MATH348. Revision.

Weeks 11 -12 Dec 6-17 2004 will be devoted to revision. Lecture times in Week 11 are a double session Monday 6 December, 9-11, and Wednesday 8 December at 10. The 2 p.m. lecture on Friday 10 December is cancelled. Office hours in Week 11 are also a bit nonstandard, in that I shall not be around on Monday 6 December after 11 . I shall have office hours 12-1 and 3-5 on Tuesday 7 December BUT I may also have a couple of Year 1 tutees, who cannot make a rearranged time calling on me between 3-5. I can also hold office hours 9-10 and 11-12 on Wednesday 8 December, but shall then be absent for the rest of the week. If anyone wants to suggest any rearrangements of lecture/office hour times in Week 12, I shall consider it. The draft exam timetable (on 1.12.04) has MATH348 on Thursday 13th January 2005, 10-12.30

Revision done in lecture times will be demand-led so be prepared to express an opinion on what you would like to go over. If you feel that for some sessions at least you would prefer to work on your own elsewhere, then of course you are free to do so, and are encouraged to do so.

The following questions are designed to help you with your revision. Although the questions are not of uniform length and difficulty, if you are eventually able to answer all of them well, you should do well in the exam. You should aim to achieve several things by your revision.

a) Understand what the key concepts are in this module, and be able to give clear definitions that you, your examiner,and hopefully anyone else who is interested, will understand. You should know what properties hold, and what the main results are.

b) Although you should not attempt to memorize proofs, you should have sufficient understanding and confidence to fill in short steps in arguments, with some prompting.

c) You should have some idea what the applications of the theory are, and be practised in standard calculations and problems. You should spend *most* of your revision time practising problems.

d) Perhaps most importantly, you should aim to have a clear head, so that you can read and digest the question in front of you, and tackle it without confusing it with somewhat similar problems that you tackled in your revision. I shall hand out solutions before the end of term, but try not to look at them more than you have to.

Good luck - here the work starts.

1a). Write down the formula for $\hat{f}(n)$ for a 2π-periodic function f which is integrable over $[-\pi, \pi]$ (or any interval of length 2π). Now write down the formula for $\hat{f}(\xi)$ for an integrable function f on $\mathbf{R}$. The formulae should look similar - but not identical.

b) Work out the Fourier $\hat{f}(n)$ for f where $f : [0, 2\pi) \to \mathbf{R}$ is defined by $f(x) = e^{-2x}$. Also, sketch the 2π-periodic extension of this function, which you should also call f.

c) Work out the Fourier transform $\hat{g}$ of g where

$$g(x) = \begin{array}{l} e^{-2x} \text{ if } 0 \leq x \leq 2\pi, \\ 0 \text{ otherwise } . \end{array}$$

There should be some similarity between $\hat{f}$ and $\hat{g}$, but, again, they will not be identical.

d) For all real x, give formulae for $\frac{1}{2}(f(x+) + f(x-))$ in terms of $\hat{f}$ and $\frac{1}{2}(g(x+) + g(x-))$ in terms of $\hat{g}$.

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2. Determine whetherthe following functions are integrable on the stated sets.

a) $f(x) + \dfrac{1}{1+x^2}$ on $\mathbf{R}$.

b) $f(x) = \dfrac{1-e^{-x}}{x}$ on $(0,1)$

c) $f(x) = \dfrac{1-e^{-x}}{x}$ on $(0,\infty)$.

d) $f(x,y) = \dfrac{\sin^2 y}{y(1+x^2y^2)}$ on $\mathbf{R}^2$.

  [Hint: it might help to use Tonelli's Theorem and
$$\int_{-\infty}^{\infty} \frac{dx}{1+a^2x^2} = \frac{\pi}{a}.]$$

3. Determine the limits of each of the following sequences and families of functions on the stated domains, and determine whether the convergence to the limit is uniform.

a) $f_n(x) = x^{-n}$ on $[1,\infty)$.

b) $g_n(x) = x^{-n}(1-x^{-1})$ on $[1,\infty)$

c) $K_\lambda(x) = \lambda^{-1}K_1(x/\lambda)$ for $\lambda \in (0,1]$, and $K_1(x) = \pi^{-1}x^{-2}\sin^2 x$ for $x \neq 0$ In this case take domain $(-\infty,0) \cup (0,\infty)$ and determine $\lim_{\lambda\to 0} K_\lambda(x)$

4. Show that, if x is not an integer multiple of 2π,
$$\sum_{n=-N}^{N} e^{inx} = \frac{\sin\left(N+\frac{1}{2}\right)x}{\sin\frac{1}{2}x}.$$

Hence show that if f is a 2π-periodic function,
$$\sum_{n=-N}^{N} \hat{f}(n)e^{inx} = \int_{-\pi}^{\pi} f(y)\frac{\sin\left(N+\frac{1}{2}\right)(x-y)}{\sin\frac{1}{2}(x-y)}dy = \int_{-\pi}^{\pi} f(x-y)\frac{\sin\left(N+\frac{1}{2}\right)y}{\sin\frac{1}{2}y}dy.$$

In order to do this properly, you need that $y \mapsto \sin\left(N+\frac{1}{2}\right)y/\sin\frac{1}{2}y$ is a 2π-periodic function for any integer N - can you see why this is true? It is not true that $y \mapsto \sin\frac{1}{2}y$ is 2π-periodic, neither is it true that $y \mapsto \sin\left(n+\frac{1}{2}\right)y$ is 2π-periodic.

5. Let f be the 2π-periodic function of question 1, that is,
$$f(x) = e^{-2x} \text{ if } 0 \leq x < 2\pi,$$
$$f(x+2n\pi) = e^{-2x} \text{ if } 0 \leq x < 2\pi, \; n \in \mathbf{Z}.$$

Now, as usual, let
$$S_N(f)(x) = \int_{x-\pi}^{x+\pi} f(x-y)\frac{\sin\left(N+\frac{1}{2}\right)y}{\sin\frac{1}{2}y}dy.$$

[The integrand is 2π-periodic, so this really is the usual definition.]

a)Let $0 < x < \pi$. By considering the formula for $f(x-y)$ separately for $x < y < x+\pi$ and $x - \pi < y < x$, or otherwise, show that
$$S_N(f)(x) = \frac{e^{-2x}}{2\pi}\int_{x-\pi}^{x} e^{2y}\frac{\sin\left(N+\frac{1}{2}\right)y}{\sin\frac{1}{2}y}dy + \frac{e^{-2x-4\pi}}{2\pi}\int_{x}^{x+\pi} e^{2y}\frac{\sin\left(N+\frac{1}{2}\right)y}{\sin\frac{1}{2}y}dy.$$

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Give the value of $\lim_{N\to\infty} I_N(x)$, according to the Fourier Series Theorem.

b) Now let

$$T_N(f)(x) = \frac{e^{-2x}}{2\pi} \int_{x-\pi}^{x} \frac{2\sin\left(N+\frac{1}{2}\right)y}{y} dy + \frac{e^{-2x-4\pi}}{2\pi} \int_{x}^{x+\pi} \frac{2\sin\left(N+\frac{1}{2}\right)y}{y} dy,$$

and let $x_N = \pi/(N+\frac{1}{2})$. Show that

$$\lim_{N\to+\infty} T_N(f)(x_N) = \frac{1}{\pi} \lim R \to +\infty \int_{-R}^{\pi} \frac{\sin t}{t} dt + \frac{1}{\pi} e^{-4\pi} \lim R \to \infty \int_{\pi}^{R} \frac{\sin t}{t} dt.$$

c) Using the fact that

$$\lim_{R\to\infty} \int_{0}^{R} \frac{\sin t}{t} dt = \frac{\pi}{2}$$

and $\sin t/t$ is an even function, show that

$$\lim_{N\to+\infty} T_N(f)(x_N) = \frac{1}{2}(1 + e^{-4\pi}) + \frac{1}{\pi}(1 - e^{-4\pi}) \int_{0}^{\pi} \frac{\sin t}{t} dt.$$

By using the fact that

$$\int_{0}^{\pi} \frac{\sin t}{t} dt > \tfrac{1}{2}\pi,$$

show that

$$\lim_{N\to\infty} T_N(f)(x_N) > 1.$$

6. This is a continuation of question 5. You might want to try another question first.

a) By using l'Hopital's rule or otherwise, show that

$$\lim_{y\to 0} \left(\frac{e^{2y}}{\sin\frac{1}{2}y} - \frac{2}{y} \right) = 4.$$

Hint. Whatever method you use, remember to write the term inside the bracket over a common denominator, before you do anything else. Explain why

$$\frac{e^{2y}}{\sin\frac{1}{2}y} - \frac{2}{y}$$

extends to a continuous function on $(-2\pi, 2\pi)$.

b) Now let $S_N(f)(x)$ and $T_N(f)(x)$ be as in question 5. By using a) show that, for any x,

$$\lim_{N\to+\infty} S_N(f)(x) - T_N(f)(x) = 0.$$

State what result (if any) you are using.

c) The limit is actually uniform in x for $x \in [0, \pi - \delta]$, for any $\delta > 0$. So $\lim_{N\to\infty} S_N(f)(x_N) > 1$ by 3c). Deduce from this that the convergence of $S_N(x)$ to its limit cannot be uniform for $x \in (0, \pi]$.

7. Let $u : [0, 2\pi] \times [0, \infty) \to \mathbf{R}$ be continuous. Let u_x, u_{xx}, u_t be defined and continuous on $(0, 2\pi) \times (0, \infty)$ and extend continuously to $[0, 2\pi] \times [0, \infty)$. Let u satisfy the heat equation

$$u_t = u_{xx}, \quad (x, t) \in (0, 2\pi) \times (0, \infty), \tag{1}$$

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$$u(x,0) = f(x), \ u(0,t) = (2\pi, t) = u_x(0, t) = u_x(2\pi, t) = 0. \tag{2}$$

Let $\hat{u}(n, t)$ be the Fourier coefficients of u with respect to x.

a) Write down (without proof) an ordinary differential equation satisfied by $\hat{u}(n, t)$ for each integer n, and the boundary conditions, implied by (1) and (2). Solve this differential equation. Then use a Fourier Series Theorem to write down a series solution for $u(x, t)$ to (1) and (2). [You will not be able to write the solution as a convolution.]

b) Use integration by parts to prove the relation $\hat{u}_x(n, t) = in\hat{u}(n, t)$.

8. Let $u : [0, 2\pi] \times [0, \infty) \to \mathbf{R}$ be continuous. Let u_x, u_{xx}, u_t, u_{tt} be defined and continuous on $(0, 2\pi) \times (0, \infty)$ and extend continuously to $[0, 2\pi] \times [0, \infty)$. For a number $\lambda > 0$, let u satisfy the damped wave equation

$$u_{tt} = u_{xx} - \lambda u_x, \ o < x < 2\pi, \ t > 0, \tag{1}$$

$$u(x, o) = f(x), \ u_t(x, 0) = 0, \ u(0, t) = u(2\pi, t) = u_x(0, t) = u_x(2\pi, t) = 0. \tag{2}$$

Let $\hat{u}(n, t)$ be the Fourier coefficients of u with respect to x.

Write down (without proof) an ordinary differential equation satisfied by $\hat{u}(n, t)$ for each integer n, and the boundary conditions, implied by (1) and (2). Solve this differential equation. Hence use the Fourier Series Theorem to show that

$$u(x, t) = \hat{f}(0) + \frac{1}{2} \sum_{n \neq 0} \hat{f}(n) e^{inx} (e^{int\sqrt{1 - i\lambda/n}} + e^{-int\sqrt{1 - i\lambda/n}}).$$

9. Calculate the Fourier transform of $f(x) = e^{-3|x|}$.

10. Calculate the Fourier transform of

$$f(x) = \frac{1}{4x^2 + 1}.$$

To do the calculation for $\xi \geq 0$ you will probably want to use a contour integral of

$$\frac{1}{4z^2 + 1}$$

round a semicircular contour in the lower half-plane. You can then avoid doing a separate contour integral for the case $\xi \leq 0$ if you can show that

$$\overline{\hat{f}(\xi)} = \hat{f}(-\xi).$$

11. Calculate the Fourier transform of

$$f(x) = \frac{1}{(x^2 + 2x + 2)^2}.$$

To some extent the method is the same as in question 8 - but in this example the integrand has *double poles*.

12. Calculate the Fourier transform of

$$f(x) = e^{-2x^2}$$

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by integrating e^{-2z^2} round a rectangular contour with corners at $-R$, R, $R+i\xi/4$, $-R+i\xi/4$. You may assume that

$$\int_{-\infty}^{\infty} e^{-2x^2}\,dx = \sqrt{\frac{\pi}{2}}.$$

13. Let

$$h(y) = \frac{1 - e^{-4\pi - 2\pi i y}}{2 + iy}.$$

Show that

$$\hat{h}(\xi) = \begin{array}{l} 2\pi e^{2\xi} \text{ if } -2\pi < \xi < 0 \\[4pt] 0 \text{ if } \xi < -2\pi, \text{ or } x > 0, \\[4pt] \pi \text{ if } \xi = 0, \\[4pt] \pi e^{-4\pi} \text{ if } \xi = -2\pi. \end{array}$$

Thus, $\hat{h}(-x) = 2\pi g(x)$ except when $x = 0$ or -2π, for g as in question 1.

Hint: you will need to consider integrals of the functions $e^{-i\xi z}/(2 + iz)$ and $e^{-i(\xi+2\pi)z}/(2+iz)$ round semicircular contours. For the first function, you will need to take a semicircular contour in the lower half plane if $\xi > 0$ and in the upper half-plane if $\xi < 0$. For the second function you will need to take a semicircular contour in the lower half-plane if $\xi + 2\pi > 0$ and in the upper half-plane if $\xi + 2\pi < 0$. You may assume that if $\xi_1 > 0$,

$$\lim_{R\to\infty} \int_{\gamma'_-(R)} \frac{e^{-i\xi_1 z}}{2 + iz}\,dz = 0,$$

(which can be applied with $\xi_1 = \xi$ or $\xi_1 = \xi + 2\pi$) and that if $\xi_1 < 0$,

$$\lim_{R\to\infty} \int_{\gamma'_+(R)} \frac{e^{-i\xi_1 z}}{2 + iz}\,dz = 0.$$

However it is good practise to check that $|e^{-i\xi_1 z}| \le 1$ on $\gamma'_-(R)$ if $\xi_1 \ge 0$ and $|e^{-\xi_1 z}| \le 1$ on $\gamma'_+(R)$ if $\xi_1 \le 0$.

The cases $\xi = 0$ and $\xi + 2\pi = 0$ have to be treated separately.

14a). State Plancherel's equality for functions f, $g \in L^1(\mathbf{R})$ such that $\hat{f}$, $\hat{g} \in L^1$.

b) Give the precise estimate which implies that $\hat{f}$, $\hat{g}$ are bounded for f, $g \in L^1(\mathbf{R})$. Explain also why the Fourier inversion formula and f, $\hat{f} \in L^1(\mathbf{R})$ imply that that f, $\hat{f}$ are bounded.

c) Show that if $h_1 \in L^1$ and h_2 is bounded then $h_1 h_2 \in L^1$.

d) State Tonelli's Theorem. Apply it to prove that

$$\int_{-\infty}^{\infty} \hat{f}(\xi) \int_{-\infty}^{\infty} e^{ix\xi}\,\overline{g(x)}\,dx\,d\xi = \int_{-\infty}^{\infty} \overline{g(x)} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{ix\xi}\,d\xi\,dx.$$

15. Let f be integrable on $\mathbf{R}$. Suppose that $\lambda > 0$ and h_λ is an integrable function with

$$\widehat{h_\lambda}(\xi) = \begin{cases} \left(e^{-\xi\lambda + i\lambda}\,\widehat{f}(\xi) & \text{if } \xi > 0 \\ e^{\xi\lambda - i\lambda}\,\widehat{f}(\xi) & \text{if } \xi < 0 \end{cases}\right).$$

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Now use the inverse Fourier formula
$$h_\lambda(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ix\xi} \widehat{h_\lambda}(\xi) d\xi$$
and interpreting this as a double integral and changing the order of integration or otherwise, show that
$$h_\lambda(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) \left(\int_0^{\infty} e^{i(x-t)-\lambda\xi+i\lambda} d\xi + \int_{-\infty}^0 e^{i(x-t)+\lambda\xi-i\lambda} d\xi \right) dt = g_\lambda * f(x),$$
where
$$g_\lambda(t) = \frac{\lambda \cos \lambda - t \sin \lambda}{\pi(t^2 + \lambda^2)}.$$

16. As in question 3, let
$$K(x) = \frac{1}{\pi} \left(\frac{\sin(x)}{x} \right)^2.$$

a) Show that $K(x)$ is integrable on $(-\infty, \infty)$.

Now you may assume that
$$\int_{-\infty}^{\infty} K(x) dx = 1.$$

[If you really want more practice in contour integration, look at part c) below.]
For $\lambda > 0$, let $K_\lambda(x) = \lambda^{-1} K(x/\lambda)$.
b) Show that K_λ ($\lambda > 0$) is an *approximate identity* as follows. It is clearly a positive function for all λ. b)(i) Show that for any $\lambda > 0$,
$$\int_{-\infty}^{\infty} K_\lambda(x) dx = 1.$$

b)(ii) Show that for any $\delta > 0$,
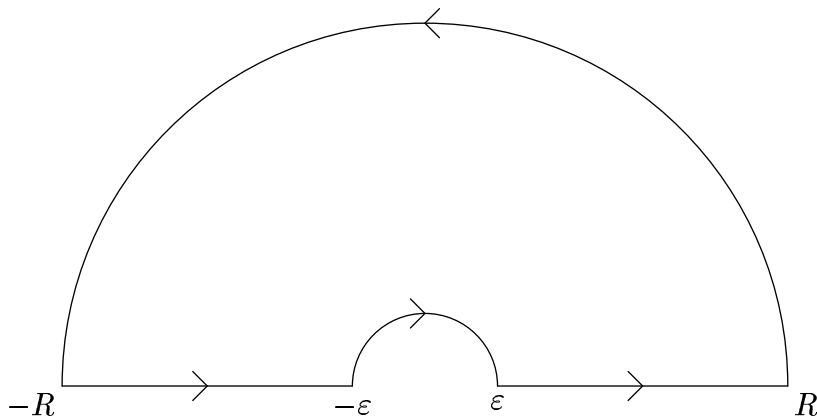$$\lim_{\lambda \to 0} \int_{|x| \geq \delta} K_\lambda(x) dx = 0.$$

b)(iii) Show that for any $\delta > 0$,
$$\lim_{\lambda \to 0} K_\lambda(x) = 0$$
uniformly on $\{x : |x| \geq \delta\}$.
c) L If you really want to do some more contour integration at this point, let $\gamma(R, \varepsilon)$ be the contour shown.

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By considering
$$\int_{\gamma(R,\varepsilon)} \frac{e^{iz}-1}{z^2}\,dz$$

or otherwise, show that
$$\lim_{R\to\infty,\varepsilon\to 0} \mathrm{Re}\left(\int_{-R}^{-\varepsilon}+\int_{\varepsilon}^{R}\right)\frac{1-e^{ix}}{x^2}\,dx = \pi.$$

Hence, or otherwise, show that
$$\int_{-\infty}^{\infty} K(x)\,dx = 1.$$

d) Show that, if f is any integrable function,
$$f(x) - K_\lambda * f(x) = \int_{-\infty}^{\infty} K_\lambda(y)(f(x)-f(x-y))\,dy.$$

17. a) Let $g(x) = f(x+a)$. Compute $\hat{g}(\xi)$ in terms of $\hat{f}(\xi)$.
b) Let $u : \mathbf{R} \times [0,\infty) \to \mathbf{C}$ be integrable in x for all $t \geq 0$. In addition, for all x, and $t > 0$, let u, u_t, u_x, u_{xx} be continuous and *locally uniformly integrable in* x, that is, for any $a > 0$. let
$$\sup_{0<t<a}\int_{-\infty}^{\infty}|u(x,t)|\,dx < +\infty$$

and
$$\lim_{\Delta\to\infty}\int_{|x|\geq\Delta}|u(x,t)|\,dx = 0$$

uniformly for $0 < t < a$, and similarly for u_t, u_x, u_{xx}.
$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 2\frac{\partial u}{\partial x}. \tag{1}$$

Show that $\hat{u}(\xi,t)$ satisfies the p.d.e.
$$\frac{\partial \hat{u}(\xi,t)}{\partial t} = -\xi^2\hat{u}(\xi,t) + 2i\xi\hat{u}(\xi,t).$$

Hence show that
$$u(x,t) = \int_{-\infty}^{\infty}\frac{u(y,0)e^{-(x-y)^2/4t+2t}\,dy}{2\sqrt{\pi t}}.$$

Show also, using the Dominated Convergence Theorem, that if n denotes any integer,
$$\lim_{n\to+\infty} u(x,n) = 0.$$

18. Show that if $f : \mathbf{R} \to \mathbf{C}$ is integrable then $\hat{f}$ is continuous.
Comment. Make sure you know what continuous means because, of course, you need to know what it means in order to have a chance of showing that a map is continuous. So: $\hat{f}$ is continuous at x if, given any $\varepsilon > 0$ there is $\delta > 0$ such that if ??? then ??? Alsao, of course, you need to know how $\hat{f}(x)$ is defined - it is in terms of an integral, of

course. You probably want to show an integral of something over $(-\infty, \infty)$ is small. It might be a good idea to break up the integral ionto the integral over $[-\Delta, \Delta]$ and $|x| \geq \Delta$, for a suitable $\Delta > 0$.

19a) Define the Laplace transform $\mathcal{L}(f)(z)$ of a function f which is integrable on $(0, \infty)$. Show that the Laplace transform is defined for $\text{Re}(z) \geq 0$, and bounded.

b) As usual, let $\chi_{[0,\infty)}(t) = 1$ for $t \geq 0$ and 0 for $t < 0$. Let f be integrable on $[0, \infty)$ and let $a \geq 0$. Let

$$g(t) = \chi_{[0,\infty)}(t - a)f(t - a).$$

Show that

$$\mathcal{L}(g) = e^{-az}\mathcal{L}(f)(z).$$

c) Suppose that f and g are integrable on $(-\infty, \infty)$ and both are equal to 0 on $(-\infty, 0)$. Write down the formula for $f * g$. Also, write down the formula for $\mathcal{L}(f * g)(z)$.

20. For each of the following functions $F = F(z)$, determine whether or not F can be the Laplace transform $\mathcal{L}(f)$ of a function $f \in L^1(0, \infty)$. Give brief reasons. If there is such an f, give it.

a) $F(z) = 1/z$.

b) $F(z) = 1/(z + 1)^2$.

c) $F(z) = 1/(\overline{z} + 1)^2$.

Now determine which of the following functions F can be the Laplace transform $\mathcal{L}(f)$ for a function $f \in L^2(0, \infty)$. Again, give brief reasons, or quote a theorem if necessary. If the answer is "yes" you need not produce f.

d) $F(z) = \left(\dfrac{1 - e^{-z}}{z}\right)^2$.

e) $F(z) = \dfrac{1 - e^{-z}}{z^2}$.

19. Find the variance of the following probability density functions:

a) $\frac{1}{2}e^{-|x|}$;

b) $\dfrac{1}{\sqrt{2\pi}}e^{-x^2/2}$; You may assume for this that

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2/2}dx = 1;$$

c) $\dfrac{1}{\sqrt{2\pi\sigma}}e^{-x^2/2\sigma}$.

d) Show that the variance of the proability density function

$$\frac{1}{\pi(1 + x^2)}$$

is infinite.

21a). Let μ be the probability measure such that for all bounded measrable functions g,

$$\int_{-\infty}^{\infty} g(x)d\mu(x) = \frac{1}{\sqrt{2\pi}}\int g(x)e^{-x^2/2}dx,$$

so that (as you are allowed to assume)

$$\hat{\mu}(\xi) = e^{-\xi^2/2}.$$

Let μ_n be the n-fold convolution of μ. Give the value of $\hat{\mu}_n$. Now let ν_n be the probablity measure defined by: for all bounded measurable functions g

$$\int_{-\infty}^{\infty} g(x) d\nu_n(x) = \int_{-\infty}^{\infty} g(x/\sqrt{n}) d\mu_n(x)$$

Find $\hat{\nu}_n$ in terms of $\hat{\mu}_n$. Hence show that $\nu_n = \mu$.

b) Now repeat the question with μ such that, for all bounded measurable functions g,

$$\int_{-\infty}^{\infty} g(x) d\mu(x) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} g(x) e^{-x^2/2\sigma} dx.$$

c) State what the Central Limit Theorem says in general, for any probability measure μ with finite varianec and zero mean, and any bounded measurable function g, about

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} g(x) d\nu_n(x)$$

where μ_n and ν_n are defined from μ in the same way as in a).

17. Now consider the heat equation in a finite strip - as in question 5 - but with different boundary conditions. Let $u : [0, \ell] \times [0, \infty) \to \mathbf{R}$ be continuous and integrable in t for all x. Let u, u_x, u_{xx}, $u_t : (0, \ell) \times (0, \infty)$ be defined, continuous and *uniformly integrable in* t, that is if $v(x, t)$ is any of the functions u, u_x, u_{xx}, u_t, then

$$\sup_{0 < x < \ell} \int_0^\infty |v(x, t)| dt < +\infty$$

$$\lim_{\Delta \to \infty} \int_\Delta^\infty |v(x, t)| dt = 0$$

uniformly in x, $0 < x < \ell$.

Let $\mathcal{L}(u)(x, z)$ denote the Laplace transform of u with respect to t.

Now let u satisfy the heat equaltion

$$u_t = u_{xx}, \ 0 < x < \ell, \ t > 0$$

and the boundary conditions

$$u(0, t) = h(t), \ u(\ell, t) = 0, \ u(x, 0) = 0.$$

a) Write down the differential equation and boundary conditions satisfied by $\mathcal{L}(u)(x, z)$. Then show that the solution to this equation is

$$\mathcal{L}(u)(x, z) = \frac{\mathcal{L}(h)(z) e^{(x - 2\ell)\sqrt{z}}}{1 - e^{-2\ell\sqrt{z}}} + \mathcal{L}(h)(z) \frac{e^{-x\sqrt{z}}}{1 - e^{-2\ell\sqrt{z}}}.$$

Now taking the square root so that for $\mathrm{Re}(z) \geq 0$, $\mathrm{Re}\sqrt{z} \geq 0$, expand this out as two infinite sums .

b) Now using the fact that $e^{-c\sqrt{z}}$ is the Laplace transform of ?

$$? \frac{e^{-c^2/4t}}{\sqrt{2\pi} t^{3/2}}?$$

find the general solution for $u(x, t)$ as infinite sum of convolutions of h with functions.

MATH348. Revision Solutions.

1b)
$$\hat{f}(n) = \int_0^{2\pi} e^{-2x-inx}dx = \left[\frac{e^{-(2+in)x}}{-(2+in)}\right]_0^{2\pi} = \frac{1-e^{-4\pi}}{2+in}$$

c)
$$\hat{g}(\xi) = \int_0^{2\pi} e^{-(2+i\xi)x}dx = \left[\frac{e^{-(2+i\xi)x}}{-(2+i\xi)}\right]_0^{2\pi} = \frac{1-e^{-4\pi-2\pi i\xi}}{2+i\xi}.$$

d)
$$\frac{f(x+)+f(x-)}{2} = \lim_{N\to\infty}\frac{1}{2\pi}\sum_{n=-N}^{N}\hat{f}(n)e^{inx} = \lim_{N\to+\infty}\frac{1}{2\pi}\sum_{n=-N}^{N}\frac{1-e^{-4\pi}}{2+in}e^{inx}.$$

$$\frac{g(x+)+g(x-)}{2} = \lim_{N\to\infty}\frac{1}{2\pi}\int_{-N}^{N}\hat{g}(\xi)e^{ix\xi}d\xi$$

$$= \lim_{N\to+\infty}\frac{1}{2\pi}\int_{-N}^{N}\frac{1-e^{-4\pi-2\pi i\xi}}{2+in}e^{ix\xi}d\xi.$$

In this formula for $(g(x+)+g(x-))/2$ the limit is taken over real number N, whereas, of course, in the corresponding limit for f, the limit is taken over positive integers N.

2a) Write $f_n(x) = \chi_{[-n,n]}(x)f(x)$. Then $0 \le f_n(x) \le f_{n+1}(x) \le f(x)$ for all x and all integers $n \ge 1$. By the Fundamental Theorem of Calculus,
$$\int f_n(x)dx = \int_{-n}^{n}\frac{1}{1+x^2}dx$$
$$= [\tan^{-1}(x)]_{-n}*n = \tan^{-1}(n) - \tan^{-1}(-n).$$

We have $\lim_{n\to\infty}f_n(x) = f(x)$ for all x. So by the Monotone Convergence Theorem,
$$\int_{-\infty}^{\infty}\frac{1}{1+x^2}dx = \int f = \lim_{n\to\infty}\int f_n$$
$$= \lim_{n\to\infty}(\tan^{-1}(n) - \tan^{-1}(-n)) = (\pi/2 - (-\pi/2)) = \pi < +\infty.$$

So f is integrable. This is not the only possible method. One could also show that f is bounded between 0 and the function $\chi_{[-1,1} + x^{-2}\chi_{(-\infty,-1)\cup(1,\infty)}$, which is integrable (and can be shown to be so very simply, for example by looking at the sequence of integrable functions $\chi_{[-1,1} + x^{-2}\chi_{(-n,-1)\cup(1,n)}$)

2b) We can define $f(0) = \lim_{x\to0}\dfrac{e^{-x}-1}{x} = -1$, using l'Hopital's Rule or a series expansion. Then f is continous on $[0,1]$. Any continuous function on a closed bounded interval is integrable. So f is integrable on $[0,1]$ - which is the same as saying that it is integrable on $(0,1)$ since the definition of the funciont at just two points $(0, 1)$ makes no difference.

Rather than use the general fact that a continuous funciton on a closed bounded interval is integrable on that interval, we can just use the fact that $\lim_{x\to0}f(x)$ exists to deduce that f is bounded on $(0,1)$ - because $|f(x)| \le \delta^{-1}$ on $[\delta,1]$, for any $\delta > 0$.

So there is M such that $|f(x)| \leq M$ for all $x \in (0,1)$ - and since the constant function M is integrable on $(0,1)$, so is f.

2c) We have $e^{-x} \leq \frac{1}{2}$ for $1 \leq x \leq 1$. So $1 - e^{-x} \geq 1 - \frac{1}{2} = \frac{1}{2}$ for $1 \leq x$. So

$$\int_0^\infty |f| \geq \int_1^\infty |f| \geq \frac{1}{2} \int_1^\infty \frac{dx}{x}$$

$$= \frac{1}{2} \lim_{n \to \infty} \int_1^n \frac{dx}{x} = \lim_{n \to \infty} (\log n - \log 1) = +\infty.$$

So f is not integrable on $(0, \infty)$.

2d) By Tonelli's Theorem, in order to show that $f(x, y) = \dfrac{\sin^2 y}{y(1 + x^2 y^2)}$ is integrable it suffices to show that

$$\int_{-\infty}^\infty \int_{-\infty}^\infty \frac{\sin^2 y \, dx dy}{|y|(1 + x^2 y^2)} < +\infty.$$

Now the inner integral, for $y \neq 0$, is

$$\lim_{\Delta \to \infty} \left[\frac{\sin^2 y}{y^2} \tan^{-1}(xy) \right]_{x=-y\Delta}^{x=y\Delta} = \pi \frac{\sin^2 y}{y^2}.$$

But $\sin^2(y) \leq y^2$ for all $y \neq 0$, and $\sin^2 y \leq 1$ for al y. So

$$\int_{-\infty}^\infty \int_{-\infty}^\infty |f(x, y)| dx dy = \int_{-\infty}^\infty \frac{\sin^2 y \, dy}{y^2} \leq$$

$$\int_{-1}^{dy} + \left(\int_{-\infty}^{-1} + \int_1^\infty \right) \frac{dy}{y^2} = 4 < +\infty.$$

So f is integrable.

3a) For $x > 1$, $\lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} x^{-n} = 0$. For $x = 1$, $f_n(1) = 1$ for all n. So if we define

$$f(x) = \begin{pmatrix} 0 & \text{if } x > 1 \\ 1 & \text{if } x = 1 \end{pmatrix}$$

then $\lim_{n \to \infty} f_n(x) = f(x)$ for all $x \geq 1$.

The convergence is not uniform. For take $x_n = 2^{1/n}$. then $x_n > 1$, so $f(x_n) = 0$. But $f_n(x_n) = x_n^{-n} = 2^{-1}$. So $f_n(x_n) - f(x_n) = 2^{-1}$. So taking $\varepsilon = \frac{1}{2}$, there is no N such that $|f_n(x) - f(x)| < \varepsilon$ for all $n \geq N$ and all $x \geq 1$.

Alternatively, f is not continuous at 1 but f_n is continuous on $[1, \infty)$ for all n. If the convergence of f_n to f was uniform on $[1, \infty)$, general theory would tell us that f was continuous on $[1, \infty)$ also.

b) For all $x > 1$,

$$\lim_{n \to \infty} x^{-n} = \lim_{n \to \infty} x^{-n-1} = 0.$$

So $\lim_{n \to \infty} g_n(x) = 0$ for all $x > 1$. But $g_n(1) = 0$ for all n. So $\lim_{n \to \infty} g_n(x) = 0$ for all $x \geq 1$. To see that the convergence is uniform, note that $g_n(x) \geq 0$ for all $x \geq 1$, because $x^{-1} \leq 1$. The maximum of $g_n(x)$ must occur when the derivative $g_n'(x) = -nx^{-n-1} + (n+1)x^{-n-2} = 0$, that is, when $x = (n+1)/n$. But

$$g_n(n+1/n) = (n+1/n)^{-n} \left(1 - \frac{n}{n+1} \right) \leq \frac{1}{n+1}.$$

So given $\varepsilon > 0$ choose N so that $\frac{1}{N+1} < \varepsilon$. then if $n \geq N$, $0 \leq g_n(x) < \varepsilon$ for all $n \geq N$ and the convergence to 0 is uniform.

c)

$$K_\lambda(x) = \frac{1}{\pi\lambda}\frac{\sin^2(x/\lambda)}{(x/\lambda)^2} = \frac{\lambda}{\pi}\frac{\sin^2(x/\lambda)}{x^2}.$$

So for all $x \neq 0$,

$$0 \leq K_\lambda(x) \leq \frac{\lambda}{\pi x^2} \to 0 \text{ as } \lambda \to 0.$$

But

$$K_\lambda(\lambda) = \frac{1}{\pi\lambda}\sin^2 1 \to \infty \text{ as } \lambda \to 0.$$

So take $\varepsilon = 1$. Then no matter how small $\lambda_0 > 0$ is we can find $\lambda < \lambda_0$ and $x = \lambda$ so that $K_\lambda(x) > 1$. So the convergence to 0 is certainly not uniform.

4. The condition that x is not an integer multiple of 2π is equivalent to the condition that $e^{ix} \neq 1$. The by the formula for the sum of a finite geometric series,

$$\sum_{n=-N}^{N} e^{inx} = e^{-iNx}\sum_{n=0}^{2N}(e^{inx})^n = e^{-iNx}\frac{1 - e^{2iNx}}{1 - e^{ix}} = \frac{e^{i(N+(1/2))x} - e^{-i(N+(1/2))x}}{e^{ix/2} - e^{-ix/2}}$$

$$= \frac{\sin(N + \frac{1}{2})x}{\sin\frac{1}{2}x}.$$

To see that this is a 2π-periodic function, of course each e^{inx} is a 2π-periodic function, so the sum from $-N$ to $-N$ must be also. To see it directly from the quotient of sines, note that

$$\sin\tfrac{1}{2}(x + 2\pi) = \sin(\tfrac{1}{2}x + \pi) = -\sin x$$

and

$$\sin(N+\tfrac{1}{2})(x+2\pi) = \sin((N+\tfrac{1}{2})x+2N\pi+\pi) = \sin((N+\tfrac{1}{2})x+\pi) = -\sin(N+\tfrac{1}{2})x.$$

So

$$\frac{\sin\frac{1}{2}(x + 2\pi)}{\sin(N + \frac{1}{2})(x + 2\pi)} = \frac{-\sin\frac{1}{2}x}{-\sin(N + \frac{1}{2})x} = \frac{\sin\frac{1}{2}x}{\sin(N + \frac{1}{2})x}.$$

Then

$$\sum_{n=-N}^{N} \hat{f}(n)e^{inx} = \sum_{n=-N}^{N} e^{inx}\int_{-\pi}^{\pi} f(y)e^{-iny}dy$$

$$= \int_{-\pi}^{\pi} f(y)\sum_{n=-N}^{N} e^{in(x-y)}dy = \int_{-\pi}^{\pi} f(y)\frac{\sin N + \frac{1}{2})(x - y)}{\sin\frac{1}{2}(x - y)}dy$$

$$= \int_{x-\pi}^{x+\pi} f(x - t)\frac{\sin(N + \frac{1}{2})t}{\sin\frac{1}{2}t}dt.$$

This last line uses the change of variable $y = x - t$, so that $t = x - y$, $dt = -dy$, when $y = \pi$, $t = x - \pi$ and when $y = -\pi$, $t = x + \pi$. So the change of sign from dy to $-dt$ cancels if we make $x - \pi$ the bottom limit of the new integral.

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5a) If $x < y < x + \pi$ then $-\pi < x - y < 0$, so $\pi < (x-y) + 2\pi < 2\pi$ and since f is 2π-periodic, $f(x-y) = f(x-y+2\pi) = e^{-2(x-y+2\pi)} = e^{-4\pi-2x+2y}$. If $x - \pi < y < x$ then $0 < x - y < \pi$ and $f(x-y) = e^{-2(x-y)} = e^{-2x+2y}$. So

$$S_N(f)(x) = \frac{e^{-2x}}{2\pi} \int_{x-\pi}^{x} e^{2y} \frac{\sin(N+\frac{1}{2})y}{\sin\frac{1}{2}y} dy + \frac{e^{-4\pi-2x}}{2\pi} \int_{x}^{x+\pi} e^{2y} \frac{\sin(N+\frac{1}{2})y}{\sin\frac{1}{2}y} dy.$$

The Fourier Series Theorem says that for all x

$$\lim_{N\to\infty} S_N(f)(x) = \frac{f(x+) + f(x-)}{2}.$$

b) Put $t = (N+\frac{1}{2})y$. Then $dt = (N+\frac{1}{2})dy$, so $dt/t = dy/y$. When $y = x_N$, $t = \pi$. When $y = x_N \pm \pi$ then $t = \pi \pm (N+\frac{1}{2})\pi$ and $\lim_{N\to+\infty} \pi \pm (N\pm\frac{1}{2})\pi = \pm\infty$. Also $\lim_{N\to+\infty} e^{-2x_N} = 1$. So

$$\lim_{N\to+\infty} TN(f)(x_N) = \frac{1}{\pi} \lim_{R\to\infty} \int_{-\infty}^{\pi} \frac{\sin t}{t} dt + \frac{e^{-4\pi}}{\pi} \lim_{R\to\infty} \int_{\pi}^{R} \frac{\sin t}{t} dt$$

$$= \frac{1}{2} + \frac{1}{\pi} \int_{0}^{\pi} \frac{\sin t}{t} dt + \frac{e^{-4\pi}}{2} - \frac{e^{-4\pi}}{\pi} \int_{0}^{\pi} \frac{\sin t}{t} dt.$$

4c) We have

$$\frac{1 - e^{-4\pi}}{\pi} \int_{0}^{\pi} \frac{\sin t}{t} dt > \frac{1 - e^{-4\pi}}{2}.$$

So

$$\lim_{N\to+\infty} TN(f)(x_N) > \frac{1 + e^{-4\pi}}{2} + \frac{1 - e^{-4\pi}}{2} = 1.$$

6a)

$$\lim_{y\to 0} \left(\frac{e^{2y}}{\sin\frac{1}{2}y} - \frac{2}{y} \right) = \lim_{y\to 0} \frac{y e^{2y} - 2\sin\frac{1}{2}y}{y \sin\frac{1}{2}y}$$

$$= \lim_{y\to 0} \frac{y(1 + 2y + (2y)^2/2 \cdots) - 2(\frac{1}{2}y - (\frac{1}{2}y)^3/6 \cdots)}{y(\frac{1}{2}y - (\frac{1}{2}y)^3/6 \cdots)} = \lim_{y\to 0} \frac{2y^2 + \cdots}{\frac{1}{2}y^2 - \cdots} = 4.$$

In the interval $(-2\pi, 2\pi)$, the denominator of the function only vanishes at $y = 0$, and both numerator and denominator are continuous and we have seen that there is a limit as $y \to 0$. So the function extends continuously to $(-2\pi, 2\pi)$, taking the value 4 at 0.

6b) Since

$$\frac{e^{2y}}{\sin\frac{1}{2}y} - \frac{2}{y}$$

extends to a continuous function on $(-2\pi, 2\pi)$ and $[x-\pi, x] \subset (-2\pi, 2\pi)$, $[x, x+\pi] \subset (-2\pi, 2\pi)$, by the Riemann Lebesgue Lemma,

$$\lim_{N\to+\infty} \int_{x-\pi}^{x} \left(\frac{e^{2y}}{\sin\frac{1}{2}y} - \frac{2}{y} \right) \sin(N+\frac{1}{2})y \, dy = 0,$$

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$$\lim_{N\to+\infty} \int_x^{x+\pi} \left(\frac{e^{2y}}{\sin\frac{1}{2}y} - \frac{2}{y} \right) \sin\left(N + \tfrac{1}{2}\right)y\, dy = 0.$$

This gives

$$\lim_{N\to+\infty} (I(N,x) - J(N,x)) = 0.$$

6c) We have

$$\lim_{N\to\infty} S_N(f)(x_N) = \lim_{N\to\infty} T_N(f)(x_N) = 1 + a.$$

for some $a > 0$. But for $x \in (0, 2\ \mathbb{C})$, $f(x) = \frac{f(x+)+f(x-)}{2} = e^{-2x}$. So if the convergence of $S_N(f)(x)$ to $\frac{f(x+)+f(x-)}{2}$ is uniform, there is N such that for all $n \geq N$, and all $x \in (0, 2\pi)$,

$$|S_n(f)(x_n) - e^{-2x_n}| \frac{a}{2}.$$

Taing limits this gives

$$|1 + a - 1| = a < \frac{a}{2},$$

which is nonsense.

7a) Equation (1) gives

$$(\partial/\partial t)\hat{u}(n,t) = -n^2 \hat{u}(n,t), \tag{3}$$

and (2) gives

$$\hat{u}(n,0) = \hat{f}(n). \tag{4}$$

The general solution is

$$\hat{u}(n,t) = \hat{f}(n)e^{-n^2 t}.$$

Since u_x extends continuously to $[0, 2\pi] \times [0, \infty)$, $u(x,t)$ is piecewise smooth in x. Since u is continuous on $[0, 2\pi] \times [0, \infty)$, and $u(0,t) = u(2\pi, t)$ for all t, u extends to a continuous 2π-periodic function in $x \in \mathbf{R}$. So by the Fourier Series Theorem,

$$u(x,t) = \lim_{N\to\infty} \frac{1}{2\pi} \sum_{n=-N}^{N} \hat{u}(n,t)e^{inx}$$

$$= \lim_{N\to\infty} \frac{1}{2\pi} \sum_{n=-N}^{N} \hat{f}(n)e^{-n^2 t + inx} = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{-n^2 t + inx}.$$

[It is not possible to write this as a convolution, at least, not for any standard function.]

b) By integration by parts

$$\hat{u}_x(n,t) = \int_0^{2\pi} u_x(x,t)e^{-inx}\, dx = \left[u(x,t)e^{-inx}\right]_0^{2\pi} + \int_0^{2\pi} inu(x,t)e^{-inx}\, dx$$

$$= u(2\pi, t)e^{-in2\pi} - u(0,t)e^{-i0} + in\hat{u}(n,t) = in\hat{u}(n,t).$$

Similarly

$$\hat{u}_{xx}(n,t) = \int_0^{2\pi} u_{xx}(x,t)e^{-inx}\, dx = \left[u_x(x,t)e^{-inx}\right]_0^{2\pi} + \int_0^{2\pi} inu_x(x,t)e^{-inx}\, dx$$

$$= 0 + in\hat{u}_x(n,t) = -n^2\hat{u}(n,t).$$

8a) The equations implied by (1), (2) are
$$(\partial^2/\partial t^2)\hat{u}(n,t) = (-n^2 - i\lambda n)\hat{u}(n,t) \tag{3}$$
$$\hat{u}(n,0) = \hat{f}(n), \ (\partial/\partial t)hat u(n,t) = 0. \tag{4}$$

The general solution to (3) is
$$\hat{u}(n,t) = A_n e^{int\sqrt{1-i\lambda/n}} + B_n e^{-int\sqrt{1-i\lambda/n}}$$

if $n \neq 0$ and
$$\hat{u}(0,t) = A_0 + B_0 t$$

The boundary conditions (4) give $B_0 = 0$, $A_0 = \hat{f}(0)$, and if $n \neq 0$,
$$A_n = B_n = \tfrac{1}{2}\hat{f}(n).$$

Then by the Fourier Series Theorem (as in question 7)
$$u(x,t) = \lim_{N\to\infty} \frac{1}{2\pi} \sum_{n=-N}^{N} (\tfrac{1}{2}\hat{f}(n)e^{inx+int\sqrt{1-i\lambda/n}} + \tfrac{1}{2}\hat{f}(n)e^{inx-int\sqrt{1-i\lambda/n}}).$$

9.
$$\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-3|x|}e^{-i\xi x}dx = \int_0^\infty e^{-(3+i\xi)x}dx + \int_{-\infty}^0 e^{(3-i\xi)x}dx$$

$$= \lim_{\Delta\to+\infty} \left[\frac{e^{-(3+i\xi)x}}{3+i\xi}\right]_0^\Delta + \lim_{\Delta\to+\infty} \left[\frac{e^{(3-i\xi)x}}{3-i\xi}\right]_{-\Delta}^0$$

$$\lim_{\Delta\to+\infty} \left(\frac{-e^{-(3+i\xi)\Delta}+1}{3+i\xi}\right) + \lim_{\Delta\to+\infty} \left(\frac{1-e^{-\Delta(3-i\xi)}}{3-i\xi}\right)$$

$$= \frac{(3-i\xi)+(3+i\xi)}{9+\xi^2} = \frac{6}{9+\xi^2}.$$

10. Since f is real valued and hence $\overline{f(x)} = f(x)$, we have, for all real ξ.
$$\overline{\hat{f}(\xi)} = \overline{\int_{-\infty}^{\infty} f(x)e^{-i\xi x}dx} = \int_{-\infty}^{\infty} f(x)e^{i\xi x}dx = \hat{f}(-\xi).$$

Now let $\xi \geq 0$, and let $\gamma(R)$ be the anticlockwise semicircular contour in the lower half-plane. Let $\gamma'(R)$ be the curved part of the contour. We have
$$4z^2 + 1 = 0 \Leftrightarrow z = \frac{\pm i}{2}.$$

The point $\frac{-i}{2}$ isinside $\gamma(R)$, while $-1+i$ is outside. So by the Residues Theorem we have
$$\int_{\gamma(R)} \frac{e^{-i\xi z}dz}{4z^2+1} = 2\pi i \operatorname{Res}\left(\frac{e^{-i\xi z}}{4(z-\frac{i}{2})(z+\frac{i}{2})}, \frac{-i}{2}\right)$$

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$$= 2\pi i \frac{e^{-\frac{\xi}{2}}}{-4i} = \frac{-\pi e^{-\frac{\xi}{2}}}{2}.$$

Now

$$\int_{\gamma(R)} \frac{dz}{4z^2 + 1} = -\int_{-R}^{R} \frac{dx}{4x^2 + 1} + \int_{\gamma'(R)} \frac{dz}{4z^2 + 1}.$$

Now if $|z| > \frac{1}{2}$,

$$|4z^2 + 1| \geq 4|z|^2 1 > 0.$$

Also, if $\operatorname{Im}(z) \leq 0$, $\xi \geq 0$ $|e^{-i\xi z}| = e^{\operatorname{Im}(\xi z)} \leq 1$. So if $R > \frac{1}{2}$

$$\left| \int_{\gamma'(R)} \frac{e^{-i\xi z} dz}{4z^2 + 1} \right| \leq \operatorname{length}(\gamma'(R)) \frac{1}{4R^2 - 1}$$

$$= \frac{\pi R}{4R^2 - 1} \to 0 \text{ as } R \to \infty.$$

So, if $\xi \geq 0$

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} \frac{e^{-i\xi x} dx}{(4x^2 + 1)} = \lim_{R \to \infty} \int_{-R}^{R} \frac{e^{-i\xi x} dx}{4x^2 + 1}$$

$$= -\int_{\gamma(R)} \frac{e^{-i\xi z}}{4z^2 + 1} = \pi e^{\frac{-\xi}{2}}.$$

Then using the expression for $\hat{f}(-\xi)$ in terms of $\hat{f}(\xi)$ we have, for all $\xi \in \mathbf{R}$,

$$\hat{f}(\xi) = \pi e^{-\frac{|\xi|}{2}}.$$

11. As in question 10, for all $\xi \in \mathbf{R}$,

$$\overline{\hat{f}(\xi)} = \overline{\int_{-\infty}^{\infty} f(x) e^{-i\xi x} dx} = \int_{-\infty}^{\infty} f(x) e^{i\xi x} dx = \hat{f}(-\xi).$$

Now let $\xi \geq 0$, and let $\gamma(R)$ be the anticlockwise semicircular contour in the lower half-plane. Let $\gamma'(R)$ be the curved part of the contour. We have

$$z^2 + 2z + 2 = 0 \Leftrightarrow z = -1 \pm i.$$

The point $-1 - i$ is inside $\gamma(R)$, while $-1 + i$ is outside. So by the Residues Theorem we have

$$\int_{\gamma(R)} \frac{e^{-i\xi z} dz}{(z^2 + 2z + 2)^2} = 2\pi i \operatorname{Res}\left(\frac{e^{-i\xi z}}{(z^2 + 2z + 2)^2}, -1 - i \right)$$

$$= 2\pi i \operatorname{Res}\left(\frac{e^{-i\xi z}}{(z+1+i)^2(z+1-i)^2}, -1 - i \right) = 2\pi i \frac{d}{dz}\left(\frac{e^{-i\xi z}}{(z+1-i)^2} \right)_{z=-1-i}$$

$$= 2\pi i \left(\frac{-i\xi e^{-i\xi z}}{(z+1-i)^2} - \frac{2e^{-i\xi z}}{(z+1-i)^3} \right)_{z=-1-i}$$

$$= 2\pi i e^{-\xi+i\xi}\left(\frac{-i\xi}{-4} - \frac{2}{8i} \right)$$

$$= \frac{\pi}{2} e^{-\xi + i\xi}(-\xi - 1).$$

Now

$$\int_{\gamma(R)} \frac{dz}{(z^2 + 2z + 2)^2} = -\int_{-R}^{R} \frac{dx}{(x^2 + 2x + 2)^2} + \int_{\gamma'(R)} \frac{dz}{(z^2 + 2z + 2)^2}.$$

Now if $|z| > 2$,

$$|z^2 + 2z + 1| \geq |z|^2 - 2|z| - 2 > 0.$$

Also, if $\operatorname{Im}(z) \leq 0$, $\xi \geq 0$ $|e^{-i\xi z}| = e^{\operatorname{Im}(\xi z)} \leq 1$. So if $R > 2$

$$\left| \int_{\gamma'(R)} \frac{e^{-i\xi z} dz}{(z^2 + 2z + 2)^2} \right| \leq \operatorname{length}(\gamma'(R)) \frac{1}{(R^2 - 2R - 2)^2}$$

$$= \frac{\pi R}{(R^2 - 2R - 2)^2} \to 0 \text{ as } R \to \infty.$$

So, if $\xi \geq 0$

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} \frac{e^{-i\xi x} dx}{(x^2 + 2x + 2)^2} = \lim_{R \to \infty} \int_{-R}^{R} \frac{e^{-i\xi x} dx}{(x^2 + 2x + 2)^2}$$

$$= -\int_{\gamma(R)} \frac{e^{-i\xi z}}{(z^2 + 2z + 2)^2} = \frac{\pi(\xi + 1)e^{-i\xi - \xi}}{2}.$$

Then using the expression for $\hat{f}(-\xi)$ in terms of $\hat{f}(\xi)$ we have, for all ξ

$$\hat{f}(\xi) = \frac{\pi(\xi + 1)e^{-i\xi - |\xi|}}{2}.$$

12. We have

$$e^{-2x^2 - ix\xi} = e^{-2(x + i\xi/4)^2 - \xi^2/8}.$$

Since e^{-2z^2} is holomorphic in the whole plane, we have

$$\int_{\gamma(R)} e^{-2z^2} dz = 0.$$

Let $\xi \geq 0$. Now let $\gamma_1(R)$ be the horizontal path from $-R$ to R, $\gamma_2(R)$ the vertical path from R to $R + i\xi/4$, $\gamma_3(R)$ the horizontal path from $R + i\xi/4$ to $-R + i\xi/4$ and $\gamma_4(R)$ the vertical path from $-R + i\xi/4$ to $-R$. Then

$$\int_{\gamma_3(R)} e^{-2z^2} dz = -\int_{-R}^{R} e^{-2(x + i\xi/4)^2} dx = -e^{\xi^2/8} \hat{f}(\xi).$$

On $\gamma_2(R)$ we have $z = R + iy$ for $0 \leq y \leq \xi/4$. So $z^2 = R^2 + 2iRy - y^2$. Then $|e^{-2z^2}| = e^{-2R^2 + 2y^2}$. So

$$\left| \int_{\gamma_2(R)} e^{-2z^2} dz \right| \leq \operatorname{length}(\gamma_2(R)) e^{-2R^2 + \xi^2/8} = \tfrac{1}{4} \xi e^{-2R^2 + \xi^2/8} \to 0 \text{ as } R \to \infty.$$

Similarly if $z = -R + iy$ then $z^2 = R^2 - 2iRy - y^2$ and

$$\lim_{R \to \infty} \int_{\gamma_4(R)} e^{-2z^2} dz = 0.$$

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So if $\xi \geq 0$

$$\hat{f}(\xi) = -\lim_{R\to\infty} e^{-\xi^2/8} \int_{\gamma_3(R)} e^{-2z^2} dz = \lim_{R\to\infty} e^{-\xi^2/8} \int_{-R}^{R} e^{-2x^2} dx$$

$$= e^{-\xi^2/8} \int_{-\infty}^{\infty} e^{-2x^2} dx = e^{-\xi^2/8} \sqrt{\frac{\pi}{2}}.$$

Since, as in question 8, we have $\hat{f}(-\xi) = \overline{\hat{f}(\xi)}$ we have, for all ξ,

$$\hat{f}(\xi) = e^{-\xi^2/8} \sqrt{\frac{\pi}{2}}.$$

13. Let $\gamma_+(R)$, $\gamma_-(R)$ be the integrals drawn, and let $\gamma'_+(R)$, $\gamma'_-(R)$ denote the curved parts of the contours. Write $z = x + iy$, and let ξ_1 be real. Then

$$\left| e^{-i\xi_1(x+iy)} \right| = e^{\xi_1 y}.$$

So

$$\left| e^{-i\xi_1 z} \right| \leq 1 \text{ on } \gamma'_-(R) \text{ if } \xi_1 \geq 0,$$

$$\left| e^{-i\xi_1 z} \right| \leq 1 \text{ on } \gamma'_+(R) \text{ if } \xi_1 \leq 0.$$

In fact we have stronger statements. If $z \in \gamma'_-(R)$ with $\mathrm{Im}(z) \leq -\sqrt{R}$ then

$$\left| e^{-i\xi_1 z} \right| \leq e^{-\xi_1 \sqrt{R}}.$$

If $z \in \gamma'_+(R)$ with $\mathrm{Im}(z) \geq \sqrt{R}$ then

$$\left| e^{-i\xi_1 z} \right| \leq e^{\xi_1 \sqrt{R}}.$$

We also have $|2 + iz| \geq R - 2$ for $|z| = R$, provided that $R \geq 2$. So breaking up the integral along $\gamma'_\pm(R)$ into the pieces where $|\mathrm{Im}(z) \leq \sqrt{R}$ and $|\mathrm{Im}(z) \geq \sqrt{R}$, we have, if $\xi_1 > 0$, $R > 2$,

$$\left| \int_{\gamma'_-(R)} \frac{e^{-i\xi_1 z}}{2 + iz} dz \right| \leq 4 \frac{\sqrt{R}}{R - 2} + \pi R e^{-\xi_1 \sqrt{R}} \to 0 \text{ as } R \to \infty,$$

and if $\xi_1 < 0$, $R > 3$,

$$\left| \int_{\gamma'_+(R)} \frac{e^{-i\xi_1 z}}{2 + iz} dz \right| \leq 4 \frac{\sqrt{R}}{R - 2} + \pi R e^{\xi_1 \sqrt{R}} \to 0 \text{ as } R \to \infty.$$

So if $\xi_1 > 0$,

$$\lim_{R\to\infty} \int_{\gamma_-(R)} \frac{e^{-i\xi_1 z}}{2 + iz} dz = -\lim_{R\to\infty} \int_{-R}^{R} \frac{e^{-i\xi_1 y}}{2 + iy} dy = -\int_{-\infty}^{\infty} \frac{e^{-i\xi_1 y}}{2 + iy} dy,$$

and if $\xi_1 < 0$,

$$\lim_{R\to\infty} \int_{\gamma_+(R)} \frac{e^{-i\xi_1 z}}{2 + iz} dz = \lim_{R\to\infty} \int_{-R}^{R} \frac{e^{-i\xi_1 y}}{2 + iy} dy = \int_{-\infty}^{\infty} \frac{e^{-i\xi_1 y}}{2 + iy} dy.$$

Now $2 + iz = 0 \Leftrightarrow z = 2i$. So by Cauchy's Theorem, for any ξ_1,

$$\int_{\gamma_-(R)} \frac{e^{-i\xi_1 z}}{2 + iz} dz = 0,$$

and by the Residues Theorem,

$$\int_{\gamma_+(R)} \frac{e^{-i\xi_1 z}}{2+iz}dz = 2\pi i \operatorname{Res}\left(\frac{e^{-i\xi_1 z}}{i(z-2i)}, 2i\right) = \frac{2\pi i e^{-i\xi_1(2i)}}{i} = 2\pi e^{2\xi_1}.$$

So

$$\int_{-\infty}^{\infty} \frac{e^{-i\xi_1 y}}{2+iy}dy = \begin{matrix} 0 & \text{if } \xi_1 > 0, \\ 2\pi e^{2\xi_1} & \text{if } \xi_1 < 0. \end{matrix}$$

So

$$\int_{-\infty}^{\infty} \frac{e^{-i\xi y} - e^{-4\pi}e^{-i(\xi+2\pi)y}}{2+iy}dy = \begin{matrix} 0 & \text{if } \xi > 0, \\ 2\pi e^{2\xi} & \text{if } -2\pi < \xi < 0, \\ 2\pi e^{2\xi} - 2\pi e^{-4\pi}e^{2(\xi+2\pi)} = 0 & \text{if } \xi < -2\pi. \end{matrix}$$

Finally we consider

$$\lim_{R\to\infty}\int_{-R}^{R}\frac{dy}{2+iy} = \lim_{R\to\infty}\int_{-R}^{R}\frac{(2-iy)dy}{4+y^2} = \lim_{R\to\infty}\int_{-R}^{R}\frac{2dy}{4+y^2},$$

because $y/(4+y^2)$ is an odd function. But

$$\lim_{R\to\infty}\int_{-R}^{R}\frac{2dy}{4+y^2} = \lim_{R\to\infty}\left[\tan^{-1}(y/2)\right]_{-R}^{R} = \pi/2 - (-\pi/2) = \pi.$$

So

$$\hat{h}(0) = \pi - 0 = \pi,$$

$$\hat{h}(-2\pi) = 2\pi e^{-4\pi} - e^{-4\pi}\pi = \pi e^{-4\pi}.$$

So altogether we have

$$\hat{h}(\xi) = \begin{matrix} 2\pi e^{2\xi} & \text{if } -2\pi < \xi < 0 \\ 0 & \text{if } \xi < -2\pi, \text{ or } x > 0, \\ \pi & \text{if } \xi = 0, \\ \pi e^{-4\pi} & \text{if } \xi = -2\pi, \end{matrix}$$

as required.

14a)

$$\int_{-\infty}^{\infty}\hat{f}(\xi)\overline{\hat{g}(\xi)}d\xi = 2\pi\int_{-\infty}^{\infty}f(x)\overline{g(x)}dx.$$

13b) One of the properties of the Fourier transform is that, if $f \in L^1$,

$$|\hat{f}(\xi)| = \left|\int_{-\infty}^{\infty}f(x)e^{-i\xi x}dx\right| \le \int_{-\infty}^{\infty}|f(x)|dx.$$

This immediately implies that if $f \in L^1$ then $\hat{f}$ is bounded. The Fourier Inversion Theorem says that if f, $\hat{f} \in L^1$ then $2\pi f = (\hat{f})\check{\ }$, where $\check{f}(\xi) = \hat{f}(-\xi)$. So if $\hat{f} \in L^1$, we have $f \in L^\infty$

13c) If $h_1 \in L^1$ and h_2 is bounded then there exists M such that $|h_2(x)| \le M$ for all x. So

$$\int_{-\infty}^{\infty} |h_1(x)h_2(x)|dx \le M \int_{-\infty}^{\infty} |h_1(x)|dx < +\infty$$

and so $h_1 h_2 \in L^1$.

d) Tonelli's Theorem: Let a measurable function f be such that at least one of the integtrals

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |F(x,y)|dxdy, \, \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |F(x,y)|dydx$$

is finite. Then both integrals

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(x,y)dxdy, \, \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(x,y)dydx$$

are defined, and they are equal. [This automatically means that

$$\int_{-\infty}^{\infty} F(x,y)dx, \, \int_{-\infty}^{\infty} F(x,y)dy$$

are also defined.] Now apply this to the function

$$F(x,\xi) = \hat{f}(\xi)e^{ix\xi}\overline{g(x)}.$$

We have

$$|F(x,\xi)| = |\hat{f}(\xi)e^{ix\xi}\overline{g(x)}| = |f(\xi)||g(x)|.$$

But

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(\xi)||g(x)|dxd\xi = \int_{-\infty}^{\infty} |f(\xi)|d\xi \int_{-\infty}^{\infty} |g(x)|dx < +\infty.$$

So Tonelli's Theorem applies and we can change the order of integration, giving

$$\int_{-\infty}^{\infty} \hat{f}(\xi) \int_{-\infty}^{\infty} e^{ix\xi}\overline{g(x)}dxd\xi = \int_{-\infty}^{\infty} \overline{g(x)} \int_{-\infty}^{\infty} \hat{f}(\xi)e^{ix\xi}d\xi dx.$$

15. We are assuming that f is integrable, which means that

$$|\hat{f}(\xi)| \le \int_{-\infty}^{\infty} |f(x)|dx < +\infty.$$

So

$$|\widehat{h_\lambda}(\xi)| \le e^{-\lambda|\xi|} \int_{-\infty}^{\infty} |f(x)|,$$

which is integrable. So we can use the following formula for $h_\lambda(x)$:

$$h_\lambda(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ix\xi}\widehat{h_\lambda}(\xi)d\xi$$

$$= \frac{1}{2\pi} \int_0^{\infty} e^{i\xi x - \xi\lambda + i\lambda}\widehat{f}(\xi)d\xi + \frac{1}{2\pi} \int_{-\infty}^0 e^{i\xi x + \xi\lambda - i\lambda}\widehat{f}(\xi)d\xi$$

$$= \frac{1}{2\pi} \int_0^{\infty} \int_{-\infty}^{\infty} e^{i\xi x - \xi\lambda + i\lambda}e^{i\xi t}f(t)dtd\xi + \frac{1}{2\pi} \int_{-\infty}^0 \int_{-\infty}^{\infty} e^{i\xi x + \xi\lambda - i\lambda}e^{i\xi t}f(t)dtd\xi.$$

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Now in both double integrals, the modulus of the integrand is $e^{-\lambda|\xi|}|f(t)|$, and

$$\int_0^\infty \int_{-\infty}^\infty e^{-\lambda|\xi|}|f(t)|\,dt\,d\xi < +\infty,$$

$$\int_{-\infty}^0 \int_{-\infty}^\infty e^{-\lambda|\xi|}|f(t)|\,dt\,d\xi < +\infty.$$

So by Tonelli's Theorem we can change the order of integration. So

$$h_\lambda(x) = \frac{1}{2\pi}\int_{-\infty}^\infty f(t)\left(e^{i\lambda}\int_0^\infty e^{i\xi(x-t)-\xi\lambda}\,d\xi\,dt + e^{-i\lambda}\int_{-\infty}^0 e^{i\xi(x-t)+\xi\lambda}\right)d\xi\,dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^\infty f(t)\left(e^{i\lambda}\left[\frac{e^{i\xi(x-t)-\xi\lambda}}{i(x-t)-\lambda}\right]_0^\infty + e^{-i\lambda}\left[\frac{e^{i\xi(x-t)+\xi\lambda}}{i(x-t)+\lambda}\right]_{-\infty}^0\right)dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^\infty f(t)\left(\frac{e^{i\lambda}}{\lambda-i(x-t)} + \frac{e^{-i\lambda}}{\lambda+i(x-t)}\right)dt$$

$$= \int_{-\infty}^\infty \frac{f(t)}{2\pi}\left(\frac{e^{i\lambda}(\lambda+i(x-t))+e^{-i\lambda}(\lambda-i(x-t))}{\lambda^2+(x-t)^2}\right)dt$$

$$= \int_{-\infty}^\infty \frac{f(t)}{\pi}\frac{\lambda\cos\lambda-(x-t)\sin\lambda}{\lambda^2+(x-t)^2}\,dt$$

$$= f + g_\lambda(x) = g_\lambda * f(x),$$

where

$$g_\lambda(t) = \frac{\lambda\cos\lambda-t\sin\lambda}{\pi(\lambda^2+t^2)}.$$

16a) For all x,

$$|\sin x| \le |x|.$$

Also $K(x)$ is an even function. So

$$\int_{-\infty}^\infty K(x)\,dx \le \frac{4}{\pi}\int_0^1 dx + \frac{4}{\pi}\int_1^\infty \frac{dx}{x^2} = 2 + 2\left[\frac{-1}{x}\right]_1^\infty = \frac{8}{\pi} < +\infty.$$

So K is integrable.

16b)(i) Making the change of variable $u = x/\lambda$, $dx/\lambda = du$,

$$\int_{-\infty}^\infty K_\lambda(x)\,dx = \int_{-\infty}^\infty \lambda^{-1}K(x/\lambda)\,dx = \int_{-\infty}^\infty K(u)\,du = 1.$$

16b)(ii) With the same change of variable as in 16b)(i), for any $\delta > 0$, $|x| \ge \delta$ if and only if $|u| \ge \delta/\lambda$. So

$$\int_{|x|\ge\delta} K_\lambda(x)\,dx = \int_{|u|\ge\delta/\lambda} K(u)\,du \to 0 \text{ as } \lambda \to 0.$$

16b)(iii) We have $0 \le K(x) \le x^{-2}$ for all $x \ne 0$. So if $|x| \ge \delta > 0$,

$$|K_\lambda(x)| = \left|\frac{K(x/\lambda)}{\lambda}\right| \le \frac{\lambda}{|x|^2}$$

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$$\le \frac{\lambda}{\delta^2} \to 0 \text{ as } \lambda \to 0.$$

So
$$\lim_{\lambda \to 0} K_\lambda(x) = 0$$

uniformly for $|x| \ge \delta$, for any $\delta > 0$.

16c) Let $\gamma'(R)$ and $\gamma'(\varepsilon)$ be as shown.

Then $|e^{iz}| \le 1$ for $z \in \gamma'(R)$. So
$$\left| \int_{\gamma'(R)} \frac{1 - e^{iz}}{z^2} dz \right| \le \text{length}(\gamma'(R) \times \frac{2}{R^2} = \frac{2\pi}{R} \to 0 \text{ as } R \to \infty.$$

The function $(1 - e^{iz})/z^2$ is holomrphic inside and on $\gamma(R, \varepsilon)$. So by Cauchy's Theorem,
$$\int_{\gamma(R,\varepsilon)} \frac{1 - e^{iz}}{z^2} dz = 0.$$

So
$$\lim_{R \to \infty, \varepsilon \to 0} \text{Re} \left(\left(\int_{-R}^{-\varepsilon} + \int_{\varepsilon}^{R} \right) \frac{1 - e^{ix}}{x^2} dx \right) = \lim_{\varepsilon \to 0} \text{Re} \left(\int_{\gamma'(\varepsilon)} \frac{1 - e^{iz}}{z^2} dz \right)$$

$$= \lim_{\varepsilon \to 0} \text{Re} \left(\int_0^\pi \frac{(1 - e^{i\varepsilon e^{i\theta}}) i\varepsilon e^{i\theta}}{\varepsilon^2 e^{2i\theta}} \right) = \lim_{\varepsilon \to 0} \text{Re} \left(\int_0^\pi \frac{1 - 1 - i\varepsilon e^{i\theta} + 2\varepsilon^2 e^{2i\theta} + \cdots)i}{\varepsilon e^{i\theta}} d\theta \right)$$

$$= \pi.$$

Now by making a change of variable $t = -x$ we see also that
$$\lim_{R \to \infty, \varepsilon \to 0} \text{Re} \left(\left(\int_{-R}^{-\varepsilon} + \int_{\varepsilon}^{R} \right) \frac{1 - e^{-it}}{t^2} dt \right) = \pi.$$

So
$$2\pi \int_{-\infty}^{\infty} K(x) dx = \lim_{R \to \infty, \varepsilon \to 0} \left(\int_{-R}^{-\varepsilon} + \int_{\varepsilon}^{R} \right) \frac{2 - e^{ix} - e^{-ix}}{x^2} dx = 2\pi.$$

16d) We have $K_\lambda * f = f * K_\lambda$. Now

$$f(x) - K_\lambda * f(x) = f(x) \int_{-\infty}^{\infty} K_\lambda(y) dy - f * K_\lambda(x) = \int_{-\infty}^{\infty} (f(x) K_\lambda(y) - f(x - y) K_\lambda((y))) dy = \int_{-\infty}^{\infty} K_\lambda(y)(f($$

17a) Using the change of variable $u = x + a$
$$\hat{g}(\xi) = \int_{-\infty}^{\infty} f(x + a) e^{-ix\xi} dx = \int_{-\infty}^{\infty} e^{-i(u-a)\xi} f(u) du$$

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$$= e^{i\xi a} \int_{-\infty}^{\infty} e^{-iu\xi} f(u)\,du = e^{i\xi a}\hat{f}(\xi).$$

17b). The Fourier transform of the righthand side of equation (1) is $(-\xi^2 + 2i\xi)\hat{u}(\xi, t)$. So the Fourier transform of (1) (with respect to x) is

$$\frac{\partial \hat{u}}{\partial t}(\xi, t) = (-\xi^2 + 2i\xi)\hat{u}(\xi, t).$$

The solution of the differential equation

$$\frac{dy}{dt} = (-\xi^2 + i\xi)y,$$

where ξ is treated as a constant, is $y(t) = y(0)e^{-\xi^2 + 2i\xi)t} = y(0)e^{2i\xi t}e^{-\xi^2 t}$. So for $y(t) = \hat{u}(\xi, t)$ we have

$$\hat{u}(\xi, t) = \hat{u}(\xi, 0)e^{i\xi t}e^{-\xi^2 t + 2i\xi t}$$

From 13a), we know that $e^{i\xi t}\hat{f}(\xi)$ is the Fourier transform of $f(x+t)$. So $e^{i\xi t}e^{-\xi^2 t}$ is the Fourier transform of $(1/2\sqrt{\pi t})e^{-(x+t)^2/4t}$, and so

$$u(x, t) = \int_{-\infty}^{\infty} u(y, 0)\frac{e^{-(x-y+2t)^2/4t}\,dy}{2\sqrt{\pi t}}.$$

Finally for any integer $n \geq 1$ we have

$$\frac{\left| u(y, 0)\frac{e^{-(x-y+2n)^2/4n}}{2\sqrt{\pi n}} \right| \leq |u(y, 0)|}{2\sqrt{\pi}.}$$

For fixed x, writing

$$f_n(y) = u(y, 0)\frac{e^{-(x-y+2n)^2/4n}}{2\sqrt{\pi n}}$$

we have $\lim_{n\to\infty} f_n(y) = 0$. The function $|u(y, 0)|$ is integrable in y. So by the Dominated Convergence Theorem we have

$$\lim_{n\to\infty} u(x, n) = \lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(y)\,dy$$

$$= \int_{-\infty}^{\infty} \lim_{n\to\infty} f_n(y)\,dy = \int_{-\infty}^{\infty} 0\,dy = 0.$$

18. We have

$$\hat{f}(x) - \hat{f}(y) = \int_{-\infty}^{\infty} (e^{-ixt} - e^{-iyt})f(t)\,dt.$$

So

$$|\hat{f}(x) - \hat{f}(y)| \leq \int_{-\infty}^{\infty} |e^{-ixt} - e^{-iyt}||f(t)|\,dt$$

$$= \left(\int_{|t|\geq\Delta} + \int_{-\Delta}^{\Delta} \right) |e^{-ixt} - e^{-iyt}||f(t)|\,dt$$

$$\leq 2\int_{|t|\geq\Delta} |f(t)|\,dt + \int_{-\Delta}^{\Delta} |e^{-ixt} - e^{-iyt}||f(t)|\,dt$$

for any $\Delta > 0$. We have used, for real x, y, t,
$$|e^{-ixt} - e^{-iyt}| \leq |e^{-ixt}| + |e^{-iyt}| = 2.$$

Now let $\varepsilon_1 > 0$ be given. Choose Δ so that
$$2 \int_{|t| \geq \Delta} |f(t)| dt < \varepsilon_1.$$

Fix x, and fix Δ as above. Choose y so that if $|x - |y < \delta$ then $e^{-ixt} - e^{-iyt}| < \varepsilon_1$ for all $|t| \leq \Delta$. For instance, this is true if $\varepsilon_1 < 1$ and $\delta = \varepsilon_1/2\Delta$ because then
$$|e^{-ixt} - e^{-iyt}| = |1 - e^{i(x-y)t}| \leq 2|t(x - y)| < \varepsilon_1.$$

Then
$$\int_{-\Delta}^{\Delta} |e^{-ixt} - e^{-iyt}||f(t)| dt < \varepsilon_1 \int_{-\infty}^{\infty} |f(t)| dt.$$

Now suppose that, given $\varepsilon > 0$, ε_1 is such that
$$\varepsilon_1 \left(1 + \int_{-\infty}^{\infty} vert f(t)| dt \right) < \varepsilon.$$

Then we can choose Δ and δ as above so that if $|x - y| < \delta$ then
$$|\hat{f}(x) - \hat{f}(y)| < \varepsilon$$

which is what continuity means.

19a)
$$\mathcal{L}(f)(z) = \int_0^{\infty} e^{-xz} f(x) dx.$$

If $\operatorname{Re}(z) \geq 0$ and $x \geq 0$ then $|e^{-xz}| = e^{-x \operatorname{Re}(z)} \leq 1$. So
$$|e^{-xz} f(x)| \leq |f(x)|$$

and, since f is integrable, $\mathcal{L}(f)(z)$ is defined for $\operatorname{Re}(z) \geq 0$, and, for all $\operatorname{Re}(z) \geq 0$,
$$|\mathcal{L}(f)(z)| \leq \int_0^{\infty} |e^{-xz} f(x)| dx \leq \int_0^{\infty} |f(x)| dx.$$

So $\mathcal{L}(f)(z)$ is bounded for $\operatorname{Re}(z) \geq 0$.

19b) $\mathcal{L}(g)(z) = \int_0^{\infty} e^{-tz} \chi_{[0,\infty)}(t - a) f(t - a) dt = \int_a^{\infty} e^{-tz} \chi_{[0,\infty)}(t - a) f(t - a) dt$

$$= \int_0^{\infty} e^{-(x+a)z} \chi_{[0,\infty)}(x) f(x) dx = e^{-az} \mathcal{L}(f)(z)$$

by making the change of variable $x = t - a$, so that $t = x + a$ and $dt = dx$.

19c) The usual formula for convolution is
$$f * g(x) = \int_{-\infty}^{\infty} f(x - y) g(y) dy = \int_{-\infty}^{\infty} f(t) g(x - t) dt$$

where the equality of the integrals is given by the change of variable $t = x - y$, so that $y = x - t$ and $dy = -dt$ (but the limits get changed and hence the minus sign

cancels out). But $f(x - y) = 0$ if $y > x$ and $g(y) = 0$ if $y < 0$. So the integral becaomes

$$f * g(x) = \int_0^x f(x - y)g(y)dy = \int_0^x f(t)g(x - t)dt.$$

The formula for the Laplace transform is

$$\mathcal{L}(f * g)(z) = \mathcal{L}(f)(z_{\mathcal{L}}(g)(z).$$

For f and g integrable, all three Laplace transforms $\mathcal{L}(f)(z)$, $\mathcal{L}(g)(z)$, $\mathcal{L}(f * g)(z)$ are defined for $\operatorname{Re}(z) \geq 0$.

20a) $F(z)$ is not bounded for $\operatorname{Re}(z) \geq 0$, in fact $F(0)$ is not defined and $\lim_{z \to 0} |F(z)| = +\infty$. So $F \neq \mathcal{L}(f)$ for any $f \in L^1(0, \infty)$. Alternatively, $F = \mathcal{L}(1)$ where 1 denotes the constant function 1, which is not in $L^1(0, \infty)$. But F can pnly be the Laplace function of at most one funciotn so there is no $f \in L^1(0, \infty)$ with $F = \mathcal{L}(f)$.

20b) $\frac{1}{z+1}$ is the Laplace transform of e^{-x}. and $(1 + z)^{-2} = -(d/dz)(1 + z)^{-1}$. So this must be the Laplace transform of xe^{-x} because

$$\frac{d}{dx} \int_0^\infty e^{-xz} g(x)dx = \int_0^\infty (-xg(x))e^{-xz} dx.$$

The function xe^{-x} iscertainly integrable on $(0, \infty)$ because (using Monotone Convergence)

$$\int_0^\infty xe^{-x} = \lim_{n \to \infty} [(-1 - x)e^{-x}]_0^n = 1.$$

20c) $(1 + \overline{z})^{-2}$ is not holomorphic on $\operatorname{Re}(z) > 0$, so cannot be of the form $\mathcal{L}(f)(z)$ for any $f \in L^1(0, \infty)$. To see that this function is not holomorphic - it is the conjugate of the holomorphic function $(1 + z)^{-2}$. The conjugate $\overline{G(z)} = u(x, y) - iv(x, y)$ of a holomorphic funciton $G(z) = u(x, y) + iv(x, y)$ (for $z = x + iy$, $u(x, y) = \operatorname{Re}(G(z))$, $v(x, y) = \operatorname{Im}(G(z))$) can never be holomorphic unless it is constant. For if we have $u_x = v_y = -v_y$, $u_y = -v_x = v_x$, then $u_x = v_y = 0 = u_y = v_x$ and G must be constant.

20d) We have $\lim_{z \to 0} \dfrac{1 - e^{-z}}{z} = 1$ (using l'Hopital's Rule, for example). So $F(z)$ is bounded by some constant M if $|z| \leq 2$ (compare with question 2) and also for all $\operatorname{Re}(z) \geq 0$, $|1 - e^{-z}| \leq 2$. So $|F(z)| \leq 2|z|^{-2}$ for all z. So

$$\int_{-\infty}^\infty |F(x + iy)|^2 dy \leq 2 \int_{-\infty}^\infty \frac{2dy}{|x + iy|^4}$$

$$= 2 \int_{-\infty}^\infty \frac{2dy}{(x^2 + y^2)^2} \leq \int_{-\infty}^\infty \frac{22}{1 + y^2} < +\infty$$

if $x \geq 1$. If $x \leq 1$ we have

$$\int_{-\infty}^\infty |F(x + iy)|^2 dy \leq M \int_{-1}^1 dy + \left(\int_{-\infty}^{-1} + \int_1^\infty \right) \frac{2dy}{y^4} < +\infty$$

By the Payley-Wiener Theorem, there must exist $f \in L^2(0, \infty)$ with $\mathcal{L}(f) = F$.

20e) For $|z| \leq 1$ we have $|(1 - e^{-z})/z| \geq C$ for some constant $C > 0$, because $\lim_{z \to 0} \dfrac{1 - e^{-z}}{z} = 1$. So if $|z| \leq 1$.

$$\left| \frac{1 - e^{-z}}{z^2} \right| \geq \frac{C}{|z|}.$$

Soif $0 < x < \frac{1}{2}$,

$$\int_{-\infty}^{\infty} |F(x + iy)|^2 dy = \int_{-\infty}^{\infty} \left| \frac{1 - e^{-x-iy}}{(x + iy)^2} \right|^2 dy$$

$$\geq \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{C^2}{|x + iy|^2}$$

$$\geq \int_{-x}^{x} \frac{C^2}{2x^2} \geq \frac{C^2}{2x} \to +\infty \text{ as } x \to 0.$$

21a) The variance is

$$\int_{-\infty}^{\infty} \frac{x^2}{2} e^{-|x|} dx = \int_{0}^{\infty} x^2 e^{-x} dx = \left[-x^2 e^{-x} \right]_0^{\infty} + \int_0^{\infty} 2x e^{-x} dx$$

$$= 0 + \left[-2xe^{-x} \right]_0^{\infty} + \int_0^{\infty} 2e^{-x} dx = 0 + \left[-2e^{-x} \right]_0^{\infty} = 2.$$

21b)
$$\int_{-\infty}^{\infty} \frac{x^2}{\sqrt{2\pi}} e^{-x^2/2} dx = \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}} . x e^{-x^2/2} dx$$

$$= \left[\frac{x}{\sqrt{2\pi}} . (-e^{-x^2/2}) \right]_{-\infty}^{\infty} + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} dx$$

$$= \lim_{\Delta \to +\infty} \frac{-1}{\sqrt{2\pi}} 2\Delta e^{-\Delta^2/2} + 1 = 0 + 1 = 1.$$

c)
$$\int_{-\infty}^{\infty} \frac{x^2}{\sqrt{2\pi\sigma}} e^{-x^2/2\sigma} dx = \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi\sigma}} . x e^{-x^2/2\sigma} dx$$

$$= \left[\frac{x}{\sqrt{2\pi\sigma}} . (-\sigma e^{-x^2/2\sigma}) \right]_{-\infty}^{\infty} + \frac{\sigma}{\sqrt{2\pi\sigma}} \int_{-\infty}^{\infty} e^{-x^2/2\sigma} dx$$

$$= \lim_{\Delta \to +\infty} \frac{-1}{\sqrt{2\pi\sigma}} 2\Delta e^{-\Delta^2/2\sigma} + \sigma = 0 + \sigma = \sigma.$$

Since $x^2/(1 + x^2 \geq \frac{1}{2}$ if $x \geq 1$,

$$\int_{-\infty}^{\infty} \frac{x^2}{\pi(1 + x^2)} dx \geq \int_1^{\infty} \frac{dx}{2\pi} = +\infty.$$

19a)
$$\hat{\mu}_n(\xi) = (\hat{\mu}(\xi))^n = (e^{-\xi^2/2})^n = e^{-n\xi^2/2}.$$

Then by the definition of ν_n and the definition of $\hat{\nu}_n$,

$$\hat{\nu}_n(\xi) = \int_{-\infty}^{\infty} e^{-ix\xi} d\nu_n(x) = \int_{-\infty}^{\infty} e^{-i(x/\sqrt{n})\xi} d\mu_n(x)$$

$$= \int_{-\infty}^{\infty} e^{-ix(\xi/\sqrt{n})} d\mu_n(x) = \hat{\mu}_n(\xi) = e^{-n(\xi/\sqrt{n})^2/2} = e^{-\xi^2/2}.$$

By the uniqueness of Fourier transforms for measures, it follows that $\nu_n = \mu$.

20b) This time we have

$$\hat{\mu}(\xi) = \frac{1}{\sqrt{2\pi\sigma}} \int_{-\infty}^{\infty} e^{-ix\xi} e^{-x^2/2\sigma} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-it(\sqrt{\sigma}\xi)} e^{-t^2/2} dt = e^{-(\sqrt{\sigma}\xi)^2/2} = e^{-\sigma\xi^2/2}$$

(by making the change of variable $x = t\sqrt{\sigma}$). Then

$$\hat{\mu}_n(\xi) = (e^{-\sigma\xi^2/2})^n = e^{-n\sigma\xi^2/2}.$$

Again

$$\hat{\nu}_n(\xi) = \hat{\mu}_n(\xi/\sqrt{n}) = e^{-\sigma\xi^2/2}.$$

Again, by uniqueness of Fourier transforms of measures, $\nu_n = \mu$ in this case also.

21c) The Central Limit Theorem says that if μ has bounded non-zero variance

$$\sigma = \int_{-\infty}^{\infty} x^2 d\mu(x),$$

and zero mean, then

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} g(x) d\nu_n(x) = \frac{1}{\sqrt{2\pi\sigma}} \int_{-\infty}^{\infty} g(x) e^{-x^2/2\sigma} dx.$$