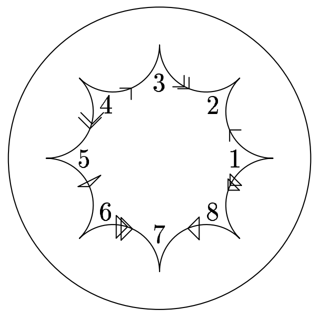
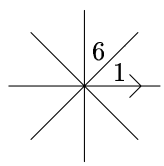
MATH553. Topology and Geometry of Surfaces
Problem Sheet 8: Hyperbolic Surfaces

Work is due in on *Thursday 1st December.*

Let $X \subset \{z : |z| < 1\}$ be an octagon with geodesic sides all of ℓ_D length equal to the same $s \in (0, \infty)$, and with all vertex angles equal to $\pi/4$. (Suppose that X does exist.) Let $X/\sim$ be the hyperbolic manifold formed by identifying sides as shown.



1. Show the order in which the angles labelled 1 to 8 occur round the point $[v]$ in $X/\sim$, where v is any vertex of X. (They are all in the same equivalence class.) The first two have been written in for you.



2. Describe an atlas for the hyperbolic manifold $X/\sim$, considering separately points $x_1 \in X$ where
(i) x_1 is an interior point of X,
(ii) x_1 is an interior point of a side of X, $[x_1] = \{x_1, x_2\}$, in which case there is $\varepsilon > 0$ such that $\{x' \in X : d_P(x_j, x') < \varepsilon\}$ does not meet any vertex of X, and only meets the side of X containing x, and you might consider your chart to have domain

$$\{[x'] : d_P(x', x_j) < \varepsilon \, x' \in X \, j = 1 \text{ or } 2\},$$

and find a chart map from this to (say) $\{z \in H : d_P(z, i) < \varepsilon\}$,
(iii) x_1 is a vertex of X, in which case there is $\varepsilon > 0$ such that $\{x' \in X : d_P(x_1, x') < \varepsilon\}$ does not meet any vertex of X apart from x_1, the set $[x_1]$ is the set of all 8 vertices of X, say x_j, $1 \leq j \leq 8$, and you might consider your chart to have domain

$$\{[x'] : d_P(x', x_j) < \varepsilon, \; x' \in X, \; 1 \leq j \leq 8\},$$

and find a chart map from this to (say) $\{z \in H : d_P(z, i) < \varepsilon\}$ (you would need to divide this disc up into 8 equal parts).

3. Now show that this octagon exists in $\{z : |z| < 1\}$, possibly as follows.

A regular octagon centred on 0 is invariant under the rotation $z \mapsto e^{\pi i/4}.z$. All vertices are the same Euclidean distance r from 0, and all vertex angles have the same value $\alpha(r)$, which is continuous in r for $r \in (0,1)$. Show that

$$\lim_{r\to 0} \alpha(r) = 7\pi/8, \ \lim_{r\to 1} \alpha(r) = 0.$$

Hint : For r near 0, the geodesics in which the sides lie are approximately diameters of the circle. For r near 1, the vertices are near the unit circle, and the geodesics in which the sides lie cut the unit circle at rightangles.
Deduce that there is $r \in (0,1)$ with $\alpha(r) = \pi/4$

4. Now show that $r \mapsto \alpha(r)$ is strictly decreasing. You could proceed as follows. Take two radii of the circle making angles $\pm\pi/8$ with the positive real axis. The geodesic segment joining points which are Euclidean distance r from 0 on these radii lies on a circle of radius R with centre on the positive real axis, and cutting the unit circle at rightangles.

Show that

$$r = \sqrt{R^2+1}\cos(\pi/8) - \sqrt{(R^2+1)\cos^2(\pi/8) - 1} = X - \sqrt{X^2-1},$$

where $X = \sqrt{R^2+1}\cos(\pi/8)$, and, by differentiating or otherwise, that r is therefore a decreasing function of X, and thus also of R (where the above equation gives a real r - which is actually for $X \geq 1$, that is, $R^2 \geq (\sqrt{2}-1)/(\sqrt{2}+1) = R_0^2$). Show, however, using the sine rule or otherwise, that

$$\frac{\sin((\alpha+\pi)/2)}{\sqrt{R^2+1}} = \frac{\sin(\pi/8)}{R}.$$

Then show that $\alpha \mapsto \sin(\alpha+\pi)/2$ is strictly decreasing for $\alpha \in (0, 3\pi/4)$, and $R \mapsto \sqrt{1+R^2}/R$ is strictly decreasing in $R \in (R_0, \infty)$ and thus that α is a strictly increasing function of R, and a strictly decreasing function of r.
Note that this is enough to show that the r with $\alpha(r) = \pi/4$ is unique.