MATH348. Harmonic Analysis. Problems 7.

Work is due in on *Wednesday 17th November.*

1. Verify Plancherel's formula

$$\int_{-\infty}^{\infty} f(x)\overline{g(x)}dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(\xi)\overline{\widehat{g}(\xi)}d\xi$$

if $f(x) = e^{-x^2}$, $g(x) = e^{-x^2/4}$. You may use the fact that if $a > 0$ and $h(x) = e^{-ax^2}$ then $\widehat{h}(\xi) = \sqrt{\pi/a}e^{-\xi^2/4a}$.

2. Prove that if $f : \mathbf{R} \to \mathbf{R}$ is continuous and integrable and $|f(x)| \leq M$ for all x, then the solution of the heat equation

$$u(x,t) = \frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty} f(y)e^{-(x-y)^2/4t}dy \ \ (t > 0)$$

satisfies

$$\lim_{t\to 0} u(x,t) = f(x).$$

Do this by first showing (i)-(v) below.

(i)
$$\frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty} e^{-y^2/4t}dy = 1,$$

[You may assume that

$$\int_{-\infty}^{\infty} e^{-w^2/2}dw = \sqrt{2\pi}]$$

(ii)
$$u(x,t) = \frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty} e^{-y^2/4t}f(x-y)dy,$$

(iii)
$$u(x,t) - f(x) = \frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty} e^{-y^2/4t}(f(x-y) - f(x)dy,$$

(iv)
$$\left|\frac{1}{2\sqrt{\pi t}}\int_{|y|\geq\delta} e^{-y^2/4t}(f(x-y) - f(x))dy\right| \leq 2M\frac{1}{2\sqrt{\pi}}\int_{|w|\geq\delta/\sqrt{t}} e^{-w^2/4}dw,$$

(v)
$$\left|\frac{1}{2\sqrt{\pi t}}\int_{-\delta}^{\delta} e^{-y^2/4t}(f(x-y) - f(x))dy\right| \leq \sup\{|f(x) - f(x-y)| : |y| \leq \delta\}.$$

3. For $u(x,t)$ as in 3, show that

$$\lim_{t\to\infty} u(x,t) = 0.$$

You may use the following version of the Dominated Convergence Theorem. Let g_t ($t \geq 1$, $t \in \mathbf{R}$) be a family of Lebesgue-measurable functions such that $|g_t(x)| \leq G(x)$ for all x and t, where G is integrable. Suppose also that $\lim_{t\to\infty} g_t(x) = g(x)$ for all x. Then g is integrable, g_t is integrable for all t and

$$\lim_{t\to+\infty}\int_{-\infty}^{\infty} g_t(x)dx = \int_{-\infty}^{\infty} g(x)dx.$$