MATH348. Harmonic Analysis. Problems 9

Work is due in on *Wednesday 1st December.*

1. Compute $\mathcal{L}(f)(z)$ for $f:(0,\infty)\to\mathbf{C}$ and a suitable set of z where a) $f(x)=x+1$ b) $f(x)=x^2e^x$. c) $f(x)=\chi_{[2,\infty)}(x)$, that is, $f(x)=1$ for $x\geq 2$ and $=0$ for $x<2$.

2. Find $f\in L^1(0,\infty)$ with $\mathcal{L}_\rangle(f)(z)=L_i(z)$ for all z with $\mathrm{Re}(z)\geq 0$, where

a) $L_1(z)=\dfrac{1}{z+2}$,

b) $L_2(z)=\dfrac{1}{(z+2)^2}$.

Hint: Is L_2 the derivative of any other function?

3. Explain, using properties of the Laplace transform, why there is no function $f\in L^1(0,\infty)$ with $\mathcal{L}(f)(z)=L_i(z)$ where

a) $L_3(z)=\dfrac{1}{z-1}$

b) $L_4(z)=e^z$

c) Explain also why there is no $f_5\in L^2(0,\infty)$ with $\mathcal{L}(f_5)(z)=\dfrac{1}{z^2+4}$ for $\mathrm{Re}(z)>0$.

4. Let $f\in L^1(0,\infty)$. a) Show that if $\mathrm{Re}(z)\geq n$, then

$$|\mathcal{L}(f)(z)|\leq\int_0^\infty e^{-nx}|f(x)|dx.$$

Why does the Monotone Convergence Theorem imply that

$$\lim_{n\to\infty}\int_0^\infty e^{-nx}|f(x)|dx=0?$$

b) Using a), and the fact that

$$\lim_{R\to\pm\infty}\int_0^\infty e^{Rix}e^{-ax}f(x)dx=0,$$

uniformly for $a\in[0,A]$ for any $A>0$, show that

$$\lim_{|z|\to\infty,\mathrm{Re}(z)\geq 0}\mathcal{L}(f)(z)=0.$$

c) Show that e^{-z} cannot be $\mathcal{L}(f)(z)$ for any $f\in L^1(0,\infty)$.

Hint: Consider e^{-1+iy} for varying real y, and use b).