MATH348.Harmonic Analysis. Problems 10.

Work is due in on *Wednesday 8th December.*

1. Find the mean and variance of the probability measure μ_j in each of the following cases.

a)
$$\mu_1(\{1\}) = \mu_1(\{0\}) = \mu(\{-1\}) = \frac{1}{3}.$$

b) μ_2 has density function f where

$$f(x) = \begin{array}{c} \frac{1}{2} \text{ if } -1 \leq x \leq 1, \\ 0 \text{ otherwise.} \end{array}$$

c) μ_3 has density function g, where

b)
$$g(x) = \frac{2}{\pi(1+x^2)^2}.$$

To compute the variance in this case, you may use

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} = 2\pi i \,\text{res}\left(\frac{1}{(1+z^2)^2}, i\right).$$

2a) Find $\hat{\mu}_j(\xi)$ for $j = 1,\ 2,\ \mu_j$ as in question 1.

3a) Let μ_1 be as in questions 1 and 2. Find the measure $\mu_1 * \mu_1 * \mu_1$ by first working out $(\hat{\mu}_j(\xi))^3$

b) Let the probability measure ν_n on $\mathbf{R}$ be defined by

$$\nu_n(A) = \int_{-\infty}^{\infty} \chi_A(x/\sqrt{n})d(*^n\mu_1)$$

Show that

$$\hat{\nu}_n(\xi) = (\hat{\mu}_1(\xi/\sqrt{n}))^n.$$

c) Find a power series expansion up to and including the ξ^4 term for

$$n\ln(\hat{\mu}_1(\xi/\sqrt{n})).$$

Hence or otherwise show that for any fixed ξ

$$\lim_{n\to\infty} \ln \hat{\nu}_n(\xi) = -\xi^2/3.$$

and

$$\lim_{n\to\infty} \hat{\nu}_n(\xi) = e^{-\xi^2/3}.$$

Relate this to what the Central limit Theorem says about

$$\lim_{n\to\infty} \nu_n(A)$$

for any measurable set $A \subset \mathbf{R}$.

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1a) The m_1 satisfies

$$m_1 = 1 \times \frac{1}{3} + 0 \times \frac{1}{3} + (-1) \times \frac{1}{3} = 0.$$

The variance σ_1 satisfies

$$\sigma_1 = 1 \times \frac{1}{3} + 0 \times \frac{1}{3} + 1 \times \frac{1}{3} = \frac{2}{3}.$$

b) The mean m_2 and variance σ_2 satisfy

1b) $$m_2 == \int_{-1}^{1} \tfrac{1}{2} x dx = \left[\frac{x^2}{2}\right]_{-1}^{1} = 0,$$

$$\sigma_2 == \int_{-1}^{1} \tfrac{1}{2}(x-0)^2 dx = \left[\frac{x^3}{6}\right]_{-1}^{1} = \frac{1}{3}.$$

c) The mean m_3 and σ_3 satisfy

1c) $$m_3 = \int_{-\infty}^{\infty} \frac{2x dx}{\pi(1+x^2)^2} = \lim_{\Delta\to+\infty} \int_{-\Delta}^{\Delta} \frac{x dx}{(1+x^2)^2} = 0$$

because the integrand is an odd function.

$$\sigma_3 = \int_{-\infty}^{\infty} \frac{2(x-0)^2}{\pi(1+x^2)^2} = \int_{-\infty}^{\infty} \frac{2dx}{\pi(1+x^2)} - \int_{-\infty}^{\infty} \frac{2dx}{\pi(1+x^2)^2}$$

$$= \lim_{\Delta\to+\infty} \left[\frac{2\arctan x}{\pi}\right]_{-\Delta}^{\Delta} - \frac{2}{\pi} \lim_{R\to\infty} \int_{\gamma(xR)} \frac{dz}{(1+z^2)^2},$$

where γ_R is the semicircular contour of radius R in the upper half-plane. To see this we need that the integral over the curved part of the contour $\gamma'(R)$ tends to 0. But

$$\left| \int_{\gamma'(R)} \frac{dz}{(1+z^2)^2} \right| \leq \frac{\pi R}{(R^2-1)^2} \to 0 \text{ as } R \to \infty.$$

The only singularity of $(1+z^2)^{-2}$ inside $\gamma(R)$ is at i, and this is a double pole, and $(z^2+1)^2 = (z+i)^2(z-i)^2$. So

$$\sigma_3 = \lim_{\Delta\to+\infty} \left[\frac{2\arctan x}{\pi}\right]_{-\Delta}^{\Delta} - 4i \operatorname{Res}\left(\frac{1}{(1+z^2)^2}, i\right)$$

$$= 2 - 4i \operatorname{Res}\left(\frac{1}{(z-i)^2(z+i)^2}, i\right) = \pi - -i\frac{d}{dz}\left(\frac{1}{(z+i)^2}\right) \Big|_{z=i}$$

$$= 2 - 4i\frac{-2}{(2i)^3} = 2 - 1 = 1.$$

2a) $$\widehat{\mu}_1(\xi) = \frac{e^{i\xi} + 1 + e^{-i\xi}}{3}$$

b). We have $\widehat{\mu}_2(\xi) = \widehat{f}(\xi)$ for all ξ. So

$$\widehat{\mu}_2(\xi) = \int_{-1}^{1} \tfrac{1}{2} e^{-ix\xi}\, dx = \frac{e^{i\xi} - e^{-i\xi}}{2i\xi}.$$

3*a*)

$$(\widehat{\mu}_1(\xi))^3 = \frac{1}{27}(e^{ix\xi}+1+e^{-ix\xi})^3 = \frac{1}{27}(e^{2ix\xi}+2e^{ix\xi}+3+2e^{-ix\xi}+e^{-3ix\xi})(e^{ix\xi}+1+e^{-ix\xi})$$

$$= \frac{1}{27}(e^{3ix\xi} + 3e^{2ix\xi} + 6e^{ix\xi} + 7 + 6e^{-ix\xi} + 3e^{-2ix\xi} + e^{-3ix\xi})$$

So, writing $\lambda = \mu_1 * \mu_1 * \mu_1$

$$\lambda(\{3\}) = \lambda(\{-3\}) = \frac{1}{27},\ \lambda(\{2\}) = \lambda(\{-2\}) = \frac{1}{9},\ \lambda(\{1\}) = \lambda(\{-1\}) = \frac{2}{9},\ \lambda(\{0\}) = \frac{7}{27}.$$

b) Let $\lambda_n = *^n \mu_1$. Then $\hat{\lambda}_n(\xi) = (\hat{\mu}_1(\xi))^n$. We have

$$\hat{\nu}_n(\xi) = \int_{-\infty}^{\infty} e^{-x\xi} d\nu_n(x)) = \int_{-\infty}^{\infty} e^{-\xi/\sqrt{n}} d\lambda_n(x)$$

$$= \hat{\lambda}_n(\xi/\sqrt{n}) = (\hat{\mu}_1(\xi/\sqrt{n}))^n.$$

c)

$$\hat{\mu}_1(\xi/\sqrt{n}) = \frac{1}{3}(1 + e^{i\xi/\sqrt{n}} + e^{-i\xi/\sqrt{n}})$$

$$= \frac{1}{3}\left(1 + 1 + \frac{i\xi}{\sqrt{n}} - \frac{\xi^2}{2n} - \frac{i\xi^3}{6n\sqrt{n}} + \frac{\xi^4}{24n^2} + \cdots + 1 - \frac{i\xi}{\sqrt{n}} - \frac{\xi^2}{2n} + \frac{i\xi^3}{6n\sqrt{n}} + \frac{\xi^4}{24n^2} + \cdots\right)$$

$$= 1 - \frac{\xi^2}{3n} + \frac{\xi^4}{36n^2} + \cdots.$$

Now

$$\ln(1+t) = t - \frac{t^2}{2} + \cdots$$

So

$$n \ln \hat{\mu}_1(\xi/\sqrt{n}) = -\frac{\xi^2}{3} + \frac{\xi^4}{36n} - \frac{n}{2}\left(-\frac{2\xi^2}{3n} + \frac{\xi^4}{36n^2} + \cdots\right)^2 + \cdots$$

$$= -\frac{\xi^2}{3} - \frac{5\xi^n}{36n} + \cdots$$

So

$$\lim_{n\to\infty} \ln \hat{\nu}_n(\xi) = \lim_{n\to\infty} n\hat{\mu}_1(\xi/\sqrt{n}) = -\frac{2\xi^2}{3}.$$

Taking exponentials,

$$\lim_{n\to\infty} (\hat{\nu}_n(\xi) = e^{-\xi^2/3}.$$

So the Fourier transform of ν_n converges to the Fourier transform of $\frac{\sqrt{3}}{2\sqrt{\pi}} e^{-3x^2/4}$, the density function of the normal density function with mean 0 and variance $2/3$. The Central limit Theorem says that for any measurable set A,

$$\lim_{n\to\infty} \nu_n(A) = \int_{-\infty}^{\infty} \frac{\sqrt{3}}{2\sqrt{\pi}} e^{-3x^2/4} dx.$$