MATH348. Harmonic Analysis. Problems 6

Work is due in on *Wednesday 10th November.*

1. Let f be integrable on $\mathbf{R}$.

a) Show that if $g(x) = e^{iax}f(x)$ for some $a \in \mathbf{R}$ and for all x, then $\hat{g}(\xi) = \hat{f}(\xi - a)$

b) Show that if $g(x) = f(ax)$ for $a > 0$ and for all x, then $\hat{g}(\xi) = a^{-1}\hat{f(\xi/a)}$.

c) Show that if $g(x) = a^{-1}f(x/a)$ for some $a > 0$ and for all x then $\hat{g}(\xi) = \hat{f}(a\xi)$.

2. Let φ, ψ be defined by

$$\varphi(x) = \frac{e^{-|x|}}{2},\ \psi(x) = \frac{1}{\pi(1+x^2)}.$$

For any $\varepsilon > 0$ let φ_ε, ψ_ε be defined by

$$\varphi_\varepsilon(x) = \varepsilon^{-1}\varphi(x/\varepsilon) = \frac{e^{-|x|/\varepsilon}}{2\varepsilon},\ \psi_\varepsilon(x) = \varepsilon^{-1}\psi(x/\varepsilon) = \frac{\varepsilon}{\pi(\varepsilon^2 + x^2)}.$$

a) Verify that

$$\int \varphi = 1 = \int \psi = 1.$$

Why is this enough to ensure that $|\hat{\varphi}(\xi)| \leq 1$, $|\hat{\psi}(\xi)| \leq 1$ for all ξ?

b) Now you may assume that (as was proved in lectures)

$$\hat{\varphi}(\xi) = \frac{1}{1+\xi^2},\ \hat{\psi}(\xi) = e^{-|\xi|}.$$

Using question 1 (or otherwise) give $\hat{\varphi}_\varepsilon(\xi)$ and $\hat{\psi}_\varepsilon(\xi)$ for all $\varepsilon > 0$. Show that $\lim_{\varepsilon\to 0}\hat{\varphi}_\epsilon(\xi) = 1$ and $\lim_{\varepsilon\to 0}\hat{\psi}_\varepsilon(\xi) = 1$.

c) Now compute $\hat{g}(\xi)$, where $g(x) = \varepsilon^{-1}\varphi_{\varepsilon^{-1}}(x) = \frac{1}{2}e^{-\varepsilon|x|}$.

3. Let f be integrable. Use the definition of $\hat{f}$, a change in the order of integration (which you should attempt to justify), a change of variable and question 2 to show that, for all $\varepsilon > 0$,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\varepsilon|\xi|}\hat{f}(\xi)e^{ix\xi}d\xi = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x-u)\int_{-\infty}^{\infty} e^{i\xi u}e^{-\varepsilon|\xi|}d\xi du$$

$$= \frac{1}{\pi}\int_{=\infty}^{\infty} f(x-u)\frac{\varepsilon du}{\varepsilon^2 + u^2} = f * \psi_\varepsilon(x).$$

Give the limit of this expression as $\epsilon \to 0$, if f is continuous. Also explain how to use the Dominated Convergence Theorem to show that if $\hat{f}$ is integrable,

$$\lim_{\epsilon\to 0}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\varepsilon|\xi|}\hat{f}(\xi)e^{ix\xi}d\xi = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\xi)e^{ix\xi}d\xi$$

This is a slightly more general version of the Dominated Convergence Theorem than in the integration notes. If $|F_\varepsilon(x)| \leq g(x)$ for all x and g is integrable and ε and $\lim_{\epsilon\to 0}F_\varepsilon(x) = F(x)$ for all x, then F is integrable and

$$\int F(x)dx = \lim_{\varepsilon\to 0}\int F_\varepsilon.$$