MATH553. Topology and Geometry of Surfaces
Problem Sheet 7: Covering Groups of Holomorphic Bijections

Work is due in on *Thursday 24th November*
1. As usual, identify $\mathbb{R}$ with the subgroup $\{x + i0 : x \in \mathbb{R}\}$ of $\mathbb{C}$. Remember that $\mathbb{C}$ is a group under addition. Let Γ be a discrete subgroup of $\mathbb{C}$, that is, there exists $\varepsilon > 0$ such that

$$\Gamma \cap \{z : |z| < \varepsilon\} = \{0\}$$

Suppose also that $\Gamma \cap \mathbb{R} = \mathbb{Z}$.
a) Let $\alpha \in \Gamma$ satisfy

$$|\mathrm{Im}(\alpha)| = \mathrm{Min}\{|\mathrm{Im}(z)| : z \in \Gamma \setminus \mathbb{R}\}.$$

Show that if $z \in \Gamma$ then $\mathrm{Im}(z) = n\mathrm{Im}(\alpha)$ for some $n \in \mathbb{Z}$, and that

$$\Gamma = \mathbb{Z} + \alpha\mathbb{Z}.$$

b) Show that α as in a) does exist, possibly by showing that, for any $M > 0$,

$$\{\mathrm{Im}(z) : z \in \Gamma \,|\mathrm{Im}(z) < M\} = \{\mathrm{Im}(z) : z \in \Gamma \,|\mathrm{Im}(z)| < M \,|\mathrm{Re}(z)| < 1\},$$

and by showing that the righthand set is finite.
Hint: a bounded sequence in $\mathbb{R}^n$ (for any n, including $n = 2$) has a convergent subsequence.

2a). Show that, for $R \in (0, \infty)$,

$$\Phi_R : S_R = \{z : 0 < \mathrm{Im}(z) < \pi R\} \to H = \{z : \mathrm{Im}(z) > 0\} : \Phi_R(z) = e^{z/R}$$

is a holomorphic bijection. Also, find $\lambda > 0$ such that

$$\Phi_R(z+1) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} .\Phi_R(z)(= \lambda^2 \Phi_R(z)).$$

b) Let

$$p_R(z) = e^{2\pi i z}.$$

Show that the image of S_R under p_R is

$$A_R = \{z : e^{-2\pi^2 R} < |z| < 1\}.$$

Assuming that p_R is a covering [which is not hard to prove], both $p_R : S_R \to A_R$ and $p_R \circ \Phi_R^{-1} : H \to A_R$ are coverings. Find the covering group, regarded as a group of holomorphic bijections of
(i) S_R (ii) H.

3. Let $U \subset \mathbb{C}$ be open. A map $\varphi : U \to \mathbb{C}$ is said to be *conformal* if either φ or the composition with conjugation, $z \mapsto \overline{\varphi(z)}$, is holomorphic, A group G of

homeomorphisms of $\mathbb{C}$ is *conformal* if φ is conformal for all $\varphi \in G$. Now let g_1 and g_2 be the homeomorphisms of $\mathbb{C}$ given by

$$g_1(x+iy) = x + \frac{1}{2} - iy \ (x,\ y \in \mathbb{R})$$

$$g_2(x+iy) = x + i(y+1) \ (x,\ y \in \mathbb{R}).$$

a) Check that g_1 and g_2 are conformal, and that g_1^2 and g_2 are holomorphic, where g_1^2 denotes $g_1 \circ g_1$. b) Show also that

$$g_1 \circ g_2 = g_2^{-1} \circ g_1,$$

and if g_1^m denotes the m-fold composition of g_1 and similarly for g_2^n,

$$g_1^m \circ g_2^n(x,y) = (x + \frac{m}{2}, (-1)^m(y+n)(\ x,\ y \in \mathbb{R}).$$

c) Now let G be the group of homeomorphisms generated by g_1, and g_2 - which, by part b), can all be written in the form $g_1^m \circ g_2^n$ for $m,\ n \in \mathbb{Z}$. Show that the action of G on $\mathbb{C}$ is free, discrete and conformal. [The quotient space $\mathbb{C}/G$ is then a complex manifold - but nonorientable. It is known as the *Klein bottle.*]
d) Let H be the subgroup of G generated by g_1^2 and g_2. Show that H is abelian. Let $\mathbb{C}/H = \{[z]_H : z \in \mathbb{C}$, where $[z]_H = \{h(z) : h \in H$ and let $\mathbb{C}/G$ be similarly defined. (This is just the Klein bottle.) Show that $\mathbb{C}/H$ is simply $\mathbb{C}/\sim_i$ where $\sim_i$ is the equivalence relation $z \sim_i z' \Leftrightarrow z' = z + m + ni$ for $m,\ n \in \mathbb{Z}$. Show also that the map $[z]_H \mapsto [z]_G : \mathbb{C}/H \to \mathbb{C}/G$ is a two-to-one covering map.

c) Find the universal covering spaces, covering maps and covering groups for

$$A_{+\infty} = \{z \in \mathbb{C} : 0 < |z| < 1\} \text{ and } A_\infty = \{z \in \mathbb{C} : z \neq 0\}.$$

Hint: the first has universal covering space H, the second has universal covering space $\mathbb{C}$.

d) Explain (possibly using the universal covering spaces and covering groups) why there are no holomorphic bijections between any of A_R ($R \in (0,\infty)$) $A_{+\infty}$, A_∞.
Hint: explain why there is no invertible P with

$$P \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} P^{-1} = \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix}$$

if $\lambda \neq \mu^{\pm 1}$. (This will be enough to show that the corresponding covering groups are not conjugate within the group of holomorphic bijections of H.)

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Problem Sheet 7: Solutions

1a) Let $z \in \Gamma$. Then there exists $n \in \mathbb{Z}$ such that

$$\text{Im}(z) \in [n|\text{Im}(\alpha)|, (n+1)|\text{Im}(\alpha))|.$$

Then

$$|\text{Im}(z - n\alpha)| < |\text{Im}(\alpha)|.$$

Then by the definition of α, $\text{Im}(z - n\alpha) = 0$, that is, $z - n\alpha \in \mathbb{R}$. Since $z - n\alpha \in \Gamma \cap \mathbb{R} = \mathbb{Z}$, this means that $z = m + n\alpha$ for $m, n \in \mathbb{Z}$. So

$$\Gamma = \mathbb{Z} + \alpha\mathbb{Z}.$$

b) Take any $\gamma \in \Gamma$ with $|\text{Im}(\gamma)| < M$. Then there is $m \in \mathbb{Z} \subset \Gamma$ with $\text{Im}(\gamma) = \text{Im}(\gamma - m)$ and $|\text{Re}(\gamma - m)| < 1$. So then $|\gamma - m| < M + 1$. There are $\leq 16M/\varepsilon 2$ $\gamma' \in \Gamma \cap ([-1,1] \times [-M, M])$ becauseotherwise there are two distinct γ', γ'' in this set with both real and imaginary parts differing by $\leq \varepsilon/2$, and then $|\gamma' - \gamma''| < \varepsilon$, a contradiction. So α does exist.

2a) $z \in S_R \Leftrightarrow z = x + iy$ with $y \in (0, \pi R) \Leftrightarrow e^{z/R} = e^{(x+iy)/R}$ has argument $\in (0, \pi) \Leftrightarrow \varphi_R(z) \in H$. The map φ_R is holomorphic because the exponential map $w \mapsto e^w$ is holomorphic, so is $z \mapsto z/R$, and a composition of holomorphic functions is holomorphic. Alternatively, one can check the Cauchy-Riemann functions.

To show that φ_R is onto: if $z = x + iy$ for x and y both real then the modulus of $e^{z/R}$ is $e^{x/R}$ and the argument is y/R. For x/R, $e^{x/R}$ takes all values > 0 and for $0 < y < \pi R$, y/R takes all values in $(0, \pi)$. So φ_R is surjective. To see injectivity, suppose that $\varphi_R(z_1) = \varphi_R(z_2)$ and write $z_j = x_j + iy_j$ forx_j and y_j real. Since the moduli $e^{x_1/R}$ and $e^{x_2/R}$ are equal and $x \mapsto e^x$ is strictly increasing on the reals, we get $x_1 = x_2$. Since the arguments y_1/R and y_2/R are also equal and between 0 and π, we get $y_1 = y_2$. So $z_1 = z_2$. So φ_R is injective. So φ_R is a holomorphic bijection. Also, $\Phi_R(z+1) = e^{1/R}.e^{z/R}$. So we can take $\lambda = e^{1/2R}$.

2b) Write $z = x + iy$ with $y \in (0, R\pi)$. Then $p_R(z) = e^{2\pi i x - 2\pi y}$. Now $|p_R(z)| = e^{2\pi y} \in (e^{-2\pi^2 R}, 1)$. So $p_R(S_R) = A_R$.

2b)(i) We have $p_R(z) = p_R(w) \Leftrightarrow w = z + n$ for some $n \in \mathbb{Z}$. So the action of the covering group on S_R is given by $n.z = z + n$ ($z \in S_R$, $n \in \mathbb{Z}$).

2b)(ii) Applying Φ_R and regarding H as the universal covering space, the action is given by $n.z = e^{-Rn}z$ ($n \in \mathbb{Z}$, $z \in H$).

3a) We have $g_1(z) = \overline{z + \frac{1}{2}}$. So $\overline{g_1(z)} = z + \frac{1}{2}$, which is holomorphic. Also, $g_2(z) = z + i$ is holomorphic, and $g_1^2(z) = \overline{\overline{z} + \frac{1}{2}} + \frac{1}{2} = z + 1$ is holomorphic. So g_1 and g_2 are conformal, and g_1^2 and g_2 are holomorphic.

b) We have

$$g_1 \circ g_2(z) = \overline{z + i} + \frac{1}{2} = \overline{z} - \frac{1}{2} = (\overline{z} + \frac{1}{2} - 1 = g_2^{-1} \circ g_1(z).$$

Also $g_2^n(x+iy) = x+iy+in$ and $g_1^m(u+iv) = u+\frac{m}{2}+(-1)^m iv$ for $u, v \in \mathbb{R}$. So

$$g_1^m \circ g_2^n(x+iy) = x + \frac{m}{2} + (-1)^m i(y+n).$$

c) Any element of G can be written in the form $\varphi = g_1^m \circ g_2^n$ and from the formula above, $g_1^m \circ g_2^n(z) = z + \frac{m}{2} + in$ if m is even and $= \overline{z} + \frac{m}{2} - in$ if m is odd. So G is conformal. Now suppose that

$$x + \frac{m}{2} + (-1)^m i(y+n) = x+iy.$$

Then $m = 0$ from equating real parts, so that $(-1)^m = 1$. Then $n = 0$ from equating imaginary parts. So the action is free. Now let $B(z, 1/4)$ denote the ball of radius $1/4$ centred on z. Then if $B(g_1^p \circ g_2^q z, 1/4) \cap B(g_1^m \circ g_2^n(z), 1/4) \neq \phi$ then

$$|g_1^p \circ g_2^q z - g_1^m \circ g_2^n(z)| < \frac{1}{2}.$$

Looking at real parts we see that if $z = x+iy$,

$$|(x + \frac{p}{2}) - (x + \frac{m}{2}| < \frac{1}{2}.$$

This only possible if $m = p$. So $(-1)^m = (-1)^p$ and looking at imaginary parts

$$|(y+q) - (y+n)| \leq \frac{1}{2}.$$

So $n = q$ and the action is discrete.

d) We have

$$g_1^2 \circ g_2 = g_1 \circ g_2^{-1} \circ g_1 = g_2 \circ g_1^2.$$

So H is abelian. Also $g_1^{2m}(z) = z + m$. So $g_1^{2m} \circ g_2^n(z) = z + m + in$, and $\mathbb{C}/H$ is indeed $\mathbb{C}/\sim_i$. We have

$$[x+iy]_G = \{x + \frac{m}{2} + (-1)^m(y+in) : m, n \in \mathbb{Z}\}$$

$$= \{x + p + in : p, n \in \mathbb{Z}\} \cup \{x + \frac{1}{2} + p + i(-y+n) : p, n \in \mathbb{Z}\}$$

$$= [z]_H \cup [\overline{z} + \frac{1}{2}]_H.$$

We also have

$$[z]_H \cap [\overline{z} + \frac{1}{2}]_H = \phi.$$

So the map $p : [z]_H \mapsto [z]_G$ is two-to-one. It is a covering map because toe sets $[B(z_0, \frac{1}{4})]_H$ and $[B(\overline{z_0} + \frac{1}{2}, \frac{1}{4})]_H$ are disjoint and both map homeomorphically to $[B(z_0, \frac{1}{4})]_G$ under the map p, for any z_0.