MATH348. Harmonic Analysis. Problems 9

Work is due in on *Wednesday 1st December.*

1. Compute $\mathcal{L}(f)(z)$ for $f : (0, \infty) \to \mathbf{C}$ and a suitable set of z where a) $f(x) = x + 1$ b) $f(x) = x^2 e^x$. c) $f(x) = \chi_{[2,\infty)}(x)$, that is, $f(x) = 1$ for $x \geq 2$ and $= 0$ for $x < 2$.

2. Find $f \in L^1(0, \infty)$ with $\mathcal{L}_\rangle(f)(z) = L_i(z)$ for all z with $\mathrm{Re}(z) \geq 0$, where

a) $L_1(z) = \dfrac{1}{z+2}$,

b) $L_2(z) = \dfrac{1}{(z+2)^2}$.

Hint: Is L_2 the derivative of any other function?

3. Explain, using properties of the Laplace transform, why there is no function $f \in L^1(0, \infty)$ with $\mathcal{L}(f)(z) = L_i(z)$ where

a) $L_3(z) = \dfrac{1}{z-1}$

b) $L_4(z) = e^z$

c) Explain also why there is no $f_5 \in L^2(0, \infty)$ with $\mathcal{L}(f_5)(z) = \dfrac{1}{z^2+4}$ for $\mathrm{Re}(z) > 0$.

4. Let $f \in L^1(0, \infty)$. a) Show that if $\mathrm{Re}(z) \geq n$, then

$$|\mathcal{L}(f)(z)| \leq \int_0^\infty e^{-nx} |f(x)| dx.$$

Why does the Monotone Convergence Theorem imply that

$$\lim_{n \to \infty} \int_0^\infty e^{-nx} |f(x)| dx = 0?$$

b) Using a), and the fact that

$$\lim_{R \to \pm\infty} \int_0^\infty e^{Rix} e^{-ax} f(x) dx = 0,$$

uniformly for $a \in [0, A]$ for any $A > 0$, show that

$$\lim_{|z| \to \infty, \mathrm{Re}(z) \geq 0} \mathcal{L}(f)(z) = 0.$$

c) Show that e^{-z} cannot be $\mathcal{L}(f)(z)$ for any $f \in L^1(0, \infty)$.
Hint: Consider e^{-1+iy} for varying real y, and use b).

MATH348. Harmonic Analysis. Problems 9 Solutions.

1a). Let $\mathrm{Re}(z) > 0$.

$$\mathcal{L}(f)(z) = \int_0^\infty (x+1)e^{-xz}dx = \lim_{N\to\infty}\left[\frac{-(x+1)}{z}e^{-xz}\right]_0^N + \lim_{N\to\infty}\int_0^N \frac{e^{-xz}}{z}dx$$

$$= \lim_{N\to\infty}\left[\frac{-(x+1)}{z}e^{-xz} - \frac{1}{z^2}e^{-xz}\right]_0^N = \frac{1}{z} + \frac{1}{z^2}.$$

1b) Let $\mathrm{Re}(z) > 1$.

$$\mathcal{L}(f)(z) = \int_0^\infty x^2 e^{x(1-z)}dx$$

$$= \lim_N \to \infty \left[\frac{x^2}{1-z}e^{x(1-z)}\right]_0^N - \int_0^\infty \frac{2x}{z-1}e^{x(1-z)}dx$$

$$= \frac{x^2}{z-1} - \lim_N \to \infty \left[\frac{2x}{(1-z)^2}e^{x(1-z)}\right]_0^N + \int_0^\infty \frac{2}{(1-z)^2}e^{x(1-z)}dx$$

$$= \lim_{N\to\infty}\left[\frac{e^{x(1-z)}}{(1-z)^3}\right]_0^N$$

$$= \frac{2}{(z-1)^3}.$$

1c) Let $\mathrm{Re}(z) > 0$.

$$\mathcal{L}(f)(z) = \int_0^\infty \chi_{(2,\infty)}(x)e^{-xz}dx = \int_2^\infty e^{-xz}dx$$

$$= \lim_{N\to\infty}\left[\frac{-e^{-xz}}{z}\right]_0^N = \frac{e^{-2z}}{z}.$$

2a) Take $f(x) = e^{-2x}$. We have

$$\mathcal{L}(f)(z) = \int_0^\infty e^{-2x-xz}dx = \left[\frac{e^{-2x-xz}}{-2-z}\right]_0^\infty = \frac{1}{z+2} = L_1(z).$$

2b) If $L = \mathcal{L}(f)(z)$ for $f \in L^1(0,\infty)$ then the derivative $L'(z) = \mathcal{L}(g)(z)$ where $g(x) = -xf(x)$. Now $L_2(z) = -L_1'(z)$. So $L_2(z) = \mathcal{L}(g)(z)$ where $g(x) = xe^{-2x}$.

3a) L_3 has a singularity at $z = 1$ and is therefore not holomorphic in $\{z : \mathrm{Re}(z) > 0\}$, as it would have to be the case for $L_3 = \mathcal{L}(f)$ for some $f \in L^1(0,\infty)$.
3b) $|e^z| = e^{\mathrm{Re}}(z) \to \infty$ as $\mathrm{Re}(z) \to \infty$. So L_4 is not bounded in $\{z : \mathrm{Re}(z) > 0\}$, as it would have to be the case for $L_4 = \mathcal{L}(f)$ for some $f \in L^1(0,\infty)$.

3c) If there is $f_5 \in L^2(0,\infty)$ with $\mathcal{L}(f_5)(z) = \dfrac{1}{z^2+4}$ then

$$\sup_{x>0} \int_{-\infty}^{\infty} \frac{1}{(x+iy)^2+4|^2} dy = \int_0^{\infty} |f_5(x)|^2 dx < +\infty.$$

But

$$\int_{-\infty}^{\infty} \frac{1}{(x+iy)^2+4|^2} dy = \int_{-\infty}^{\infty} \frac{1}{|(x+iy+2i)(x+iy-2i)|^2} dy$$

$$= \int_{-\infty}^{\infty} \frac{1}{(x^2+(y+2)^2)(x^2+(y-2)^2)} dy \geq \frac{1}{x^2+25} \int_1^3 \frac{1}{x^2+(y-2)^2} dy$$

$$\geq \frac{1}{x^2+25} \int_{2-x}^{2+x} \frac{1}{x^2+(y-2)^2} dy \text{ (if } 0 < x \leq 1)$$

$$\geq \frac{1}{x^2+25} \int_{2-x}^{2+x} \frac{1}{2x^2} dy = \frac{1}{(x^2+25)x^2} \to \infty \text{ as } x \to 0.$$

So there is no such $f_5 \in L^2(0,\infty)$.

Alternatively, $\frac{1}{z^2+4} = \{(z)$ for $\mathrm{Re}(z) > 0$ where $f(t) = \frac{1}{2}\sin 2t$, as can be checked by direct calculation, as follows. We have

$$\sin 2t = \frac{1}{2i}(e^{2it} - e^{-2it}),$$

so

$$\int_0^{\infty} \sin 2t e^{-zt} dt = \frac{1}{2i} \int_0^{\infty} (e^{(2i-z)t} - e^{-(2i+z)t}) dt$$

$$= \frac{1}{2i}\left(\frac{1}{z-2i} - \frac{1}{z+2i}\right) = \frac{2}{z^2+4}.$$

Now

$$\int_{n\pi}^{(n+1)\pi} \sin^2 2t dt = \int_0^{\pi} \sin^2 2t dt > 0$$

for any integer $n > 0$. So

$$\int_0^{\infty} \sin^2 2t dt = \sum_{n=0}^{\infty} \int_{n\pi}^{(n+1)\pi} \sin^2 2t dt = +\infty >$$

So $f \notin L^2(0,\infty)$. So since a function is uniquely determined by its Laplace transform, there is no $f_5 \in L^2(0,\infty)$ with $\mathcal{L}(f_5)(z) = \mathcal{L}(f)(z)$ for all $\mathrm{Re}(z) > 0$.

4a) If $\mathrm{Re}(z) \geq n$,

$$|\mathcal{L}(f)(z)| \leq \int_{-\infty}^{\infty} |e^{-xz} f(x)| dx$$

$$= \int_{-\infty}^{\infty} e^{-\mathrm{Re}(z)x}|f(x)|dx \le \int_{-\infty}^{\infty} e^{-nx}|f(x)|dx.$$

Put $f_n(x) = e^{-nx}|f(x)|$. Then $f_{n+1}(x) \le f_n(x)$ for all $x > 0$ and all n, f_n is integrable, and
$\lim_{n\to\infty} f_n(x) = 0$ for all $x > 0$. So by the Monotone Convergence Theorem

$$\lim_{n\to\infty} \int_0^{\infty} e^{-nx}|f(x)|dx = \int_0^{\infty} 0dx = 0.$$

4b) Given $\epsilon > 0$ choose N so that, for all $n \ge N$,

$$\int_0^{\infty} e^{-nx}|f(x)|dx < \epsilon, \tag{1}$$

and choose $R_0 \ge N$ such that for all real R with $|R| \ge R_0$, and all $a \in [0, N]$

$$\left|\int_0^{\infty} e^{-ax+iRx} f(x)dx\right| < \epsilon. \tag{2}$$

Now take any z with $\mathrm{Re}(z) \ge 0$ and $|z| \ge 2R_0$. Then $\mathrm{Re}(z) + |\mathrm{Im}(z)| \ge |z| \ge 2R_0$. So either $\mathrm{Re}(z) \ge N$, in which case

$$|\mathcal{L}(f)(z)| \le \int_0^{\infty} e^{-\mathrm{Re}(z)x}|f(x)|dx \le \int_0^{\infty} e^{-Nx}|f(x)|dx < \epsilon$$

by (1), or $\mathrm{Re}(z) \in [0, N]$ and $|\mathrm{Im}(z)| \ge R_0$ and putting $\mathrm{Re}(z) = a$ and $\mathrm{Im}(z) = -R$ in (2), we again obtain $|\mathcal{L}(f)| < \epsilon$. So we do indeed have

$$\lim_{|z|\to\infty, \mathrm{Re}(z)\ge 0} \mathcal{L}(f)(z) = 0.$$

4c) Put $z = 1 + iy$. Then $|e^{-(1+iy)}| = e^{-1}$. So e^{-z} does not tend to 0 as $|z| \to \infty$ for $\mathrm{Re}(z) \ge 0$. So by b) e^{-z} cannot be $\mathcal{L}(f)(z)$ for any $f \in L^1(0, \infty)$.