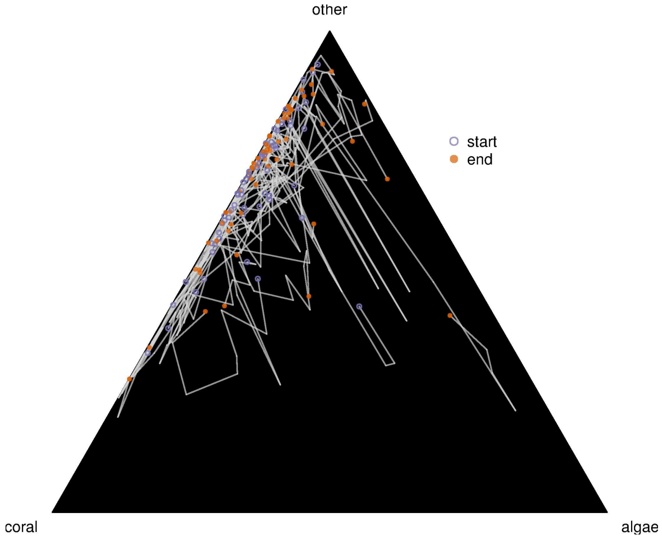


Photos: Diaz-Pulido et al. 2009, PLoS ONE 4:e5239

Coral reef video surveys

Image: Australian Institute of Marine Science

What do the surveys show?



46 sites, surveyed between 1996 and 2006.

Satellite measurements of sea surface temperature

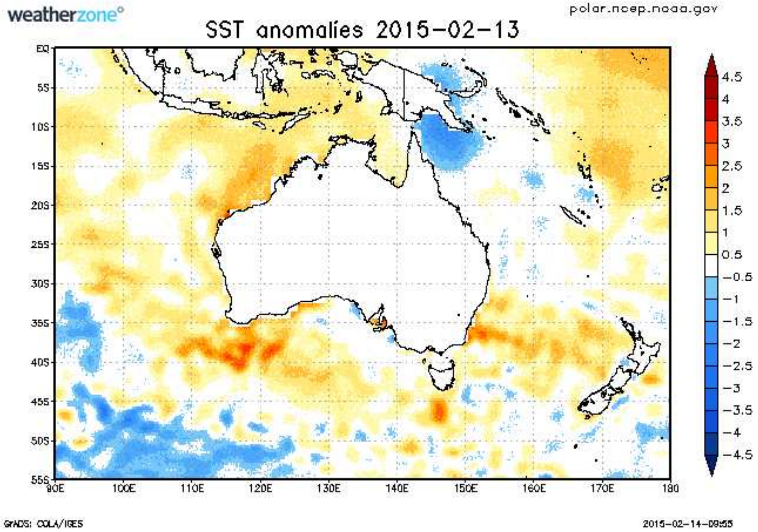
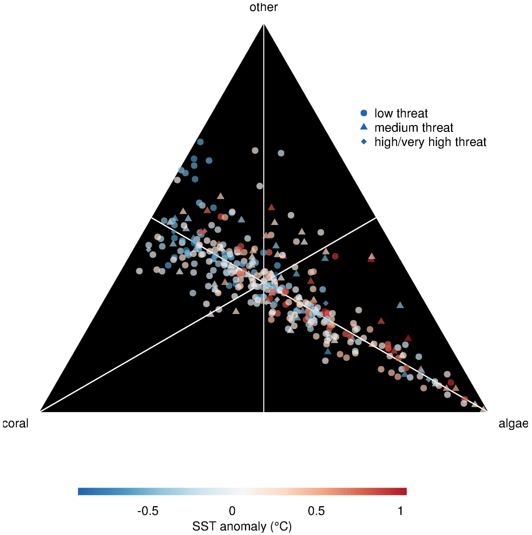


Image: www.weatherzone.com.au

Year-to-year changes depend on sea surface temperature



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- A very simple model for changes in reef composition is

$$x_{t+1} = a(z) + bx_t,$$

where $x_t = \log(\text{proportion coral}/\text{proportion not coral})$ at time t, $a(z)$ is a function of sea surface temperature z, b is the effect of a unit increase in x_t on x_{t+1}.

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- You may have recognized that this is a *linear recurrence sequence*.

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Suppose we know the value of x at time 0, and sea surface temperature has some constant value z (representing climate).

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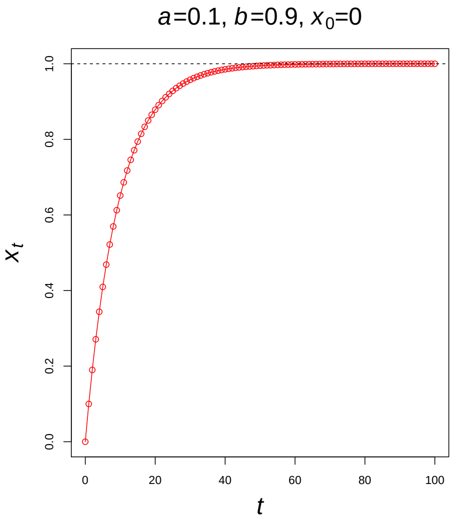
- Also, if b is between -1 and 1, then as $t \to \infty$, $b^t \to 0$.
- Thus, if b is between -1 and 1, the long-term value of x will tend to

$$x^* = \frac{a(z)}{1-b}.$$

Behaviour of linear recurrence sequences



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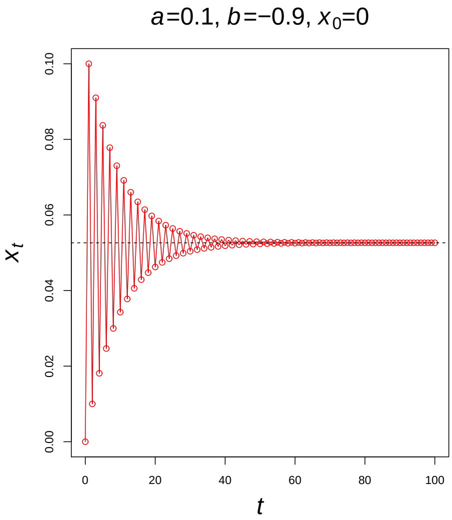

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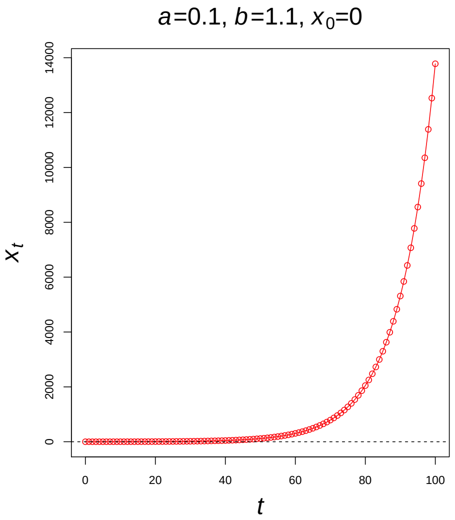

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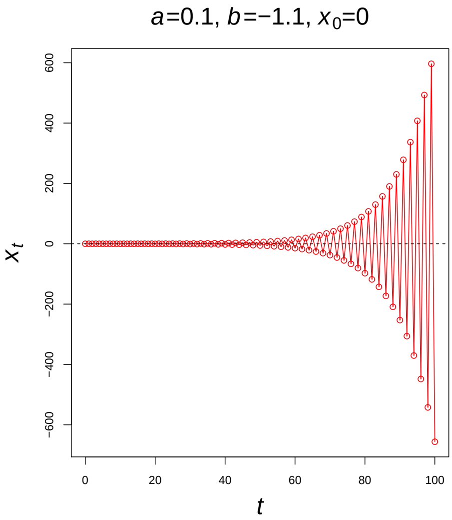

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- The *chain rule* says that this is the product of how x_{t+1} changes with respect to tiny changes in $a(z)$, and how $a(z)$ changes with respect to tiny changes in z:

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- If we increase $a(z)$ by some tiny amount, x_{t+1} increases by the same amount, so $\frac{\mathrm{d}x_{t+1}}{\mathrm{d}a(z)} = 1$ and $\frac{\mathrm{d}x_{t+1}}{\mathrm{d}z} = \frac{\mathrm{d}a(z)}{\mathrm{d}z}$.

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- If we increase $a(z)$ by a tiny amount, x^* increases by $\dfrac{1}{1-b}$ times that amount, so $\dfrac{\mathrm{d}x_{t+1}}{\mathrm{d}a(z)} = \dfrac{1}{1-b}$ and

$$\frac{\mathrm{d}x^*}{\mathrm{d}z} = \frac{1}{1-b} \times \frac{\mathrm{d}a(z)}{\mathrm{d}z}.$$

Short- versus long-term effects of sea surface temperature

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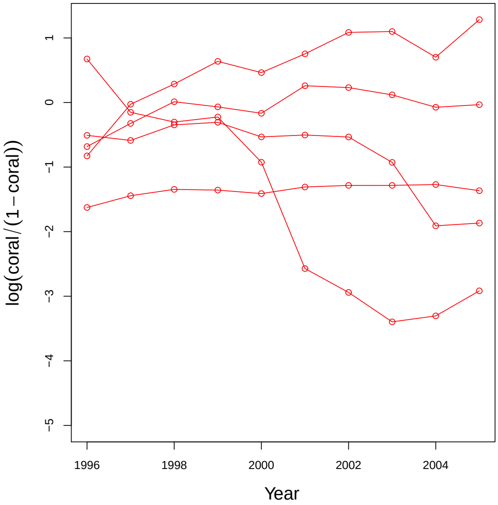
- Which is bigger? Remember we assumed $-1 < b < 1$.

Some real data



Adding a random component

- In the real world, unpredictable things happen. To describe this in our model, we need to add a random component:

$$x_{t+1} = a(z) + bx_t + \epsilon_t,$$

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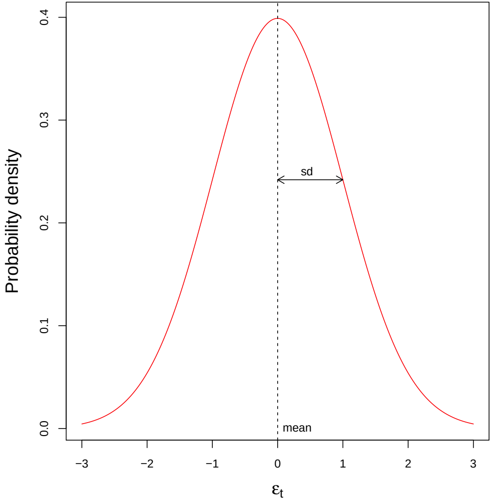
- We'll assume that ϵ_t has a normal distribution with mean 0 and variance σ^2.

The normal distribution



The standard deviation is the square root of the variance.

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Long-term behaviour

- For $t = 0, 1, 2, \ldots,$

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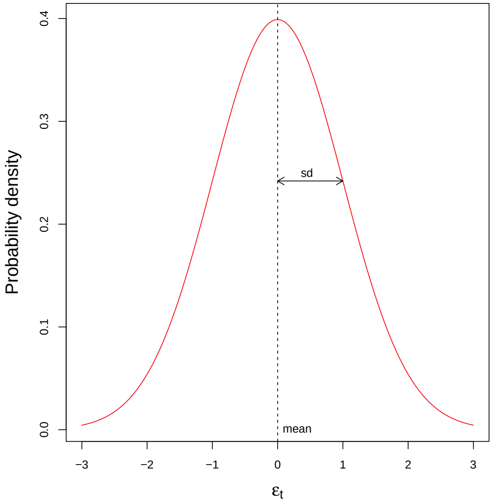
- We need to figure out how the noise term behaves as t gets large.

What happens to the noise terms?



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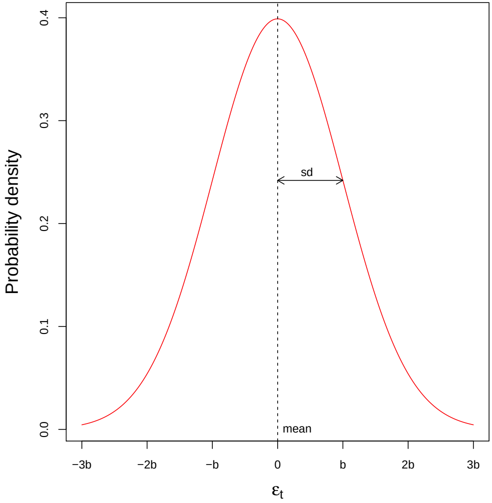

Multiplying by b keeps the mean at zero. The standard deviation is also multiplied by b, so the variance (the square of the standard deviation) is multiplied by b^2.

Adding up the noise terms

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- The variances are σ^2, $b^2\sigma^2$, $b^4\sigma^2$, ...
- If we assume each noise term is independent of the others, the variances also add up, to give

$$\begin{aligned} V &= \sigma^2 + b^2\sigma^2 + b^4\sigma^2 + \ldots + b^{2(t-1)}\sigma^2 \\ &= \sigma^2(1 + b^2 + b^4 + \ldots + b^{2(t-1)}) \end{aligned}$$

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- It's really exactly the same, but with b^2 instead of b. So if b^2 is between -1 and 1 (when will this be true?), then as $t \to \infty$,
$$V \to \frac{\sigma^2}{1-b^2}$$

Long-term behaviour

- For $t = 0, 1, 2, \ldots,$

$$x_t = a(z)(1 + b + b^2 + \ldots + b^{t-1}) + b^t x_0$$
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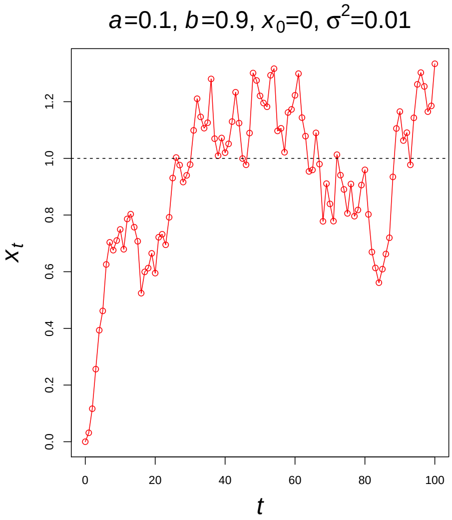
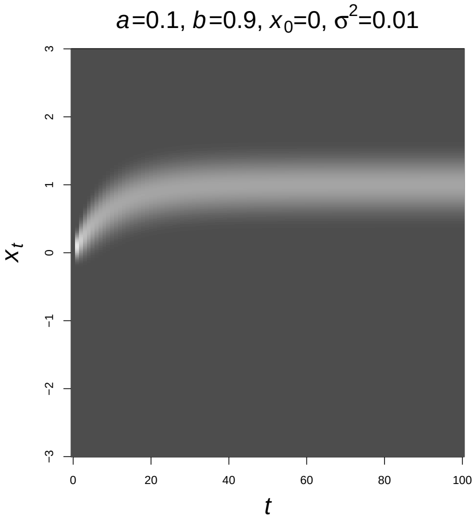
- As $t \to \infty$, x_t will approach mean $a/(1-b)$ (the same as before) and variance $\sigma^2/(1-b^2)$.

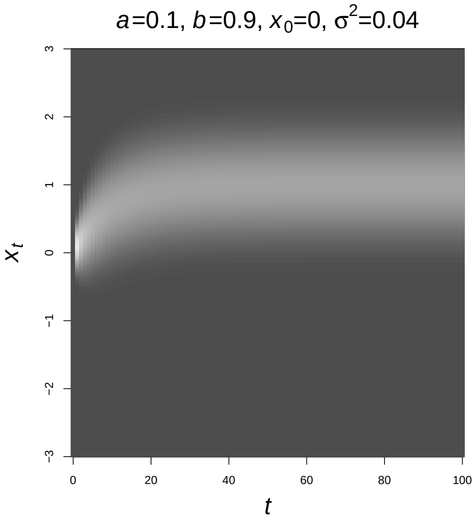
Running this model once



What do we expect from this model?



Increasing the variance



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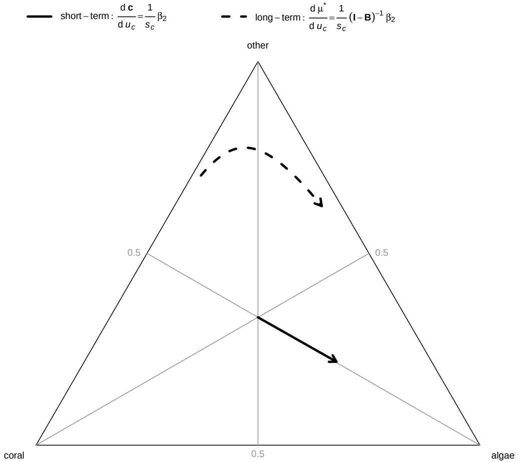
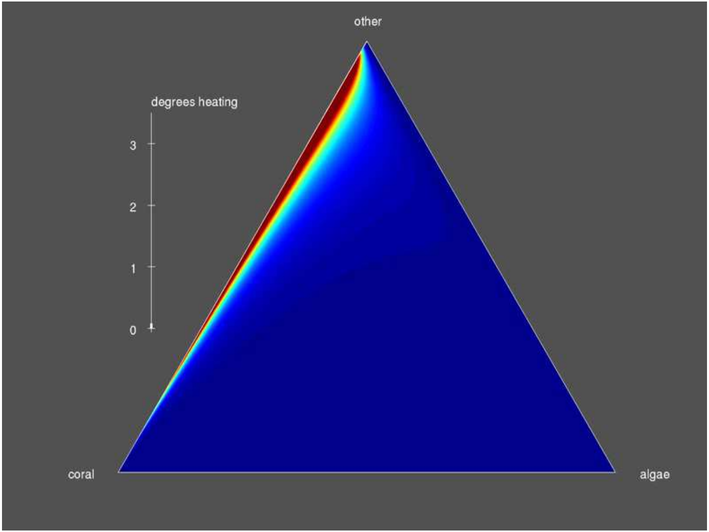
- This means that sea surface temperature effects will have a direction as well as a magnitude.
- We have to account for other variables such as fishing and nutrients running off the land.
- We have to check the condition equivalent to b being between -1 and 1: it was true for the Great Barrier Reef.

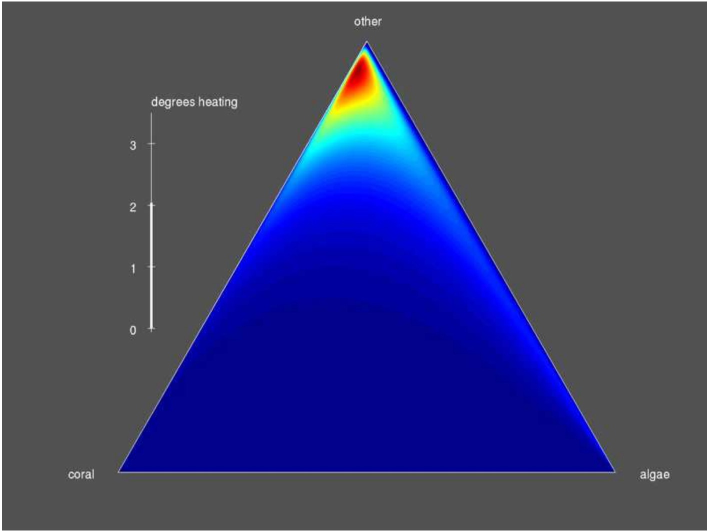
Direction of short- and long-term effects of a warmer climate



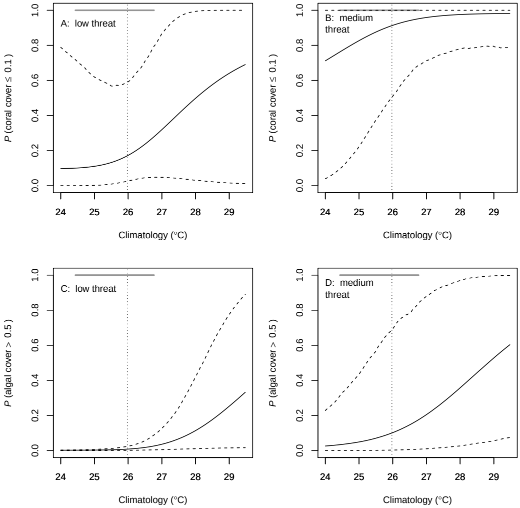
Long-term behaviour under current climate



Long-term behaviour in a warmer climate



Probabilities of low coral cover and high algal cover



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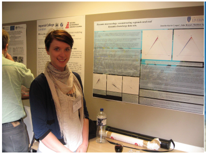
- The reason for the difference can be at least partly understood with A-level mathematics (using ideas that included sequences, calculus and probability).
- You can read the full results at `http://www.liv.ac.uk/~matts/stochasticgbr.html`.
- Lots of great pictures and videos of coral reefs: `http://catlinseaviewsurvey.com/`.

Acknowledgements

- Jennifer Cooper, undergraduate student, University of Liverpool (now at James Cook University, Australia).
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