MATH553. Topology and Geometry of Surfaces
Problem Sheet 4: Möbius Transformations.

Please hand in your solutions to 1-5 in class on *Monday 31st October*. Question no 5 is part of the Continuous Assessment and is worth 3 marks.

Throughout, we consider the action of

$$SL(2,\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1 \right\}$$

on $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} .z = \frac{az+b}{cz+d}.$$

Of course this is really an action of $PSL(2,\mathbb{C}) = SL(2,\mathbb{C})/\pm I$, since $A.z = -A.z$ for all $A \in SL(2,\mathbb{C})$.

1. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq \pm I$, show that $z \mapsto A.z$ has 1 fixed point in $\overline{\mathbb{C}}$ if $|a+d| = \pm 2$, and two fixed points in $\overline{\mathbb{C}}$ if $a + d \neq \pm 2$.

2. Show that the action of $SL(2,\mathbb{R}) \leq SL(2,\mathbb{C})$ on $\overline{\mathbb{C}}$ preserves each of the sets

$$H = \{z : \text{Im}(z) > 0\}, \ \mathbb{R} \cup \{\infty\}, \ \{z : \text{Im}(z) < 0\}.$$

3. Find the *stabilizer* H in $SL(2,\mathbb{R})$, that is, find

$$K = \{A \in SL(2,\mathbb{R}) : A.i = i\}.$$

4. Show that

$$L = \left\{ \begin{pmatrix} \lambda & \mu \\ 0 & \lambda^{-1} \end{pmatrix} : \lambda, \ \mu \in \mathbb{R}, \ \lambda > 0 \right\} \leq SL(2,\mathbb{R})$$

acts *transitively* on H, that is, for all z_1, $z_2 \in H$, there is $A \in L$ with $A.z_1 = z_2$.

5. Find centre and radius of the ccircle which passes the points z_1 and z_2 and has a diameter along the real axis, and hence find a Möbius transformation of H which maps z_1 and z_2 to points on the positive imaginary axis.

Freddie: $z_1 = i$, $z_2 = 3 + i$.
Cian Be: $z_1 = -1 + i$, $z_2 = 2 + 2i$
Paul: $z_1 = 1 + 2i$, $z_2 = 2 + i$
Joel: $z_1 = -5 + i$, $z_2 = 3 + i$.

6. Compute the derivative of $z \mapsto A.z$ at $z \in \mathbb{C}$. Using the fact that each map $z \mapsto A.z$ preserves the set of circles and straight lines, and preserves angles between smooth curves, show that, given any two curves γ_1, γ_2 through i which are circles or straight lines and intersect $\mathbb{R}$ at rightangles, there is $h \in K$ (as in question 3) such that $h.\gamma_1 = \gamma_2$.

7. Let $\text{Im}(z_1) > 0$, $\text{Im}(z_2) > 0$. Let γ_j ($j = 1, 2$) be a curve through z_j which is a circle or straight line, and cuts $\mathbb{R}$ at right angles. Show (possibly using 4 and 6) that there is $A \in SL(2,\mathbb{R})$ such that $A.z_1 = z_2$ and $A.\gamma_1 = \gamma_2$.

8. The *centraliser* of an element $A \in SL(2,\mathbb{R})$ is the subgroup

$$\{B \in SL(2,\mathbb{R}) : AB = BA\}.$$

Find the centraliser of A where

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \ (\lambda > 0,\ \lambda \neq \pm 1),\ A = \begin{pmatrix} 1 & \pm 1 \\ 0 & 1 \end{pmatrix},$$

$$A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \ (a,\ b \neq 0,\ a^2 + b^2 = 1).$$

9. Consider A of each of the forms given in question 8. Show that for such A, the centralizer leaves invariant either a vertical line, or all horizontal lines, or the point i. Show also that for A of ther first type, the action of the centraliser on the intersection of the invariant vertical line with H is transitive. Show also that for A of the second type, the action on of the centraliser on any horizintal straight line is transitive.

10. Now every matrix in $SL(2,\mathbb{R})$ apart from $\pm I$ is of the form $\pm PAP^{-1}$ where $P \in SL(2,\mathbb{R})$ and A is one of the matrices given in question 8. So now let $B \in SL(2,\mathbb{R})$, $B \neq \pm I$. Using the fact just given, or otherwise, show that the centraliser of B either fixes a point in H, or leaves invariant a circle or straight line which intersect $\mathbb{R}$ at rightangles, or leaves invariant a horizontal line in H, or leaves invariant a circle in H which is tangent to $\mathbb{R}$.

Hint : First consider A of each of the forms given in question 8. Show that for such A, the centralizer fixes either a vertical line, or all horizontal lines, or the point i. Then use the fact that maps of the form $z \mapsto P.z$ $(P \in SL(2,\mathbb{R}))$ preserve H, preserve the set of straight lines and circles cutting $\mathbb{R}$ at rightangles, and preserve the set of horizontal straight lines and circles tangent to $\mathbb{R}$.

MATH553. Topology and Geometry of Surfaces
Problem Sheet 4: Solutions

1.
$$\frac{az+b}{cz+d} = z \Leftrightarrow az+b = cz^2+dz \Leftrightarrow cz^2+(d-a)z-b=0$$
$$\Leftrightarrow z = \frac{a-d\pm\sqrt{a^2+2ad+d^2-4(ad-bc)}}{2c},$$
for $c\neq 0$, or, if $c=0$, $z=\infty$ or $z=b/(d-a)$. For $c\neq 0$, this gives
$$z=\frac{a-d\pm\sqrt{(a+d)^2-4}}{2c}.$$
So there is a repeated root $\Leftrightarrow a+d=\pm 2$. If $c=0$, there is a repeated root at $\infty \Leftrightarrow d-a=0$. Since $ad=1$, this happens $\Leftrightarrow a=d=1$ or $a=d=-1$, which again gives $a+d=\pm 2$.
2.
$$\mathrm{Im}(A.z)=\mathrm{Im}\left(\frac{az+b}{cz+d}\right)=\mathrm{Im}\left(\frac{(az+b)\overline{cz+d}}{|cz+d|^2}\right)$$
$$=\mathrm{Im}\left(\frac{ac|z|^2+bd+adz+bc\overline{z}}{|cz+d|^2}\right)=\frac{\mathrm{Im}(z)}{|cz+d|^2}.$$
So
$$\mathrm{Im}(A.z)\begin{array}{c}>0\\=0\\<0\end{array}\Leftrightarrow \mathrm{Im}(z)\begin{array}{c}>0\\=0\\<0\end{array}$$

3. Write
$$A=\begin{pmatrix}a & b\\ c & d\end{pmatrix}.$$
Then $A.i=i \Leftrightarrow ai+b=i(ci+d) \Leftrightarrow a=d$ and $b=-c$. Then $ad-bc=1\Rightarrow a^2+b^2=1$. So $(a,b)=(\cos\theta,\sin\theta)$ for some $\theta\in\mathbb{R}$ and
$$A=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}.$$

4. Write $z_1=x_1+iy_1$, $z_2=x_2+iy_2$,
$$A=\begin{pmatrix}\lambda & \mu\\ 0 & \lambda^{-1}\end{pmatrix}$$
We want to solve $A.z_1=z_2$ for real λ and μ with $\lambda\neq 0$. But this gives the two equations
$$\lambda^2x_1+\mu\lambda=x_2,\ \lambda^2y_1=y_2.$$

Since $y_1,\, y_2 > 0$, we can solve these with

$$\lambda = \sqrt{\frac{y_2}{y_1}},\ \mu = x_2\sqrt{\frac{y_1}{y_2}} - \lambda x_1.$$

5. The centre lies on the line

$$\frac{z_1 + z_2}{2} + ti(z_1 - z_2)$$

and is given by t such that

$$\mathrm{Im}\left(\frac{1}{2}(z_1 + z_2) + ti(z_1 - z_2)\right) = 0,$$

or

$$t = -\frac{1}{2}\frac{\mathrm{Im}(z_1 + z_2 0}{\mathrm{re}(z_1 - z_2)}.$$

The centre of the circle is then at

$$\frac{1}{2}\mathrm{Re}(z_1 + z_2) - t\mathrm{im}(z_1 - z_2).$$

So the centres and radii are Freddie: 2, $\sqrt{5}$
Cian Be: 1, $\sqrt{5}$
Paul: 0, $\sqrt{5}$
Joel: -1, $\sqrt{17}$

Joel's geodesic is disjoint from Paul's but intersects both Freddie's and Cian Be's. Freddie's, Cian Be's and Paul's geodesics all intersect pairwise.

If the centre is at $c \in \mathbb{R}$ and the radius is $R > 0$ then the Möbius transformation

$$z \mapsto \frac{z - (c - R)}{c + R - z}$$

maps H to H and the semicircle in H with centre at c and radius R to the positive imaginary axis. The point z_j gets mapped to

$$\frac{-|z_j|^2 + 2c\mathrm{Re}(z_j) - 2R\mathrm{Im}(z_j)}{|c + r - z_j|^2}.$$

6. Write $f(z) = A.z$ where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

$ad - bc = 1$. Then

$$f'(z) = \frac{ad - bc}{cz + d)^2} = \frac{1}{(cz + d)^2}.$$

Now let $A.i = i$, so that A is as in question 3. Then

$$f'(z) = \frac{1}{(-i\sin\theta + \cos\theta)^2} = e^{2i\theta}.$$

This means that for any differentiable curve γ at i (such as a semicircle arc) $f(\gamma)$ makes and angle 2θ with γ at i. A geodesic through i - that is, a semicircle with centre on the real axis, or the imaginary axis - is uniquely determined by the direction of its tangent vector at i. So given any two geodesics γ_1, γ_2 through i, we can choose f - by choosing θ - so that $f(\gamma_1) = \gamma_2$.

7. Choose A_1, $A_2 \in SL(2, mathbbR)$ so that $A_1.z_1 = i$ and $A_2.z_2 = i$ (using 4). Then choose B with $B.i = i$ and $B.(A_1.\gamma_1) = A_2.\gamma_2$. Then

$$(A_2^{-1}BA_1).z_1 = z_2, \ (A_2^{-1}BA_1^{-1}).\gamma_1 = \gamma_2.$$

8. Let B be a matrix in the centraliser, and write

$$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

(i)

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$\Leftrightarrow b\lambda = b\lambda^{-1}$ and $c\lambda^{-1} = c\lambda$. Since $\lambda \neq \pm 1$ this implies $b = c = 0$. Then $ad = 1$ $\Rightarrow d = a^{-1}$ (and $a \neq 0$). So then

$$B = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}.$$

(ii)

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$\Leftrightarrow a + b = d + b$, $a = a + c$ and $d + c = d$, which gives $c = 0$, $a = d$, which gives $a = d = \pm 1$. So, for $b' = \pm b$,

$$B = \pm \begin{pmatrix} 1 & b' \\ 0 & 1 \end{pmatrix}.$$

The case of $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ is similar.

(iii) Write

$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$\Leftrightarrow$

$$-b\sin\theta = c\sin\theta \text{ and } a\cos\theta = d\cos\theta$$

$\Leftrightarrow a = d$ and $b = -c \Leftrightarrow$, for some $t \in \mathbb{R}$,

$$B = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

9.(i)

$$B.(ti) = a^2 ti$$

for all $t > 0$. So $z \mapsto B.z$ preserves the positive imaginary axis. Also, given any $t_1,\ t_2 > 0$ we can find $a > 0$ with $a^2 t_1 i = t_2 i$, by taking $a = \sqrt{t_2/t_1}$. So the action of the centraliser on the positive imaginary axis is transitive.

(ii)

$$B.(t+iy) = \begin{pmatrix} 1 & b' \\ 0 & 1 \end{pmatrix}.(t+iy) = t + b' + iy$$

for all $t \in \mathbb{R}$, $b' \in \mathbb{R}$, $y > 0$. So $z \mapsto B.z$ preserves any horzintal line $\{t + iy : t \in \mathbb{R}\}$. Given any $t_1,\ t_2 \in \mathbb{R}$ we can take $b' = t_2 - t_1$ and then $B.(iy + t_1) = iy + t_2$. So tha action of the centraliser on any horizontal line is transitive.

(iii) From question 6, we already know that $z \mapsto B.z$ fixes i for all B.

10. Take any $B \in SL(2, \mathbb{R})$. Then $B = PAP^{-1}$ for $P \in SL(2, \mathbb{R})$ and A one of the matrices in question 8. (Actually, in the second case A is pf the form

$$\pm \begin{pmatrix} 1 & \pm 1 \\ 0 & 1 \end{pmatrix}.)$$

Then the centraliser of B is

$$\{PA'P^{-1} : A' \in \text{centraliser}(A)\},$$

because $(PA'P^{-1})(PAP^{-1}) = PA'AP^{-1}$ and $(PAP^{-1})(PA'P^{-1}) = PAA'P^{-1}$.

Then the centraliser of B leaves invariant a set $P.\ell \Leftrightarrow$ the centraliser of A leaves invariant a set ℓ, because $(PA'P^{-1}).P\ell = P(A'.\ell)$. A map $z \mapsto P.z$ maps the positive imaginary axis to a geodesic, that is a semicircle with centre on $\mathbb{R}$, or another vertical half-line. A map $z \mapsto P.z$ maps a horizontal line to another horizontal line, or to a circle in the upper half plane H which is tangent to $\mathbb{R}$. A map $z \mapsto P.z$ maps i to a point in H. So the set ℓ is: a geodesic; a circle in H tangent to $\mathbb{R}$ or horizontal line; or a point, depending on whether A is

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \ (\lambda \neq 0), \pm \begin{pmatrix} 1 & \pm 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$