**MATH348. Harmonic Analysis. Problems 3.**

Work due on *Wednesday 20th October.* Formal office hours are 11-1 on mondays and 3-5 on Tuesdays.

1. Compute $f * g$ for $2\pi$-periodic functions on $\mathbf{R}$, where

a) $f(x) = e^{ix} = g(x)$,

b) $f(x) = e^{ix}$, $g(x) = e^{2ix}$.

c) $f(x) = e^{inx}$, $g(x) = e^{imx}$, any $n$, $m\mathbf{Z}$.

*Hint:* the cases $n = m$ and $n \neq m$ are probably best treated separately.

2. Let $f : [0, 2\pi) \to \mathbf{R}$ be defined by

$$f(x) = \begin{matrix} x \text{ if } 0 \leq x \leq \pi, \\ \pi \text{ if } \pi \leq x < 2\pi. \end{matrix}$$

a) Sketch the $2\pi$-periodic extension of $f$, which, in future, will also be called $f$.

b) Let $0 < x < \pi$. Write down the formula for $f(x - u)$ for $x - \pi < u < \pi + x$. You will need to consider separately the cases $x - \pi \leq u \leq x$, $x < u \leq \pi + x$.

c) Let $x_n = \pi/(n + \frac{1}{2})$, and show that

$$\lim_{n\to\infty} \int_{x_n}^{x_n+\pi} \frac{\sin(n + \frac{1}{2})u}{u} du = \int_{\pi}^{\infty} \frac{\sin t}{t} dt.$$

d) Now, (as usual) let

$$s_n(u) = \frac{\sin(n + \frac{1}{2})u}{2\pi \sin \frac{1}{2}u}, \quad S_n(f)(x) = \int_{x-\pi}^{x+\pi} f(x-u)s_n(u)du \left(= \int_0^{2\pi} f(x - u)s_n(u)du.\right)$$

Using c), show that

$$\lim_{n\to\infty} S_n(f)(x_n) = \int_{\pi}^{\infty} \frac{\sin t}{t} dt +$$

$$\lim_{n\to\infty} \left( \int_{x_n-\pi}^{x_n} (x_n - u)s_n(u)du + \int_{x_n}^{x_n+\pi} \sin(n + \tfrac{1}{2})u \left( \frac{1}{2\sin \frac{1}{2}u} - \frac{1}{u} \right) du \right).$$

[The point of this question is that the righthand limit term is $0$, and the first term on the right is $< 0$. So this shows that $\lim_{n\to\infty} S_n(f)(x_n) < 0$, despite the facts that $f(0) = 0$, $(f(0+) + f(0-)) > 0$ and $\lim_{n\to\infty} x_n = 0$.]

3. Show that the function

$$\lim_{u\to 0} \frac{1}{2\sin \frac{1}{2}u} - \frac{1}{u} = 0$$

and hence that the functon is continuous and bounded on $[-\pi, 0) \cup (0, \pi]$. What does the Riemann Lebesgue Lemma then say about

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} \sin(n + \tfrac{1}{2})u \left( \frac{1}{2\sin \frac{1}{2}u} - \frac{1}{u} \right) du?$$

**MATH348. Harmonic Analysis. Solutions 3.**

1a)
$$f * g(x) = \int_{-\pi}^{\pi} e^{i(x-y)} e^{iy} dy = 2\pi e^{ix}.$$

b)
$$f * g(x) = \int_{-\pi}^{\pi} e^{i(x-y)} e^{2iy} dy = e^{ix} \left[ \frac{e^{iy}}{i} \right]_{-\pi}^{\pi} = 0.$$

c) If $n = m$,
$$f * g(x) = \int_{-\pi}^{\pi} e^{in(x-y)} e^{iny} dy = e^{inx} \int_{-\pi}^{\pi} dy = 2\pi e^{inx}.$$

If $n \neq m$,
$$f * g(x) = \int_{-\pi}^{\pi} e^{in(x-y)} e^{imy} dy = e^{inx} \int_{\pi}^{\pi} e^{i(m-n)y} dy = 0.$$

2a) The $2\pi$-periodic extension is as shown.

b)
$$f(x-u) = \begin{array}{l} x - u \text{ if } x - \pi \le u \le x, \\ \pi \text{ if } x < u \le x + \pi. \end{array}$$

c) Let $t = (n + \frac{1}{2})u$. Then $du/u = dt/t$. If $u = x_n = \pi/(n+\frac{1}{2})$ then $t = \pi$. If $u = x_n + \pi$ then $t = \pi + (n + \frac{1}{2})\pi \to \infty$ as $n \to \infty$. So
$$\lim_{n\to\infty} \int_{x_n}^{x_n+\pi} \frac{\sin{(n+\frac{1}{2})u}}{u} du = \int_{\pi}^{\infty} \frac{\sin t}{t} dt.$$

d)
$$S_n(f)(x_n) = \int_{x_n-\pi}^{x_n} (x_n - u) s_n(u) du + \int_{x_n}^{x_n+\pi} \pi \sin{(n+\tfrac{1}{2})u} \left( \frac{1}{2\pi \sin \frac{1}{2}u} - \frac{1}{\pi u} \right)$$
$$+ \int_{x_n}^{x_n+\pi} \frac{\sin{(n+\frac{1}{2})u}}{u} du.$$

By c), the limit of the last term on the right is
$$\int_{\pi}^{\infty} \frac{\sin t}{t} dt.$$

So

$$\lim_{n\to\infty} S_n(f)(x_n) = \int_\pi^\infty \frac{\sin t}{t} dt +$$

$$\lim_{n\to\infty} \left( \int_{x_n-\pi}^{x_n} (x_n - u) s_n(u) du + \int_{x_n}^{x_n+\pi} \sin(n+\tfrac{1}{2})u \left( \frac{1}{2\sin\frac{1}{2}u} - \frac{1}{u} \right) du \right),$$

as required.

3. By l'Hôpital's rule

$$\lim_{u\to 0} \frac{1}{2\sin\frac{1}{2}u} - \frac{1}{u}$$

$$= \lim_{u\to 0} \frac{u - 2\sin\frac{1}{2}u}{2u\sin\frac{1}{2}u} = \lim_{u\to 0} \frac{1-\cos\frac{1}{2}u}{u\cos\frac{1}{2}u + 2\sin\frac{1}{2}u}$$

$$= \lim_{u\to 0} \frac{\sin\frac{1}{2}u}{4\cos\frac{1}{2}u - 2u\sin\frac{1}{2}u} = 0.$$

Alternatively, we can use power series:

$$\lim_{u\to 0} \frac{u - 2\sin\frac{1}{2}u}{2u\sin\frac{1}{2}u} = \lim_{u\to 0} \frac{u - 2(\frac{u}{2} - \frac{u^3}{8\times 24}\cdots)}{2u(\frac{u}{2} - \frac{u^3}{8\times 24}\cdots)}$$

$$= \lim_{u\to 0} \frac{\frac{u^3}{4\times 24} + \cdots}{u^2 - \frac{u^4}{4\times 24} + \cdots} = \lim_{u\to 0} \frac{\frac{u}{4\times 24} + \cdots}{1 - \frac{u^2}{4\times 24} + \cdots} = 0.$$

The denominator of

$$\frac{u - 2\sin\frac{1}{2}u}{2u\sin\frac{1}{2}u}$$

is nonzero on $[-\pi, \pi]$ except at $0$. So the function is continuous on $[-\pi . \pi]$ if we define it to be $0$ at $0$. Then is is integrable, because a continuous funciton on a closed bounded interval is always integrable. Then the Riemann Lebesgue Lemma says that

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} \sin(n+\tfrac{1}{2}u) \left( \frac{1}{2\sin\frac{1}{2}u} - \frac{1}{u} \right) du = 0$$