MATH348. Harmonic Analysis. Problems 3.

Work due on *Wednesday 20th October.* Formal office hours ar 11-1 on Mondays and 3-5 on Tuesdays.

1. Compute $f * g$ for 2π-periodic functions on $\mathbf{R}$, where

a) $f(x) = e^{ix} = g(x)$,

b) $f(x) = e^{ix}$, $g(x) = e^{2ix}$.

c) $f(x) = e^{inx}$, $g(x) = e^{imx}$, any n, $m\mathbf{Z}$.

Hint: the cases $n = m$ and $n \neq m$ are probably best treated separately.

2. Let $f : [0, 2\pi) \to \mathbf{R}$ be defined by

$$f(x) = \begin{matrix} x \text{ if } 0 \leq x \leq \pi, \\ \pi \text{ if } \pi \leq x < 2\pi. \end{matrix}$$

a) Sketch the 2π-periodic extension of f, which, in future, will also be called f.

b) Let $0 < x < \pi$. Write down the formula for $f(x - u)$ for $x - \pi < u < \pi + x$. You will need to consider separately the cases $x - \pi \leq u \leq x$, $x < u \leq \pi + x$.

c) Let $x_n = \pi/(n + \frac{1}{2})$, and show that

$$\lim_{n\to\infty} \int_{x_n}^{x_n+\pi} \frac{\sin{(n + \frac{1}{2})u}}{u} du = \int_{\pi}^{\infty} \frac{\sin t}{t} dt.$$

d) Now, (as usual) let

$$s_n(u) = \frac{\sin{(n + \frac{1}{2})u}}{2\pi \sin\frac{1}{2}u}, \; S_n(f)(x) = \int_{x-\pi}^{x+\pi} f(x-u) s_n(u) du \left(= \int_0^{2\pi} f(x - u) s_n(u) du. \right)$$

Using c), show that

$$\lim_{n\to\infty} S_n(f)(x_n) = \int_{\pi}^{\infty} \frac{\sin t}{t} dt +$$

$$\lim_{n\to\infty} \left(\int_{x_n-\pi}^{x_n} (x_n - u) s_n(u) du + \int_{x_n}^{x_n+\pi} \sin{(n + \tfrac{1}{2})u} \left(\frac{1}{2\sin\frac{1}{2}u} - \frac{1}{u} \right) du \right).$$

[The point of this question is that the righthand limit term is 0, and the first term on the right is < 0. So this shows that $\lim_{n\to\infty} S_n(f)(x_n) < 0$, despite the facts that $f(0) = 0$, $(f(0+) + f(0-)) > 0$ and $\lim_{n\to\infty} x_n = 0$.]

3. Show that the function

$$\lim_{u\to 0} \frac{1}{2\sin\frac{1}{2}u} - \frac{1}{u} = 0$$

and hence that the functon is continuous and bounded on $[-\pi, 0) \cup (0, \pi]$. What does the Riemann Lebesgue Lemma then say about

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} \sin(n + \tfrac{1}{2})u \left(\frac{1}{2\sin\frac{1}{2}u} - \frac{1}{u} \right) du?$$