MATH348.Harmonic Analysis. Problems 10.

Work is due in on *Wednesday 8th December.*

1. Find the mean and variance of the probability measure μ_j in each of the following cases.

a)
$$\mu_1(\{1\}) = \mu_1(\{0\}) = \mu(\{-1\}) = \frac{1}{3}.$$

b) μ_2 has density function f where

$$f(x) = \begin{matrix} \frac{1}{2} \text{ if } -1 \le x \le 1, \\ 0 \text{ otherwise.} \end{matrix}$$

c) μ_3 has density function g, where

b)
$$g(x) = \frac{2}{\pi(1+x^2)^2}.$$

To compute the variance in this case, you may use

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} = 2\pi i \,\text{res}\left(\frac{1}{(1+z^2)^2}, i\right).$$

2a) Find $\hat{\mu}_j(\xi)$ for $j = 1,\ 2,\ \mu_j$ as in question 1.

3a) Let μ_1 be as in questions 1 and 2. Find the measure $\mu_1 * \mu_1 * \mu_1$ by first working out $(\hat{\mu}_j(\xi))^3$

b) Let the probability measure ν_n on $\mathbf{R}$ be defined by

$$\nu_n(A) = \int_{-\infty}^{\infty} \chi_A(x/\sqrt{n})d(*^n\mu_1)$$

Show that

$$\hat{\nu}_n(\xi) = (\hat{\nu}(\xi/\sqrt{n}))^n.$$

c) Find a power series expansion up to and including the ξ^4 term for

$$n\ln(\hat{\nu}(\xi/\sqrt{n})).$$

Hence or otherwise show that for any fixed ξ

$$\lim_{n\to\infty} \ln \hat{\nu}_n(\xi) = -\xi^2/3.$$

and

$$\lim_{n\to\infty} \hat{\nu}_n(\xi) = e^{-\xi^2/3}.$$

Relate this to what the Central limit Theorem says about

$$\lim_{n\to\infty} \nu_n(A)$$

for any measurable set $A \subset \mathbf{R}$.