MATH348. Harmonic Analysis. Problems 6

Work is due in on *Wednesday 10th November.*

1. Let f be integrable on $\mathbf{R}$.

a) Show that if $g(x) = e^{iax}f(x)$ for some $a \in \mathbf{R}$ and for all x, then $\hat{g}(\xi) = \hat{f}(\xi - a)$

b) Show that if $g(x) = f(ax)$ for $a > 0$ and for all x, then $\hat{g}(\xi) = a^{-1}\hat{f(\xi/a)}$.

c) Show that if $g(x) = a^{-1}f(x/a)$ for some $a > 0$ and for all x then $\hat{g}(\xi) = \hat{f}(a\xi)$.

2. Let φ, ψ be defined by

$$\varphi(x) = \frac{e^{-|x|}}{2},\ \psi(x) = \frac{1}{\pi(1+x^2)}.$$

For any $\varepsilon > 0$ let φ_ε, ψ_ε be defined by

$$\varphi_\varepsilon(x) = \varepsilon^{-1}\varphi(x/\varepsilon) = \frac{e^{-|x|/\varepsilon}}{2\varepsilon},\ \psi_\varepsilon(x) = \varepsilon^{-1}\psi(x/\varepsilon) = \frac{\varepsilon}{\pi(\varepsilon^2 + x^2)}.$$

a) Verify that

$$\int \varphi = 1 = \int \psi = 1.$$

Why is this enough to ensure that $|\hat{\varphi}(\xi)| \leq 1$, $|\hat{\psi}(\xi)| \leq 1$ for all ξ?

b) Now you may assume that (as was proved in lectures)

$$\hat{\varphi}(\xi) = \frac{1}{1+\xi^2},\ \hat{\psi}(\xi) = e^{-|\xi|}.$$

Using question 1 (or otherwise) give $\hat{\varphi}_\varepsilon(\xi)$ and $\hat{\psi}_\varepsilon(\xi)$ for all $\varepsilon > 0$. Show that $\lim_{\varepsilon\to 0}\hat{\varphi}_\epsilon(\xi) = 1$ and $\lim_{\varepsilon\to 0}\hat{\psi}_\varepsilon(\xi) = 1$.

c) Now compute $\hat{g}(\xi)$, where $g(x) = \varepsilon^{-1}\varphi_{\varepsilon^{-1}}(x) = \frac{1}{2}e^{-\varepsilon|x|}$.

3. Let f be integrable. Use the definition of $\hat{f}$, a change in the order of integration (which you should attempt to justify), a change of variable and question 2 to show that, for all $\varepsilon > 0$,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\varepsilon|\xi|}\hat{f}(\xi)e^{ix\xi}d\xi = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x-u)\int_{-\infty}^{\infty} e^{i\xi u}e^{-\varepsilon|\xi|}d\xi du$$

$$= \frac{1}{\pi}\int_{=\infty}^{\infty} f(x-u)\frac{\varepsilon du}{\varepsilon^2 + u^2} = f * \psi_\varepsilon(x).$$

Give the limit of this expression as $\epsilon \to 0$, if f is continuous. Also explain how to use the Dominated Convergence Theorem to show that if $\hat{f}$ is integrable,

$$\lim_{\epsilon\to 0}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\varepsilon|\xi|}\hat{f}(\xi)e^{ix\xi}d\xi = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\xi)e^{ix\xi}d\xi$$

This is a slightly more general version of the Dominated Convergence Theorem than in the integration notes. If $|F_\varepsilon(x)| \leq g(x)$ for all x and g is integrable and ε and $\lim_{\epsilon\to 0}F_\varepsilon(x) = F(x)$ for all x, then F is integrable and

$$\int F(x)dx = \lim_{\varepsilon\to 0}\int F_\varepsilon.$$

1a)
$$\hat{g}(\xi) = \int_{-\infty}^{\infty} f(x)e^{ix(a-\xi)}dx = \hat{f}(\xi - a).$$

b) Using the change of variable $u = ax$, $x = u/a$ an $dx = du/a$. So
$$\hat{g}(\xi) = \int_{-\infty}^{\infty} f(xa)e^{-x\xi}dx = \int_{-\infty}^{\infty} f(u)e^{-u\xi/a)}\frac{du}{a} = a^{-1}\hat{f}(\xi/a).$$

c) This actually follows from b) replacing a by a^{-1} since if $f_1 = a^{-1}f_2$, $\hat{f}_1 = a^{-1}\hat{f}_2$. So writing $g(x) = a^{-1}f_1(x)$ where $f_1(x) = f(x/a)$, we have $\hat{f}_1(\xi) = a\hat{f}(a\xi)$ and $\hat{g}(\xi) = \hat{f}(a\xi)$. Alternatively, we can prove it directly using a change of variable $v = x/a$, and proceed similarly to 1b).

2a)
$$\int \varphi = 2\int_0^{\infty} \frac{e^{-x}dx}{2} = \left[-e^{-x}\right]_0^{\infty} = 1,$$
$$\int \psi = \int_{-\infty}^{\infty} \frac{dx}{\pi(1+x^2} = [\text{arc}\ \tan(x)]_{-\infty}^{\infty} = \frac{\pi/2 + \pi/2}{\pi} = 1.$$

We have
$$|\hat{\varphi}(\xi)| = \left| \int_{-\infty}^{\infty} e^{-ix\xi}\varphi(x)dx \right| \leq \int \varphi(x)dx = 1$$

Here, we used the fact that $\varphi(x) \geq 0$ for all x. Similarly we see that $|\hat{\psi}(\xi)| \leq 1$ for all $\xi \in \mathbf{R}$.

2b) Using 1c) and the given formula for $\hat{\varphi}(\xi)$
$$\lim_{\varepsilon\to 0} \hat{\varphi}_\varepsilon(\xi) = \lim_{\varepsilon\to 0} \frac{1}{(1+\xi^2\varepsilon^2)} = 1.$$

Similarly, using 1c) and the formula for $\hat{\psi}(\xi)$,
$$\lim_{\varepsilon\to 0} \hat{\psi}_\varepsilon(\xi) = \lim_{\varepsilon\to 0} e^{-\varepsilon|\xi|} = 1.$$

2c) Using 1b), if $g(x) = \frac{1}{2}e^{-\varepsilon|x|}$ we have
$$\hat{g}(\xi) = \varepsilon^{-1}\hat{\varphi}(\xi/\varepsilon) = \frac{\varepsilon^{-1}}{1+(\xi/\varepsilon)^2} = \frac{\varepsilon}{\varepsilon^2+\xi^2}.$$

3.
$$I = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\varepsilon|\xi|}\hat{f}(\xi)e^{ix\xi}d\xi = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\varepsilon|\xi|}\int_{-\infty}^{\infty} f(y)e^{i(x-y)\xi}dyd\xi$$
$$= \frac{1}{2\pi y}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\varepsilon|\xi|}f(y)e^{i(x-y)\xi}d\xi dy$$

by Tonelli's Theorem: We can change the order of integration because
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |e^{-\varepsilon|\xi|}f(y)e^{i(x-y)\xi}|d\xi dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} te^{-\varepsilon|\xi|}|f(y)|d\xi dy < +\infty.$$

Then

$$I = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\varepsilon|\xi|}\int_{-\infty}^{\infty} f(x-u)e^{iu\xi}d\xi du$$

making the change of variable $u = x - y$, $y = x - u$, $dy = -du$, $u \to -\infty$ as $y \to +\infty$ and $u \to +\infty$ as $y \to -\infty$

Using 2c) to work out the inner integral,

$$I = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x-u)\int_{-\infty}^{\infty} e^{-ixu}e^{-\epsilon|\xi|}d\xi du = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x-u)\frac{\epsilon du}{\epsilon^2+x^2} = f * \psi_\epsilon(x).$$

If f is continuous the limit as $\epsilon \to 0$ is $f(x)$. Also, since $0 \le e^{-\epsilon|\xi|} \le 1$ for all ξ, ϵ,

$$|e^{-\epsilon|\xi|}\hat{f}(\xi)e^{ix\xi}| \le |\hat{f}(\xi)|$$

. So if $\hat{f}$ is integrable, by Dominated Convergence, since $\lim_{\epsilon\to 0} e^{-\epsilon|\xi|} = 1$ for all ξ,

$$\lim_{\epsilon\to 0}\int_{-\infty}^{\infty} e^{-\epsilon|\xi|}\hat{f}(\xi)e^{ix\xi d\xi} = \int_{-\infty}^{\infty}\hat{f}(\xi)e^{ix\xi}d\xi.$$