MATH553. Topology and Geometry of Surfaces
Problem Sheet 1: Topology

This module is run both as part of the MMath course, an option which can be taken in eiyther Years 3 and 4, and an M.Sc. course. Lectures are on Mondays at 2, Thursdays at 3 and Fridays at 12. There is a further scheduled contact hour on Monday at 3. I would prefer to have this as an informal office hour, but shall seek the class's opinion on this. The usual day for handing in homework will be Thursdays. It seems sensible to have another office hour in the days before homework hand-in. I suggest either Friday (some time near to the lecture) or Tuesday.

Please hand in your solutions to the following problems in class on *Thursday 6th October* — or earlier if you wish

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1. Show that $[0,1) = \{x : 0 \leq x < 1\}$ is *not* open in $\mathbb{R}$ (in the usual topology).
Hint: consider the point 0.

2. Show that $U \subset \mathbb{R}^2$ is open in the usual topology on $\mathbb{R}^2$ by finding, for $\overline{x} \in U$, an $\varepsilon > 0$ with $\{\overline{y} : \|\overline{y} - \overline{x}\| < \varepsilon\} \subset U$ where:

a) $U = \{(x,y) \in \mathbb{R}^2 : x > 0\}$,   b) $U = \{(x,y) \in \mathbb{R}^2 : x + y > 0\}$.

3. This question is about characterizing open sets in $\mathbb{R}$, in the usual topology. Let U be open in $\mathbb{R}$. For $x \in U$, let

$$I_x = \{y \geq x : [x,y] \subset U\} \cup \{y \leq x : [y,x] \subset U\}.$$

Show that if I_x is not bounded above then $[x, \infty) \subset U$ and that if I_x is not bounded below then $(-\infty, x] \subset U$. Let $b_x = \sup(I_x)$ or ∞, depending on whether or not I_x is bounded above, and similarly let $a_x = \inf(I_x)$ or $-\infty$. Show that $I_x = (a_x, b_x)$ and that a_x, $b_x \notin U$. (Of course, the last statement only makes sense when a_x, $b_x \in \mathbb{R}$.) Finally deduce that U is a *countable* disjoint union of intervals.

Hint: you can consider intervals of the form I_x, and remember that the rationals are countable and dense in $\mathbb{R}$.

4. Take $Y = \mathbb{R} \times \{0,1\} \subset \mathbb{R}^2$, where Y is given the subspace topology. Show that any set of the form $(a,b) \times \{j\}$ (any a, $b \in \mathbb{R}$, and $j = 0$ or 1) is open in Y. Then show that $U \subset Y$ is open in $Y \Leftrightarrow U_j$ is open in $\mathbb{R}$ for $j = 0$ and 1, where

$$U_j = \{x : (x,j) \in U\}.$$