MATH553. Topology and Geometry of Surfaces
Problem Sheet 3: Manifolds

Please hand in your solutions in class on *Monday 24th October.*

1. Let
$$S^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\},$$
and let
$$U_1 = \{(x,y) \in S^1 : y > 0\},\ U_2 = \{(x,y) \in S^1 : y < 0\},$$
$$U_3 = \{(x,y) \in S^1 : x > 0\},\ U_4 = \{(x,y) \in S^1 : x < 0\}.$$
a) Sketch the sets U_j on the circle.
Let chart maps $\varphi_j : U_j \to \mathbb{R}$ be defined by
$$\varphi_1(x,y) = x,\ \varphi_2(x,y) = x,\ \varphi_3(x,y) = y,\ \varphi_4(x,y) = y.$$
b) Compute the transition functions $\varphi_3 \circ \varphi_1^{-1} : (0,1) \to \mathbb{R}$, $\varphi_3 \circ \varphi_2^{-1} : (0,1) \to \mathbb{R}$, $\varphi_4 \circ \varphi_1^{-1} : (-1,0) \to \mathbb{R}$.

2. Fix any $\alpha \in \mathbb{C} \setminus \mathbb{R}$. Define an equivalence relation $\sim_\alpha$ on $\mathbb{C}$ by: $z' \sim_\alpha z \Leftrightarrow z' = z + m + n\alpha$ for some m, $n \in \mathbb{Z}$.
a) You might like to check that this is an equivalence relation.
Let
$$B(w,\varepsilon) = \{w' : |w' - w| < \varepsilon\}.$$
b) Find an $\varepsilon > 0$ such that, for any $z \in \mathbb{C}$, the sets $B(z+m+n\alpha,\varepsilon)$ $(m,\ n \in \mathbb{Z})$ are all disjoint.

Now consider $\mathbb{C}/\sim_\alpha$ with the quotient topology, write $[z]_\alpha = \{z' : z' \sim_\alpha z\}$ and for $B \subset \mathbb{C}$ let
$$[B]_\alpha = \{[z]_\alpha : z \in B\}.$$
Fix $\varepsilon \leq \min(1/4, |\mathrm{Im}(\alpha)|/4)$. For $z \in \mathbb{C}$, define $\varphi_z : [B(z,\varepsilon)]_\alpha \to \mathbb{C}$ by
$$\varphi_z([z']_\alpha = z' \text{ if } |z' - z| < \varepsilon.$$
c) Find the transition function $\varphi_{z_2} \circ \varphi_{z_1}^{-1}$ if
(i) $|z_1 - z_2| < 2\varepsilon$, (ii)$|z_1 - z_2 - 1| < 2\varepsilon$, (iii) $|z_1 - z_2 - \alpha| < 2\varepsilon$.

3a). Again, let $\alpha \in \mathbb{C} \setminus \mathbb{R}$.Show that $\mathbb{C}/\sim_\alpha$ is compact and Hausdorff.
b) Show that the function $[x+iy]_i \mapsto [x+\alpha y]_\alpha : \mathbb{C}/\sim_i \to \mathbb{C}/\sim_\alpha$ $(x,\ y \in \mathbb{R})$ is well-defined, a bijection, continuous and a homemorphism. (To show that the map is continuous it suffices to look at the map $x+iy \mapsto x+\alpha y : \mathbb{C} \to \mathbb{C}$ and write this in coordinate form, identifying $\mathbb{C}$ with $\mathbb{R}^2$. For this, write $\alpha = \alpha_1 + i\alpha_2$ with $\alpha_2 \neq 0$. For a homeomorphism, you could use a fact about continuous bijections between compact Hausdorff spaces.)

4. Let
$$U = \{z : 1 < |z| < 2\},\ A_1 = \{z : 1 < |z| < 5/4\},\ A_2 = \{z : 7/4 < |z| < 2\}.$$
Let the equivalence relation $\sim$ be defined on $U \times \{1,2\}$ by $(z,j) \sim (z,k) \Leftrightarrow$ either $z = z' \in A_1 \cup A_2$ or $(z,j) = (z',k)$. Now let $(U \times \{1,2\})/\sim$ be given the

quotient topology. Show that this space is not Hausdorff, possibly by showing that it is impossible to find open sets separating points $[(z,1)]$ and $[(z,2)]$ if $|z| = 5/4$ (or $7/4$).

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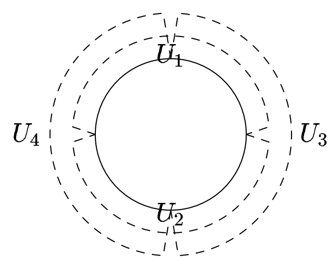
Problem Sheet 3: Solutions

1a) The sets U_j are as shown.



This makes it clear that $\varphi_3 \circ \varphi_1^{-1}$, $\varphi_3 \circ \varphi_2^{-1}$, $\varphi_4 \circ \varphi_1^{-1}$ are defined on $(0,1)$, $(0,1)$, $(-1,0)$ respectively.
b) If $\varphi_1(x,y) = x$ then $y = +\sqrt{1-x^2}$ and if $\varphi_2(x,y) = x$ then $y = -\sqrt{1-x^2}$. So

$$\varphi_3 \circ \varphi_1^{-1}(x) = +\sqrt{1-x^2}\, \varphi_3 \circ \varphi_2^{-1}(x) = -\sqrt{1-x^2},\ \varphi_4 \circ \varphi_1^{-1}(x) = +\sqrt{1-x^2}.$$

2a) (i)$z = z + 0 + 0\alpha$, so $z \sim_\alpha z$. (Reflexive)
(ii) $z' \sim_\alpha z \Leftrightarrow z' = z + m + n\alpha \Leftrightarrow z = z' + (-m) + (-n)\alpha \Leftrightarrow z \sim_\alpha z'$. (Symmetric)
(iii) If $z' \sim_\alpha z$ and $z'' \sim_\alpha z'$, then $z' = z + m + n\alpha$, $z'' = z' + p + q\alpha$ (m, n, p, $q \in \mathbb{Z}$) and so $z'' = z + (m+p) + (n+q)\alpha$ and $z'' \sim_\alpha z$. (Transitive)
b) Take $\varepsilon = \min(1/2, |\mathrm{Im}(\alpha)|/2)$. If $|z' - (z+m+n\alpha)| < \varepsilon$ and $|(z+p+q\alpha) - z'| < \varepsilon$ then

$$|(z+p+q\alpha) - (z+m+n\alpha)| < 2\varepsilon = \min(1, |\mathrm{Im}(\alpha)|).$$

Then

$$|\mathrm{Im}((q-n)\alpha| < |\mathrm{Im}(\alpha)|,$$

so $q = n$. Then $|p - m| < 1$. So $p = m$. So the sets $B(z+m+n\alpha, \varepsilon)$ are all disjoint (m, $n \in \mathbb{Z}$).
c) (i) If $[z]_\alpha \in [B(z_1,\varepsilon)]_\alpha \cap [B(z_2,\varepsilon)]_\alpha$, then we can choose z in its equivalence class so that $\varphi_{z_1}([z]_\alpha) = z$, that is, z is in the image of φ_{z_1}, that is, $|z - z_1| < \varepsilon$. If $\varphi_{z_1}^{-1}(z) = [z]_\alpha$ is in the domain of φ_{z_2} then $|z - (z_2 + m + n\alpha)| < \varepsilon$ for some m, $n \in \mathbb{Z}$, which gives

$$|z_1 - (z_2 + m + n\alpha)| \leq |z_1 - z| + |z - (z_2 + m + n\alpha < 2\varepsilon.$$

But

$$|z_1 - (z_2+m+n\alpha)| \geq |z_2 - (z_2+m+n\alpha| - |z_1 - z_2| \geq 4\varepsilon - 2\varepsilon\varepsilon \geq 2\varepsilon \text{ if } (m,n) \neq (0,0).$$

So we have $(m,n) = (0,0)$ and $|z - z_2| < \varepsilon$, and

$$\varphi_{z_2} \circ \varphi_{z_1}^{-1}(z) = \varphi_{z_2}([z]_\alpha) = z.$$

(ii) If $[z]_\alpha \in [B(z_1,\varepsilon)]_\alpha \cap [B(z_2,\varepsilon)]_\alpha$ then again we can choose z in its equivalence class so that $|z - z_1| < \varepsilon$. Then $|z - (z_2 + m + n\alpha)| < \varepsilon$ for some m, $n \in \mathbb{Z}$, which gives

$$|z_1 - (z_2 + m + n\alpha)| \le |z - z_1| + |z - (z_2 + m + n\alpha| < 2\varepsilon.$$

But

$$|z_1-(z_2+m+n\alpha)| \ge |z_2-(z_2+(m-1)+n\alpha|-|z_1-(z_2+1)| \ge 4\varepsilon-2\varepsilon \ge 2\varepsilon \text{ if } (m,n) \ne (1,0).$$

So $(m,n) = (1,0)$ and $|z - (z_2 + 1)| < \varepsilon$, that is, $|(z - 1) - z_2| < \varepsilon$. Then

$$\varphi_{z_2} \circ \varphi_{z_1}^{-1}(z) = \varphi_{z_2}([z]_\alpha) = \varphi_{z_2}([z - 1]_\alpha) = z - 1.$$

(iii) Again, if $[z]_\alpha \in [B(z_1,\varepsilon)]_\alpha \cap [B(z_2,\varepsilon)]_\alpha$ then we can choose z in its equivalence class so that $|z - z_1| < \varepsilon$. Then $|z - z_2 + m + n\alpha| < \varepsilon$ for some $(m,n) \in \mathbb{Z}^2$.

$$|z_1 - (z_2 + m + n\alpha)| \le |z - z_1| + |z - (z_2 + m + n\alpha| < 2\varepsilon.$$

But

$$|z_1-(z_2+m+n\alpha)| \ge |z_2-(z_2+m+(n-1)\alpha|-|z_1-(z_2+\alpha)| \ge 4\varepsilon-2\varepsilon \ge 2\varepsilon \text{ if } (m,n) \ne (0,1).$$

So we have $(m,n) = (0,1)$. Then $|z - (z_2 + \alpha)| < \varepsilon$, that is, $|(z - \alpha) - z_2| < \varepsilon$. So

$$\varphi_{z_2} \circ \varphi_{z_1}^{-1}(z) = \varphi_{z_2}([z]_\alpha) = \varphi_{z_2}([z - \alpha]_\alpha) = z - \alpha.$$

3a) To show Hausdorff: take any $[z]_\alpha \ne [z']_\alpha$. Then $z' \ne z + m + n\alpha$ for any m, $n \in \mathbb{Z}$. The distance between any 2 points $z+m+n\alpha$ is $\ge \min(1/2, |\mathrm{Im}(\alpha)|/2) = \delta_0$. So there can be at most one point $z + m + n\alpha$ distance less than $\delta_0/2$ of z' because if there are two of them the distance between them is less than δ_0. If there is no point $z + m + n\alpha$ within $\delta_0/2$ of z', take $\delta = \delta_0/4$. If there is one such point $z + m_0 + n_0\alpha$, take $\delta = \frac{1}{2}|z + m_0 + n_0\alpha - z'|$. Then such that

$$|z+m+n\alpha-(z'+p+q\alpha)| = |z+(m-p)+(n-q)\alpha-z'| \ge 2\delta \text{ for all } m,\ n\, p,\ q \in \mathbb{Z}.$$

Then $[B(z,\delta)]_\alpha \cap [B(z',\delta)]_\alpha = \phi$ and these are open sets containing $[z]_\alpha$, $[z']_\alpha$. So
To show compact: Take

$$K = \{z : |z| \le 1 + |\alpha|\}.$$

For any $z \in \mathbb{C}$ we can find $m, n \in \mathbb{Z}$ such that $z-(m+n\alpha) \in K$. Then $z \mapsto [z]_\alpha$ is continuous and maps K onto $\mathbb{C}/\sim_\alpha$. So $\mathbb{C}/\sim_\alpha$ is a continuous image of a compact set, and hence compact.
b) We have $x+iy \sim_i x'+iy' \Leftrightarrow x-x' \in \mathbb{Z}$ and $y-y' \in \mathbb{Z} \Leftrightarrow x+\alpha y \sim_\alpha x'+\alpha y'$. So $[x + iy]_i \mapsto [x + \alpha y]_\alpha$ is well-defined and injective. The map is surjective because any number in $\mathbb{C}$ can be written in the form $x + \alpha y$ for some x, $y \in \mathbb{R}$, because 1, α are linearly independent over the reals and hence form a basis of $\mathbb{C}$ over $\mathbb{R}$. (This also implies that the representation as $x + \alpha y$ is unique.)

To show continuity, it suffices to show continuity of $x + iy \mapsto x + \alpha y$. (This is a general result about quotient topology.) Write $\alpha = \alpha_1 + i\alpha_2$ with α_1, $\alpha_2 \in \mathbb{R}$, $\alpha_2 \neq 0$. Then the map becomes

$$(x, y) \mapsto (x + \alpha_1 y, \alpha_2 y).$$

This is continuous. In fact, it is also clear that the inverse is continuous since this is given by

$$(X, Y) \mapsto (X - \alpha_1 Y/\alpha_2, Y/\alpha_2).$$

Alternatively, a continuous bijection between the compact Hausdorff spaces $\mathbb{C}/\sim_i$ and $\mathbb{C}/\sim_\alpha$ is automatically a homeomorphism.

4. Consider $[(z, 1)]$ and $[(z, 2)]$ for any $|z| = 5/4$. THen we have $[(z, 1)] \neq [(z, 2)]$. Let U_1 and U_2 be any open sets containing $[(z, 1)]$ and $[(z, 2)]$ respectively. Let $\pi : U \times \{1, 2\} \to (U \times \{1, 2\})/\sim$ be the quotient map $\pi(w, j) = [(w, j)]$. Then $\pi^{-1}(U_1)$ and $\pi^{-1}(U_2)$ are open sets in $U \times \{1, 2\}$ containing $(z, 1)$ and $(z, 2)$ respectively, Then there is $\delta > 0$ such that

$$\{(z', j) : |z - z'| < \delta\} \subset \pi^{-1}(U_j).$$

But $\pi^{-1}(U_j)$ is a union of equivalence classes. So if $|z - z'| < \delta$ and $|z'| < 5/4$, then $(z', 1)$ and $(z', 2)$ are both in $\pi^{-1}(U_1)$ and both in $\pi^{-1}(U_2)$. So

$$[(z', 1)] = [(z', 2)] \in U_1 \cap U_2.$$

There are such points z'. So $U_1 \cap U_2 \neq \phi$ and $(U \times \{1, 2\})/\sim$ is not Hausdorff.