MATH553. Topology and Geometry of Surfaces
Problem Sheet 4: Möbius Transformations.

Please hand in your solutions to 1-5 in class on *Monday 31st October*. Question no 5 is part of the Continuous Assessment and is worth 3 marks.

Throughout, we consider the action of

$$SL(2,\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1 \right\}$$

on $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} .z = \frac{az+b}{cz+d}.$$

Of course this is really an action of $PSL(2,\mathbb{C}) = SL(2,\mathbb{C})/\pm I$, since $A.z = -A.z$ for all $A \in SL(2,\mathbb{C})$.

1. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq \pm I$, show that $z \mapsto A.z$ has 1 fixed point in $\overline{\mathbb{C}}$ if $|a+d| = \pm 2$, and two fixed points in $\overline{\mathbb{C}}$ if $a + d \neq \pm 2$.

2. Show that the action of $SL(2,\mathbb{R}) \leq SL(2,\mathbb{C})$ on $\overline{\mathbb{C}}$ preserves each of the sets

$$H = \{z : \mathrm{Im}(z) > 0\}, \ \mathbb{R} \cup \{\infty\}, \ \{z : \mathrm{Im}(z) < 0\}.$$

3. Find the *stabilizer* H in $SL(2,\mathbb{R})$, that is, find

$$K = \{A \in SL(2,\mathbb{R}) : A.i = i\}.$$

4. Show that

$$L = \left\{ \begin{pmatrix} \lambda & \mu \\ 0 & \lambda^{-1} \end{pmatrix} : \lambda, \ \mu \in \mathbb{R}, \ \lambda > 0 \right\} \leq SL(2,\mathbb{R})$$

acts *transitively* on H, that is, for all z_1, $z_2 \in H$, there is $A \in L$ with $A.z_1 = z_2$.

5. Find centre and radius of the ccircle which passes the points z_1 and z_2 and has a diameter along the real axis, and hence find a Möbius transformation of H which maps z_1 and z_2 to points on the positive imaginary axis.

Freddie: $z_1 = i$, $z_2 = 3 + i$.
Cian Be: $z_1 = -1 + i$, $z_2 = 2 + 2i$
Paul: $z_1 = 1 + 2i$, $z_2 = 2 + i$
Joel: $z_1 = -5 + i$, $z_2 = 3 + i$.

6. Compute the derivative of $z \mapsto A.z$ at $z \in \mathbb{C}$. Using the fact that each map $z \mapsto A.z$ preserves the set of circles and straight lines, and preserves angles between smooth curves, show that, given any two curves γ_1, γ_2 through i which are circles or straight lines and intersect $\mathbb{R}$ at rightangles, there is $h \in K$ (as in question 3) such that $h.\gamma_1 = \gamma_2$.

7. Let $\mathrm{Im}(z_1) > 0$, $\mathrm{Im}(z_2) > 0$. Let γ_j $(j = 1, 2)$ be a curve through z_j which is a circle or straight line, and cuts $\mathbb{R}$ at right angles. Show (possibly using 4 and 6) that there is $A \in SL(2,\mathbb{R})$ such that $A.z_1 = z_2$ and $A.\gamma_1 = \gamma_2$.

8. The *centraliser* of an element $A \in SL(2,\mathbb{R})$ is the subgroup

$$\{B \in SL(2,\mathbb{R}) : AB = BA\}.$$

Find the centraliser of A where

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \ (\lambda > 0,\ \lambda \neq \pm 1),\ A = \begin{pmatrix} 1 & \pm 1 \\ 0 & 1 \end{pmatrix},$$

$$A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \ (a,\ b \neq 0,\ a^2 + b^2 = 1).$$

9. Consider A of each of the forms given in question 8. Show that for such A, the centralizer leaves invariant either a vertical line, or all horizontal lines, or the point i. Show also that for A of ther first type, the action of the centraliser on the intersection of the invariant vertical line with H is transitive. Show also that for A of the second type, the action on of the centraliser on any horizintal straight line is transitive.

10. Now every matrix in $SL(2,\mathbb{R})$ apart from $\pm I$ is of the form $\pm PAP^{-1}$ where $P \in SL(2,\mathbb{R})$ and A is one of the matrices given in question 8. So now let $B \in SL(2,\mathbb{R})$, $B \neq \pm I$. Using the fact just given, or otherwise, show that the centraliser of B either fixes a point in H, or leaves invariant a circle or straight line which intersect $\mathbb{R}$ at rightangles, or leaves invariant a horizontal line in H, or leaves invariant a circle in H which is tangent to $\mathbb{R}$.

Hint : First consider A of each of the forms given in question 8. Show that for such A, the centralizer fixes either a vertical line, or all horizontal lines, or the point i. Then use the fact that maps of the form $z \mapsto P.z$ ($P \in SL(2,\mathbb{R})$) preserve H, preserve the set of straight lines and circles cutting $\mathbb{R}$ at rightangles, and preserve the set of horizontal straight lines and circles tangent to $\mathbb{R}$.