MATH553. Topology and Geometry of Surfaces
Problem Sheet 9: Euler Characteristic

Work is due in on *Monday 12th December.*
1. Compute the Euler characterstic of
a) a sphere minus three discs,
b) a torus minus three discs.

2. a) Let S be
Paul: a two-holed torus minus two open discs;
Joel: a sphere minus 6 open discs;
Bian Ce: a one-holed torus minus 3 open discs;
Freddie: a three-holed torus minus an open disc. Let A be an open annulus such that the closure $\overline{A} \subset (S \setminus \partial S)$ is a closed annulus which is homotopically nontrivial, that is, not homotopic to a point, not homotopic to a boundary component. By using $\chi(S) = \chi(S \setminus A)$ and quoting any necessary results about Euler characteristic, find all possibilities, up to homeomorphism, for the surface-with-boundary $S \setminus A$.
Caution. A may, or may not, disconnect S.
b) Now do the same if S is as in a) and $\overline{A_1} \cup \overline{A_2} \subset (S \setminus \partial S)$, where A_1 and A_2 are open annului with disjoint closures which are closed annuli, both homotopically nontriviali, not homotopic to boundary components, and not homotopic to each other.
I'll do this for the three-holed torus minus three discs. There are many more cases for this than for any of the exercises set
for a) $\chi(S) = -7$, as can be checked. In any case the Euler characteristic of a 3-holed torus is -4, subtracting 3 for the three discs gives -7. So $\chi(S \setminus A) = -7$. Now $S \setminus A$ has one or two components, each of which has at least one boundary component, and altogether $S \setminus A$ has 5 boundary components, one for eaach disc and one for each component of ∂A.
Case 1. First suppose that $S \setminus A$ has one component. [For one of you, this case cannot arise.] The only orientable compact surface with boundary with Euler characteristic -7 and 5 boundary components is the two-holed torus minus 5 open discs. So $S \setminus A$ must be this up to homeomorphism in this case, and this completely determines (S, A) up to homeomorphism. Now suppose that $S \setminus A$ has two components S_1 and S_2. Renumbering if necessary, either
Case 2.S_1 has one boundary component and S_2 has 4, or
case 3.S_1 has 2 boundary components and S_2 has 3.
Case 2. $\chi(S_i) \leq -1$ because S_i is not a disc or an annulus. Since S_1 has one boundary component its Euler characteristic is odd and must be -1 or -3 or -5. So S_1 is a torus minus one disc - in which case (2(i)) S_2 is a two- holed torus minus 4 discs - or a two-holed torus minus one disc - in which case (2(ii)) S_2 is a one-holed minus 4 discs - or a three-holed torus minus one disc - in which case (2(iii)) S_2 is a sphere minus 4 discs.
Case 3. Again, $\chi(S_i) \leq -1$ because S_i is not a disc or an annulus. Since S_1 has two boundary components its Euler characteristic is even and must be -2

or -4 or -6. So S_1 is a torus minus two discs - in which case (3(i)) S_2 is a two holed torus minus 3 discs - or a two-holed torus minus two discs - in which case (3(ii)) S_2 is a torus minus 3 discs - or a three-holed torus minus one disc - in which case (3(iii)) S_2 is a sphere minus 3 discs.

This gives 7 cases altogether. None of you has more than three.

Part b): Either Case 1(i): $A_1 \cup A_2$ does not separate S, (1 possibility) or Case 1(ii): neither A_1 nor A_2 separates S but $A_1 \cup A_2$ does (1(ii), 5 possibilities) The possibilitiies for $(\chi(S_1), \chi(S_2))$ are $(-6, -1)$, $(-5, -2)$, $(-4, -3)$. The possibility $(-6, -1)$ just gives just a two-holed torus minus four disc and a sphere minus three discs (i.e. pair of pants). The possibility $(-5, -2)$ gives either (two-holed torus minus three discs and a sphere minus four discs) or (one-holed torus minus five discs and one-holed torus minus two discs). The case $(-4, -3)$ also splits into two. [Freddie, Bian Ce and Paul have to contend with this sort of thing]

Case 2/3: one of A_1 and A_2, say A_1 separates S.

Rather than consider cases 2 and 3 above for $S \setminus A_1$, it seems best to start afresh. For A_1 separating S, one can consider separately the cases when A_2 separates $S \setminus A_1$ and the cases when it does not. For the cases when A_2 does not separate $S \setminus A_1$, we return to a) and consider cases 2/3(i)-(iii) for A_1 replacing A. I think this gives 10 cases altogether. In the case when A_2 does separate $S \setminus A_1$ it seems best to start afresh. If A_2 does separate $S \setminus A_1$ then A_2 also separates S. So both A_1 and A_2 separate S and $S \setminus (A_1 \cup A_2)$ has three components, one of which has a boundary component in common with each of A_1 and A_2, and the other two have a boundary component in common with exactly one of A_1, A_2. The Euler characteristic of each component of $S \setminus (A_1 \cup A_2)$ is ≤ -1, the Euler characteristics sum to -7 and the total number of boundary components is 7. I think the total number of cases arising in this way is 10, giving 26 in all. I think that some of you have only two. One of you has more .. but less than 10.

3. Show that no subset of the two-sphere is homeomorphic to a torus minus a disc.

Hint: if this were possible, we would have

$$S^2 = M_1 \cup M_2,$$

for compact orientable surfaces-with-boundary M_1, M_2 intersecting only in complete boundary components (which are homeomorphic to circles) with M_1 homeomorphic to a torus minus a disc. We would then have $\chi(S^2) = \chi(M_1) + \chi(M_2)$. If you don't know what $\chi(S^2)$ and $\chi(M_1)$ are, you can work them out. If M is any compact manifold-with-boundary, with b boundary components, then

$$\chi(M) + b \leq 2.$$

4. Find all connected orientable compact-surfaces-with-boundary M up to homeomorphism with $\chi(M) = -5$.

Hint: The Euler characteristic, together with the number of boundary components, completely determine M up to homeomorphism.