MATH553. Topology and Geometry of Surfaces
Problem Sheet 5: Lifts, Homotopy, Coverings

Please hand in your solutions to 1-5 in class on *Thursday 10th November.*
1. For each of the following closed paths in $S^1 = \{z : |z| = 1\}$, find a lift to $\mathbb{R}$ with first endpoint at $0 \in mathbbR$. Hence, determine which (if any) of the paths in S^1 are homotopic.
a) $t \mapsto e^{2\pi it} : [0,1] \to S^1$, b) $t \mapsto e^{-2\pi it} : [0,1] \to S^1$,
c) $t \mapsto -e^{\pi i \cos \pi t} : [0,1] \to S^1$

2. Show that your letter is path-connected. In all cases let S_1 be the topological space represented by your letter.

Freddie, Paul and Joel: find a point s_0 which can be removed from your letter so that $S_1 \setminus \{s_0\}$ is not path connected. To show this find a continuous function on $S_1 \setminus \{s_0\}$ which takes exactly two values (for Freddie's letter), and exactly 3 values (for Joel's letter) and exactly 4 values (for Paul's letter).

Cian Be: for your letter, it is not possible to find just a single point s_0 such that $S_1 \setminus \{s_0\}$ is not path-connected, but you do not need to prove this. Instead, find two points s_1, s_2 such that $S_1 \setminus \{s_1, s_2\}$ is path-connected, by producing a continuous function on $S_1 \setminus \{s_1, s_2\}$ which takes exactly 3 values.

Here is how to tackle the problem for the letter X which, as in Problem Sheet 2, is represented by the set

$$S_1 = \{(t,t) : t \in [-1,1]\} \cup (t,-t) : t \in [-1,1]\}.$$

To show this set is path connected: a path from (s,s) to (t,t) is given by a continuous map $f : [0,1] \to S_1$ defined by

$$f(u) = ((1-u)s + ut, (1-u)s + ut),$$

a path from $(s,-s)$ to $(t,-t)$ is given by a continuous map $f : [0,1] \to S_1$ defined by

$$f(u) = ((1-u)s + ut, -(1-u)s - ut),$$

and a path from (s,s) to $(t,-t)$ is given by a continuous map $f : [0,2] \to S_1$ defined by

$$f(u) = \begin{matrix} ((1-u)s, (1-u)s) & \text{if } u \in [0,1], \\ ((u-1)t, (1-u)t) & \text{if } u \in [1,2] \end{matrix}$$

Note that the two definitions of $f(1)$ in this formula coincide, so f is indeed continuous. Also, in the last formula, $u \mapsto f(2-u) : [0,2] \to S_1$ is a path from $(t,-t)$ to (s,s). So any two points in S_1 can be connected by a path.

The set $S_1 \setminus \{(0,0)\}$ is not path-connected. We can find a sontinous surjective function $f : S_1 \setminus \{(0,0)\} \to \{1,2,3,4\}$ defined by

$$\begin{array}{ll} f(t,t) = 1 & \text{if } t < 0 \\ f(t,t) = 2 & \text{if } t > 0 \\ f(t,-t) = 3 & \text{if } t < 0 \\ f(t,-t) = 4 & \text{if } t > 0 \end{array}$$

Each of the sets $f^{-1}(\{j\})$ is open in S_1 so f is continuous. But by the Intermediate Value Theorem the set of values of a continuous real-valued function on a path-connected space must be an interval.

3. Let $\mathbb{Z}$ act on $\mathbb{R} \times (-1, 1)$ by

$$n.(x, y) = \varphi_n(x, y) = (x + n, (-1)^n y).$$

a) Check that $\varphi_n \circ \varphi_m = \varphi_{n+m}$ for all n, $m \in \mathbb{Z}$.

b) Let

$$B(\underline{x}_0, r) = \{\underline{x} \in \mathbb{R} \times (-1, 1) : \|underlinex - \underline{x}_0\| < r\},$$

where $\|.\|$ denotes Euclidean norm. Show that if $r \leq \frac{1}{2}$, the sets $\varphi_n(B(\underline{x}_0, r))$ ($n \in \mathbb{Z}$) are all disjoint, for any $\underline{x}_0 \in \mathbb{R} \times (-1, 1)$ and $n \in \mathbb{Z}$.

c) Write $[\underline{x}] = \{n.\underline{x} : n \in \mathbb{Z}\}$. Then by using the sets

$$[B(\underline{x}_0, r)] = \{[\underline{x}] : \underline{x} \in B(\underline{x}_0, r)\}, \ r \leq \tfrac{1}{2},$$

or otherwise, show that $\pi : \underline{x} \mapsto [\underline{x}] : \mathbb{R} \times (-1, 1) \to (\mathbb{R} \times (-1, 1))/\mathbb{Z}$ is a covering. (This then shows that $\mathbb{R} \times (-1, 1)/\mathbb{Z}$ is a manifold: it is, in fact, the Möbius strip.)

4a) Show that $\mathbb{R} \times (-1, 1)$ is path-connected and simply-connected.

b) Find lifts of the following closed paths in $(\mathbb{R} \times (-1, 1))/\mathbb{Z}$ to $\mathbb{R} \times (-1, 1)$ with first endpoint at $(0, 0)$. Determine which (if any) of the closed paths in $(\mathbb{R} \times (-1, 1))/\mathbb{Z}$ are homotopic.

(i) $t \mapsto [(t + \sin \pi t, 0)] : [0, 1] \to (\mathbb{R} \times (-1, 1))/\mathbb{Z}$,

(ii) $t \mapsto [(-t, 0)] : [0, 1] \to (\mathbb{R} \times (-1, 1))/\mathbb{Z}$,

(iii) $t \mapsto [(\cos \pi t, 0)] : [0, 1] \to (\mathbb{R} \times (-1, 1))/\mathbb{Z}$,

(iv) $t \mapsto [(-2t, 0)] : [0, 1] \to (\mathbb{R} \times (-1, 1))/\mathbb{Z}$.

5. Produce a homeomorphism between $(\mathbb{R} \times (-1, 1))/2\mathbb{Z}$ and $S^1 \times (-1, 1)$, where $S^1 = \{z : |z| = 1\}$ and $2\mathbb{Z} = \{2n : n \in \mathbb{Z}\}$. It might be slightly simpler to produce a map which sends $(\mathbb{R} \times [-1 + \delta, 1 - \delta])/2\mathbb{Z}$ to $S^1 \times [-1 + \delta, 1 - \delta]$ for each $\delta > 0$, because a continuous bijection from a compact space to a Hausdorff space is automatically a homeomorphism.

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Problem Sheet 5: Solutions

1. Lifts with first endpoints at 0 in cases a) and b) c) are given by
a) $t \mapsto t + \sin \pi t$ b) $t \mapsto -t$. c) $t \mapsto \frac{-1}{2} + \frac{1}{2} \cos \pi t$ The paths of a) and b) are not homotopic, because the second endpoints of the lifted paths are 1, -1 respectively. The path in c) has the same endpoints as thepath in b), so these are homotopic.

3a)

$$\varphi_n \circ \varphi_m(x, y) = \varphi_n(x + m, (-1)^m y) = (x + n + m, (-1)^{n+m} y) = \varphi_{n+m}(x, y)$$

b) If $\underline{x} \in \varphi_n(B(\underline{x}_0, r)) \cap \varphi_m(B(\underline{x}_0, r))$ then $\|\varphi_{-n}(\underline{x}) - \underline{x}_0\| < r$ and $\varphi_{-m}(\underline{x}) - \underline{x}_0\| < r$. So $\|\varphi_{-n}(\underline{x}) - \varphi_{-m}(\underline{x})\| < 2r \leq 1$ if $r \leq \frac{1}{2}$. Writing $\underline{x} = (x, y)$, we have $\|(x + n, (-1)^n y) - (x + m, (-1)^m y)\| < 1$. So $|(x + n) - (x + m)| < 1$, and $|m - n| < 1$, which implies $m = n$.
c) Now for any $\underline{x}_0 \in \mathbb{R} \times (-1, 1)$ and $r \leq \frac{1}{2}$,

$$\pi^{-1}([B(\underline{x}_0, r)] = \coprod_{n \in \mathbb{Z}} \varphi_n(B(\underline{x}_0, r)),$$

so the set $[B(\underline{x}_0, r)]$ is open, and the map $\pi|\varphi_n(B(\underline{x}_0, r))$ is continuous with a well-defined inverse

$$[\underline{x}] \mapsto \varphi_n(\underline{x}) \text{ for } \underline{x} \in B(\underline{x}_0, r).$$

This is well-defined because if $[\underline{x}] = [\underline{x}']$ for $\underline{x}$, $\underline{x}' \in B(\underline{x}_0, r)$, $r \leq \frac{1}{2}$, then $\underline{x} = \underline{x}'$. It is continuous because the map without square brackets is continuous. So $\pi|\varphi_n(B(\underline{x}_0, r))$ is a homeomorphism for all $n \in \mathbb{Z}$. So π is a covering.

4a) Take any $\underline{x}$, $\underline{y} \in \mathbb{R} \times (-1, 1)$ Then a path from $\underline{x}$ to $\underline{y}$ is given by

$$t \mapsto (1 - t)\underline{x} + t\underline{y} : [0, 1] \to \mathbb{R} \times (-1, 1).$$

b) Lifts of the paths with first endpoints at $(0, 0)$ are given by
(i) $t \mapsto (t + \sin \pi t, 0)$ (or at least that was awhat was intended - there was a mistake in the question so it actually comes out as $t \mapsto (t + \sin t, 0)$, (ii)$t \mapsto (-t, 0)$)
(iii) $t \mapsto (-1 + \cos \pi t, 0)$, (iv) $t \mapsto (-2t, 0)$
where the second endpoints of these paths are: $(1, 0)$ (in (i) as corrected), $(-1, 0)$ (in (ii)), $(-2, 0)$ (in (iii) and (iv)). It follows that the only two paths that are hom0topic are those in (iii) and (iv).

5. Let $[\underline{x}]_2 = \{\underline{x} + (2n, 0) : n \in \mathbb{Z}\}$, and $(\mathbb{R} \times (-1, 1))/2\mathbb{Z} = \{[\underline{x}]_2 : \underline{x} \in \mathbb{R} \times (-1, 1)\}$. Define $\Phi : (\mathbb{R} \times (-1, 1))/2\mathbb{Z} \to S^1 \times (-1, 1)$ by $\Phi([(x, y)]_2) = (e^{\pi i x}, y)$. Then

$$\Phi([(x, y)]_2) = \Phi([(x', y')]_2) \Leftrightarrow (x', y') = (x + 2n, y) \text{ for some } n \in \mathbb{Z} \Leftrightarrow [(x', y')]_2 = [(x, y)]_2.$$

So Φ is well-defined and injective, and surjective since $x \mapsto e^{\pi i x} : \mathbb{R} \to S^1$ is surjective. It is continuous since the map without square brackets is continuous.

The map is a homeomorphism restricted to $(\mathbb{R} \times [-1+\delta, 1-\delta])/2\mathbb{Z}$ for any $\delta > 0$, (because then we have a map from a Hausdorff space to the compact space $S^1 \times [-1+\delta, 1-\delta])$ and hence is a homeomorphism.