# Estimates on Pleated Surfaces

Mary Rees

Warwick Symposium July 2007

**Pleated surface: determination**

A pleated surface is determined by a continuous map

$$f : S \to N$$

from a finite type surface $S$ to a hyperbolic 3-manifold $N$, together with a topological lamination $\ell$ on $S$, the *pleating locus*, such that

- the (lifted) image of each leaf of $\ell$ is geodesic, with $f$ lifting to a homeomorphism on the lift of each leaf and
- the(lifted) image of each component of $S \setminus \ell$ is totally geodesic, with $f$ lifting to a homeomorphism on this component.
- If $S$ has cusps we also require cusps to map to cusps.

**Domain and Target**

A pleated surface $Imp(f) : S \to N$ determines

- A hyperbolic surface $S(f)$, together with a homeomorphism

  $$f : S \to S(f),$$

  which determines an element $[f]$ of the Teichmüller space $\mathcal{T}(S)$
- a map

  $$Imp_1(f) : S(f) \to N$$

  which is distance decreasing with respect to the hyperbolic metrics on $S(f)$ and $N$

  such that

  $$Imp(f) = Imp_1(f) \circ f.$$

**Pleating locus restrictions**

- A *maximal multicurve on* $S$ is a set of isotopically distinct, disjoint closed simple nontrivial nonperipheral loops on $S$, such that $S \setminus (\cup\Gamma)$ is a union of pairs of pants, with some pant ends at cusps, if $S$ has cusps.
- We shall only consider pleating loci $\ell$ which consist of a maximal multicurve $\Gamma$ and geodesics asymptotic to loops of $\Gamma$ and/or cusps at both ends, so that each component of $S \setminus \ell$ is a topological triangle.

**Existence**

- Fix a continuous map $f_0 : S \to N$ and a maximal multicurve $\Gamma$ on $S$. Then any pleating locus extending $\Gamma$ is the pleating locus of some $f : S \to N$ in the homotopy class of $f_0$ if and only if $f_0(\gamma)$ is homotopically nontrivial and nonperipheral for each $\gamma \in \Gamma$.
- (Peripheral images are sometimes allowed.)
- For fixed $f_0 : S \to N$, and any nontrivial nonperipheral loop $\gamma$ on $S$, we write $\gamma_*$ for the closed geodesic freely homotopic to $f_0(\gamma)$.

**Length**

- Given $f : S \to S(f)$, and a nontrivial nonperipheral closed loop $\gamma \subset S$, we write $|f(\gamma)|$ for the hyperbolic length of the (unique) geodesic on $S(f)$ which is freely homotopic to $f(\gamma)$.
- We write $|\gamma_*|$ for the length of $\gamma_*$ in $N$.
- Then $|\gamma_*| \leq |f(\gamma)|$

**Injectivity Radius Lemma**

- *Fix a Margulis constant $\varepsilon_0$.*
- *There are constants $L_1$ and $\varepsilon_1$ depending only on $\varepsilon_0$ and $S$ such that the following holds.*
- *Let $f : S \to N$ be a pleated surface. Suppose that $f(\zeta)$ is nontrivial in $N$ for any nontrivial closed loop $\zeta$ on $S$ with $|f(\zeta)| \leq L_1$ .*

*Then for any $\varepsilon \leq \varepsilon_0$,*

$$f((S(f))_{<\varepsilon}) \subset N_{<\varepsilon} \tag{1}$$

*and*

$$f((S(f))_{\geq\varepsilon_0}) \subset N_{\geq\varepsilon_1} \tag{2}$$

**Proof of (2)**

- If $K$ is a component of $(S(f))_{\geq\varepsilon_0}$, then its diameter is $\leq C_1$, where $C_1$ is a constant depending only on $S$ and $\varepsilon_0$.
- So $f(K)$ has diameter $\leq C_1$.
- So if

  $$f(K) \cap N_{<\varepsilon_1} \neq \emptyset$$

  we have

  $$f(K) \subset N_{<\varepsilon_0}$$

  which is impossible.

**Theorem 1**

*Fix a finite type surface $S$ and a Margulis constant $\varepsilon_0$.*

*The following holds for a sufficiently large $L_1$, and for a sufficiently large $L_3$ given $L_2$.*

*For $j = 1$, $2$, let $f_j : S \to N$ be homotopic pleated surfaces with pleating loci including including maximal multicurves $\Gamma_j$. Suppose that*

1. *$|\gamma_*| \geq \varepsilon_0$ for all $\gamma \in \Gamma_1 \cup \Gamma_2$;*
2. *$\#(\Gamma_1 \cap \Gamma_2) \leq L_2$;*
3. *$f_1(\zeta)$ is nontrivial in $N$ for any nontrivial closed loop $\zeta$ on $S$ with $|f_1(\zeta)| \leq L_1$ .*

*Then*

- *$d([f_1], [f_2]) \leq L_3$, where $L_3$ denotes Teichmüller distance,*
- *There is a homotopy in $N$ between $f_1$ and $f_2$ with tracks of hyperbolic length $\leq L_3$.*

**Theorem 2** *Fix a finite type surface $S$ and a Margulis constant $\varepsilon_0$.*

*The following holds for a sufficiently large $L_1$ given $L_2$.*

- *Let $f : S \to N$ be a pleated surface with pleating locus including maximal multicurve $\Gamma$.*
- *Let $\zeta$ be a nontrivial loop on $S$ such that $f(\zeta)$ is trivial in $N$.*
- *Suppose that*
  1. *$|\gamma_*| \geq \varepsilon_0$ for all $\gamma \in \Gamma$;*
  2. *$\#(\Gamma \cap \zeta) \leq L_2$;*

*Then there is a nontrivial closed loop $\zeta'$ on $S$ such that $f(\zeta')$ is trivial in $N$ and*

$$|f(\zeta')| \leq L_1.$$

**Proof of Theorem 1.**

- Concentrate on the bound on homotopy tracks.
- Bound the distance between $f_1(\gamma) \subset f_1(S(f_1)) \subset N$ and $\gamma_* = f_2(\gamma) \subset N$, for $\gamma \in \Gamma_2$.
- We can write $\gamma$ as a union of $\leq 6L_2$ arcs in the pleating locus of $f_1$ which map under $f_1$ to geodesic arcs in $N$, and $\leq 6L_2$ short arcs in $S(f_1)$.

  Call such a short arc $\tau$.
- Then $f_1(\gamma)$ is a bounded distance from a geodesic if , for any homotopic image $f(\tau')$ of $f(\tau)$, keeping endpoints in the pleating locus, such that the endpoints of $f_1(\tau')$ are close in $N$, the path $\tau'$ is bounded in $S(f_1)$.

**Proof of Theorem 2.**

- $f(\zeta)$ can be written as a union of $\leq 6L_2$ geodesic arcs in $S(f)$, and $\leq 6L_2$ short paths in $S(f)$.
- $f(\zeta)$ is the boundary of a disc in $N$, made up of $\leq 12L_2 - 2$ geodesic triangles in $N$, which hence has the structure of a hyperbolic surface with piecewise geodesic boundary.

  Consider a short path $\tau$ across the disc.
- Then $|f(\zeta)|$ is bounded if, for any homotopic image of $f(\tau')$ of $f(\tau)$, keeping endpoints in the pleating locus, such that the endpoints of $f(\tau')$ are close in $N$, the path $\tau'$ is bounded in $S(f)$.

**Comparing hypotheses**

- The first two hypotheses of Theorems 1 and 2 are rather similar
- The third hypothesis of Theorem 1 is the opposite of the conclusion of Theorem 2.
- So we can assume this hypothesis of Theorem 1 in the hypotheses of Theorem 2.
- In the notation of Theorem 2 this becomes:$f(\zeta')$ *is nontrivial in* $N$ *for any non-trivial closed loop* $\zeta'$ *on* $S$ *with* $|f(\zeta')| \leq L_1$.

BothTheorems 1 and 2 can be deduced from *The Short Bridge Arc Lemma*.
**Short Bridge Arc Lemma** *The following holds for* $L_1$ *sufficiently large given* $L_2$ *and* $S$.

- *Let* $f : S \to N$ *be a pleated surface.*
- *For* $j = 1, 2$*, let* $t \mapsto \gamma_j(t) : [0, T] \to S$ *be continuous, such that* $f \circ \gamma_j$ *is a geodesic in* $S(f)$*, not transverse to the pleating locus, with length parameter* $t$.
- *Fix lifts* $\tilde{f} : H^2 \to H^3$, $\tilde{\gamma_j} : [0, T] \to H^2$.
- *Let* $d_2$ *and* $d_3$ *denote the hyperbolic metrics in* $H^2$ *and* $H^3$.

*Suppose that:*

- $|\gamma_*| \geq \varepsilon_0$ *whenever* $\gamma$ *is a closed loop in the pleating locus of* $f$*;*
- $f(\zeta)$ *is nontrivial in* $N$ *for any nontrivial closed loop in* $\zeta$ *on* $S$ *with* $|f(\zeta)| \leq L_1$ *;*
- $d_2(\tilde{\gamma_1}(0), \tilde{\gamma_2}(0)) \leq L_2$*;*
- $d_3(\tilde{f} \circ \tilde{\gamma_1}(t), \tilde{f} \circ \tilde{\gamma_2}(t)) \leq L_2$ *for all* $t \in [0, T]$.

*Then*

$$d_2(\tilde{\gamma_1}(t), \tilde{\gamma_2}(t)) \leq L_1 \text{ for all } t \in [0, T]$$

**Ideas of proof**

- Suppose for simplicity that $\tilde{\gamma_1}$ and $\tilde{\gamma_2}$ are closest at $t = 0$, and distance at least 1 apart at $t = 0$.
- For a constant $C_1$ the following holds. For all $t \in [0, T]$ and $j = 1, 2$, $\tilde{\gamma_j}(t)$ is distance $\leq C_1$ from the geodesic with endpoints $\tilde{\gamma_1}(T)$ and $\tilde{\gamma_2}(T)$.
- The whole geodesic must project into $(S(f))_{\geq\varepsilon_0}$ if $\varepsilon_0$ is sufficiently small given $L_2$.
- For a constant $L_3$ depending only on $\varepsilon_0$, and a fixed basepoint $x_0$ in $H^2$, projecting to the component of $(S(f))_{\geq\varepsilon_0}$, any point $\tilde{\gamma_j}(t)$ is distance $\leq L_3$ from $g_{t,j}.x_0$ for some element $g_{t,j}$ of the covering group.
- $$d_3(g_{t,1}\tilde{f}(x_0), g_{t,2}.\tilde{f}(x_0)) \leq L_2 + 2L_3.$$
- For a $T_1$ depending only on $L_2$ and some $t$, $s \leq T_1$, $g_{t,1}^{-1}g_{t,2}g_{s,1}g_{s,2}^{-1}$ is trivial in $\pi_1(N)$ but not in $\pi_1(S_1)$.
- This gives the required contradiction if $L_1$ is large enough.

**Removing the hypothesis $|\gamma_*| \geq \varepsilon_0$.**

**Lemma 1.** *The following holds for sufficiently large $L_1$, and for sufficiently large $L_2$ given $L_0$.*

- *Let $f : S \to N$ be a continuous map. Let $\Gamma_1$ be a maximal multicurve*
- *Let $[\varphi] \in (\mathcal{T}(S))_{\varepsilon_0}$ with $|\varphi(\Gamma_1)| \leq L_0$.*
- *Suppose that $f(\zeta)$ is nontrivial in $N$ whenever $\zeta$ is a nontrivial closed loop in $S$ with $|\varphi(\zeta)| \leq L_1$.pause*

*Then there is a maximal multicurve $\Gamma_2$ such that*

$$|\gamma_*| \geq \varepsilon_0 \text{ for all } \gamma \in \Gamma_2$$

*and*

$$|\varphi(\Gamma_2)| \leq L_2.$$

**How to bound geometry of the Scott core**

- These results are instrumental in obtaining biLipschitz bounds on the non-interval-bundle part of the cusp-relative Scott core of a hyperbolic 3-manifold $N$ with finitely generated fundamental group.
- The biLipschitz constant is bounded in terms of the topological type of $N$ and and a constant $c$ which is $> 0$ for any set of end invariants

  (This constant is not needed if all ends are incompressible.)

- In fact there are constants $c_1$ and $c_2$ which are $> 0$ for any set of end invariants, where $c_1$ gives Lipschitz bounds, while the constant $c_2$ gives biLipschitz bounds.
- In general $c_2$ is much smaller than $c_1$, although the two are boundedly proportional in the case of combinatorial bounded geometry.

**The idea**

- The main hypothesis of Theorem 1:

  *$f(\zeta)$ is nontrivial in $N$ whenever $\zeta$ is a nontrivial closed loop on $S$ with $|f(\zeta)| \leq L_1$*

  holds whenever $f : S \to N$ is a pleated surface sufficiently far out in an end.
- We can then apply Theorem 1 and a theory of Teichmüller geodesics to show that a suitably defined family of pleated surfaces give rise to a Lipschitz map (with bounded constants) from ends of a model manifold to ends of $N$,
- and use Theorem 2 and the Teichmüller geodesic theory to show that the Lipschitz map is defined on all but a bounded part (depending on $c$) of the model end manifolds.
- If there are compressing discs in the core with boundary in the boundary of the core manifold, we then have bounds on their geometry, using Theorem 2.
- Another theorem, similar to Theorem 2, can then be used to help bound the geometry of the Scott core.

**Theorem 3** *The following holds for a sufficiently large constant $L_1$ given $S_1$, $S_2$, $\varepsilon_0$ and $L_2$.*

- *Let $W \subset N$ with be compact connected and an essential submanifold of $N$ with boundary $S_1$, with $S_1$ incompressible in $W$, not necessarily connected.*
- *Let $S_2 \subset W$ be a compact subsurface with boundary with $S_2 \cap S_1 = \partial S_2$.*
- *Let $S_2$ be incompressible and boundary incompressible*
- *Let $f_1 : S_1 \to N$ be a pleated surface homotopic to inclusion with pleating locus including a maximal multicurve $\Gamma_1$.*
- *Suppose that $\#(\partial S_2 \cap \Gamma_1) \leq L_2$.*
- *Suppose that $|\gamma_*| \geq \varepsilon_0$ for all $\gamma \in \Gamma_1$.*

*Then one of the following holds:*

1. $|f_1(\partial S_2)| \leq L_1$.
2. *There is an essential annulus $S_3 \subset W$ with $\partial S_3 = S_3 \cap W$ and $|f_1(\partial S_3)| \leq L_1$.*

**Idea of proof of Theorem 3**

- Triangulate $S_2$ with boundary consisting of segments from the pleating locus of $S_1$, and some short arcs. This gives $S_2$ the structure of a complete hyperbolic structure, covered by a subset of $H^2$. Take a thick -thin decomposition of this surface. The thin part (if nonempty) includes cylinders with short core and long thin rectangles.
- Consider long thin rectangles.

**Idea continued**

- The main difference from Theorem 2 is that we need to look at lifts to $H^3$ of geodesics in different lifts of $f_1(S_1)$ which bound a thin rectangle in $H^3$ for along a sufficiently long length.

  Rectangles which lift with boundaries in the same lift of $f_1(S_1)$ are impossible by the Short Bridge Arc Lemma
- We can assume that these close geodesic segments $\tilde{\gamma_1}(t)$ and $\tilde{\gamma_2}(t)$ project to $(S(f_1))_{\geq\varepsilon}$ for $\varepsilon$ depending on $L_2$.
- We can find $t_1$ and $t_2$ with $t_1 - t_2$ bounded in terms of $L_2$, and $g_1$ and $g_2$ covering group of $S(f_1)$ such that $d_2(\tilde{\gamma_j}(t_2), g_j.\tilde{\gamma_j}(t_1) < \varepsilon/10$.
- Then use the Annulus Theorem to get an embedded essential annulus in $S(f_1)$ with boundary of bounded length.

**Condition on the Masur constant to give the $L_1$ condition**

The condition: $f(\zeta)$ *is nontrivial in* $N$ *for any nontrivial closed loop in* $\zeta$ *on* $S$ *with* $|f(\zeta)| \leq L_1$ is satisfied for all pleated surfaces in the corresponding to an end $e$ of $N_d$, for all pleated surfaces determined by points in the model manifold at least a certain distance from the core, with this distance depending on $c_1 > 0$, if the (geodesic lamination) ending invariant $\mu(e)$ satisfies the following condition:

- For any simple closed loop $\zeta$ on $S(e)$ sufficiently close to $\mu(e)$, any compressible simple loop $\gamma$ on $S(e)$ and any normalised measured foliation $\mu_1$ on $S(e)$either

  $$i(\zeta, \mu_1) \geq c_1|\zeta|$$

  or

  $$i(\gamma, \mu_1) \geq c_1|\gamma|.$$
- It is weaker than the condition for $\mu(e)$ to be in the Masur domain: for some constant $c_2 > 0$:

  For any simple closed loop $\zeta$ on $S(e)$ sufficiently close to $\mu(e)$, any compressible simple loop $\gamma$ on $S(e)$ and any normalised measured foliations $\mu_1$ and $\mu_2$ on $S(e)$ with $i(\mu_1, \mu_2) = 0$:

either

$$i(\zeta, \mu_1) \geq c_2|\zeta|$$

or

$$i(\gamma, \mu_2) \geq c_2|\gamma|.$$

- The constant $c_1$ gives Lipschitz bounds in terms of $c_1$.
- The constant $c_2$ gives biLipschitz bounds.
- It is possible to choose $\mu(e)$ with $c_1$ bounded from $0$ while $c_2$ is arbitrarily close to $0$.

  But this does not happen in the case of combinatorial bounded geometry.