MATH553. Topology and Geometry of Surfaces
Problem Sheet 3: Manifolds

Please hand in your solutions in class on *Monday 24th October.*

1. Let
$$S^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\},$$
and let
$$U_1 = \{(x,y) \in S^1 : y > 0\},\ U_2 = \{(x,y) \in S^1 : y < 0\},$$
$$U_3 = \{(x,y) \in S^1 : x > 0\},\ U_4 = \{(x,y) \in S^1 : x < 0\}.$$
a) Sketch the sets U_j on the circle.
Let chart maps $\varphi_j : U_j \to \mathbb{R}$ be defined by
$$\varphi_1(x,y) = x,\ \varphi_2(x,y) = x,\ \varphi_3(x,y) = y,\ \varphi_4(x,y) = y.$$
b) Compute the transition functions $\varphi_3 \circ \varphi_1^{-1} : (0,1) \to \mathbb{R}$, $\varphi_3 \circ \varphi_2^{-1} : (0,1) \to \mathbb{R}$, $\varphi_4 \circ \varphi_1^{-1} : (-1,0) \to \mathbb{R}$.

2. Fix any $\alpha \in \mathbb{C} \setminus \mathbb{R}$. Define an equivalence relation $\sim_\alpha$ on $\mathbb{C}$ by: $z' \sim_\alpha z \Leftrightarrow z' = z + m + n\alpha$ for some $m,\ n \in \mathbb{Z}$.
a) You might like to check that this is an equivalence relation.
Let
$$B(w,\varepsilon) = \{w' : |w' - w| < \varepsilon\}.$$
b) Find an $\varepsilon > 0$ such that, for any $z \in \mathbb{C}$, the sets $B(z+m+n\alpha,\varepsilon)$ $(m,\ n \in \mathbb{Z})$ are all disjoint.

Now consider $\mathbb{C}/\sim_\alpha$ with the quotient topology, write $[z]_\alpha = \{z' : z' \sim_\alpha z\}$ and for $B \subset \mathbb{C}$ let
$$[B]_\alpha = \{[z]_\alpha : z \in B\}.$$
Fix $\varepsilon \le \min(1/4, |\mathrm{Im}(\alpha)|/4)$. For $z \in \mathbb{C}$, define $\varphi_z : [B(z,\varepsilon)]_\alpha \to \mathbb{C}$ by
$$\varphi_z([z']_\alpha = z' \text{ if } |z' - z| < \varepsilon.$$
c) Find the transition function $\varphi_{z_2} \circ \varphi_{z_1}^{-1}$ if
(i) $|z_1 - z_2| < 2\varepsilon$, (ii)$|z_1 - z_2 - 1| < 2\varepsilon$, (iii) $|z_1 - z_2 - \alpha| < 2\varepsilon$.

3a). Again, let $\alpha \in \mathbb{C} \setminus \mathbb{R}$.Show that $\mathbb{C}/\sim_\alpha$ is compact and Hausdorff.
b) Show that the function $[x+iy]_i \mapsto [x+\alpha y]_\alpha : \mathbb{C}/\sim_i \to \mathbb{C}/\sim_\alpha$ $(x,\ y \in \mathbb{R})$ is well-defined, a bijection, continuous and a homemorphism. (To show that the map is continuous it suffices to look at the map $x+iy \mapsto x+\alpha y : \mathbb{C} \to \mathbb{C}$ and write this in coordinate form, identifying $\mathbb{C}$ with $\mathbb{R}^2$. For this, write $\alpha = \alpha_1 + i\alpha_2$ with $\alpha_2 \ne 0$. For a homeomorphism, you could use a fact about continuous bijections between compact Hausdorff spaces.)

4. Let
$$U = \{z : 1 < |z| < 2\},\ A_1 = \{z : 1 < |z| < 5/4\},\ A_2 = \{z : 7/4 < |z| < 2\}.$$
Let the equivalence relation $\sim$ be defined on $U \times \{1,2\}$ by $(z,j) \sim (z,k) \Leftrightarrow$ either $z = z' \in A_1 \cup A_2$ or $(z,j) = (z',k)$. Now let $(U \times \{1,2\})/\sim$ be given the

quotient topology. Show that this space is not Hausdorff, possibly by showing that it is impossible to find open sets separating points $[(z, 1)]$ and $[(z, 2)]$ if $|z| = 5/4$ (or $7/4$).