MATH348. Harmonic Analysis. Problems 7.

Work is due in on *Wednesday 17th November.*

1. Verify Plancherel's formula

$$\int_{-\infty}^{\infty} f(x)\overline{g(x)}dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(\xi)\overline{\widehat{g}(\xi)}d\xi$$

if $f(x) = e^{-x^2}$, $g(x) = e^{-x^2/4}$. You may use the fact that if $a > 0$ and $h(x) = e^{-ax^2}$ then $\widehat{h}(\xi) = \sqrt{\pi/a}e^{-\xi^2/4a}$.

2. Prove that if $f : \mathbf{R} \to \mathbf{R}$ is continuous and integrable and $|f(x)| \leq M$ for all x, then the solution of the heat equation

$$u(x,t) = \frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty} f(y)e^{-(x-y)^2/4t}dy \ \ (t > 0)$$

satisfies

$$\lim_{t\to 0} u(x,t) = f(x).$$

Do this by first showing (i)-(v) below.

(i)
$$\frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty} e^{-y^2/4t}dy = 1,$$

[You may assume that

$$\int_{-\infty}^{\infty} e^{-w^2/2}dw = \sqrt{2\pi}]$$

(ii)
$$u(x,t) = \frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty} e^{-y^2/4t}f(x-y)dy,$$

(iii)
$$u(x,t) - f(x) = \frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty} e^{-y^2/4t}(f(x-y) - f(x)dy,$$

(iv)
$$\left|\frac{1}{2\sqrt{\pi t}}\int_{|y|\geq\delta} e^{-y^2/4t}(f(x-y) - f(x))dy\right| \leq 2M\frac{1}{2\sqrt{\pi}}\int_{|w|\geq\delta/\sqrt{t}} e^{-w^2/4}dw,$$

(v)
$$\left|\frac{1}{2\sqrt{\pi t}}\int_{-\delta}^{\delta} e^{-y^2/4t}(f(x-y) - f(x))dy\right| \leq \sup\{|f(x) - f(x-y)| : |y| \leq \delta\}.$$

3. For $u(x,t)$ as in 3, show that

$$\lim_{t\to\infty} u(x,t) = 0.$$

You may use the following version of the Dominated Convergence Theorem. Let g_t $(t \geq 1,\ t \in \mathbf{R})$ be a family of Lebesgue-measurable functions such that $|g_t(x)| \leq G(x)$ for all x and t, where G is integrable. Suppose also that $\lim_{t\to\infty} g_t(x) = g(x)$ for all x. Then g is integrable, g_t is integrable for all t and

$$\lim_{t\to+\infty}\int_{-\infty}^{\infty} g_t(x)dx = \int_{-\infty}^{\infty} g(x)dx.$$

MATH348. Harmonic Analysis. Solutions 7.

1. Using the given formula,

$$\widehat{f}(\xi) = \sqrt{\pi}e^{-\xi^2/4}, \ \widehat{g}(\xi) = 2\sqrt{\pi}e^{-\xi^2}.$$

So

$$\int_{-\infty}^{\infty} f(x)\overline{g(x)}dx = \int_{-\infty}^{\infty} e^{-5x^2/4}dx = \frac{2\pi}{2\pi}\int_{-\infty}^{\infty} e^{-5\xi^2/4}d\xi.$$

2(i) Making the change of variable $w = y/\sqrt{2t}$, we have $dw = dy/\sqrt{2t}$, which gives

$$\frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty} e^{-y^2/4t}dy = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-w^2/2}dw = 1.$$

(ii) Making the change of variable $w = x - y$ we have $dw = -dy$ and $y = x - w$, as $y \to +\infty$, $w \to -\infty$ and as $y \to -\infty$, $w \to +\infty$. So although the sign of the integral changes, the limits change too. So

$$\int_{-\infty}^{\infty} f(y)e^{-(x-y)^2/4t}dy = \int_{-\infty}^{\infty} f(x-w)e^{-w^2/4t}dw.$$

Since w is an integration variable, we can replace it by y.

(iii) From (ii) we have

$$u(x,t) = \frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty} f(x-y)e^{-y^2/4t}dy.$$

From (i) we have

$$f(x) = \frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty} f(x)e^{-y^2/4t}dy.$$

Subtracting these, the result follows.

(iv) We have $|f(x)| \le M$, $|f(x-y)| \le M$. So

$$\left|\frac{1}{2\sqrt{\pi t}}\int_{|y|\ge\delta}(f(x-y) - f(x))e^{-y^2/4t}dy\right| \le \frac{1}{2\sqrt{\pi t}}\int_{|y|\ge\delta} 2Me^{-y^2/4t}dy.$$

Now making the change of variable $w = y/\sqrt{t}$, $dy/\sqrt{t} = dw$ and when $y = \pm\delta$, $w = \pm\delta/\sqrt{t}$, so this becomes

$$2M\frac{1}{2\sqrt{\pi}}\int_{|w|\ge\delta/\sqrt{t}} e^{-w^2/4}dw.$$

(v) $$\left|\frac{1}{2\sqrt{\pi t}}\int_{-\delta}^{\delta} e^{-y^2/4t}(f(x-y) - f(x))dy\right| \le \frac{1}{2\sqrt{\pi t}}\int_{-\delta}^{\delta} |f(x-y) - f(x)|e^{-y^2/4t}dy$$

$$\le \sup\{|f(x-y) - f(x)| : |y| \le \delta\}\frac{1}{2\sqrt{\pi t}}\int_{-\delta}^{\delta} e^{-y^2/4t}dy.$$

Then the result follows since by (i) the last integral is < 1 (because the integrand is strictly positive).

Now to show that $\lim_{t\to 0} u(x,t) = f(x)$: fix x, and given $\varepsilon > 0$, choose $\delta > 0$ so that $|f(x-y) - f(x) < \varepsilon/2$ for all $|y| \le \delta$. Then choose t_0 so that

$$\frac{1}{2\sqrt{\pi}} \int_{|w|\ge \delta/\sqrt{t_0}} e^{-w^2/4} dw < \varepsilon/4M.$$

Then by (iii), for this δ,

$$|u(x,t) - f(x)| \le \frac{1}{2\sqrt{\pi t}} \int_{-\delta}^{\delta} |f(x-y) - f(x)| e^{-y^2/4t} dy + \frac{1}{2\sqrt{\pi t}} \int_{|y|\ge\delta} |f(x-y) - f(x)| e^{-y^2/4t} dy$$

and by (iv) and (v) we get, for $0 < t \le t_0$,

$$|u(x,t) - f(x)| \le \sup\{|f(x-y) - f(x)| : |y| \le \delta\} + 2M \frac{1}{2\sqrt{\pi}} \int_{|w|\ge \delta/\sqrt{t_0}} e^{-w^2/4} dw$$

$$< \varepsilon/2 + 2M\varepsilon/4M = \varepsilon$$

as required.

3. For a fixed x, put

$$g_t(y) = \frac{1}{2\sqrt{\pi t}} e^{-(x-y)^2/4t} f(y).$$

Now $e^{-(x-y)^2/4t} \le 1$ for all x, $y \in \mathbf{R}$ and for all $t > 0$. So for all $t \ge 1$,,

$$|g_t(y)| \le \frac{1}{2\sqrt{\pi t}} |f(y)| \le \frac{1}{2\sqrt{\pi}} |f(y)|.$$

The righthand side is integrable. Also

$$\lim_{t\to\infty} g_t(y) = \lim_{t\to\infty} \frac{1}{2\sqrt{\pi t}} |f(x-y)| = 0.$$

So then by the Dominated Convergnce Theorem given

$$\lim_{t\to\infty} u(x,t) = \lim_{t\to\infty} \int_{-\infty}^{\infty} g_t(y) dy = \lim_{t\to\infty} \int_{-\infty}^{\infty} 0 dy = 0,$$

as required.