MATH348. Harmonic Analysis. Problems 4.

Office hours are 11-1 Monday and 3-5 Tuseday. I am also in my office from 9 on Wed Work due on *Wednesday 27th October.*

1. Work out the Fourier coefficients $\widehat{f}(n)$, $\widehat{g}(n)$, $\widehat{h}(n)$ of the following functions on $[-\pi, \pi]$.

a)
$$f(x) = \begin{matrix} 1 \text{ if } 0 \leq x \leq \pi, \\ 0 \text{ if } -\pi < x < 0, \end{matrix}$$

b) $g(x) = x$,

c) $h(x) = |x|\pi - \frac{1}{2}\pi^2$.

2. Regard the functions f and g of question 1 as 2π-periodic functions on $\mathbf{R}$. Show that $f * g = h$. You may find it helpful to note that if $0 \leq x \leq \pi$ then

$$f * g(x) = \int_{x-\pi}^{x} y dy$$

and if $-\pi \leq x \leq 0$ then

$$f * g(x) = \int_{-\pi}^{x} y dy + \int_{x+\pi}^{\pi} y dy.$$

Verify that $\widehat{h}(n) = \widehat{f}(n)\widehat{g}(n)$.

3. Consider the Laplace equation in $\{(x, y) \in \mathbf{R}^2 : x^2 + y^2 > 1\}$ in polar coordinates:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0,$$

assume that $u(r, \theta)$ is continuous for $1 \leq r < \infty$ and $\theta \in \mathbf{R}$, and write

$$\widehat{u}(r, n) = \int_0^{2\pi} e^{-in\theta} u(r, \theta) d\theta.$$

a) Show that if u is bounded then

$$|\widehat{u}(r, n)| \leq 2\pi \sup\{|u(r, \theta)| : r > 1,\ \theta \in \mathbf{R}\}.$$

b) Assuming that $\widehat{u}(r, n) = A_n r^{-|n|} + B_n r^{|n|}$ if $n \neq 0$ and $\widehat{u}(r, 0) = A_0 + B_0 \log r$, show that $B_n = 0$ for all n.

c) Show that

$$\sum_{n=1}^{\infty} r^{-n}(e^{in\theta} + e^{-in\theta}) + 1 = \frac{1 - r^{-2}}{|1 - r^{-1}e^{i\theta}|^2}.$$

d) Deduce that

$$u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^{-2}}{|1 - r^{-1}e^{i\theta - t}|^2} u(1, t) dt.$$