MATH348 Solutions

January 4, 2005

1.(i)

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-ix\xi} dx.$$

[2 marks]

Now let

$$f(x) = \frac{1}{(x^2 + a^2)}.$$

To compute the Fourier transform, we consider the function

$$f(z) = \frac{e^{-iz\xi}}{(z^2 + a^2)}.$$

[2 marks]

Let $\xi \geq 0$. If $\mathrm{Im}(z) \leq 0$ then $|e^{-iz\xi}| = e^{\mathrm{Im}(z)\xi} \leq 1$. So let $\gamma_R = \gamma_1(R) \cup \gamma_2(R)$ be the anticlockwise contour in the lower half plane, with $\gamma_1(R)$ being the straightline from R to $-R$ and $\gamma_2(R)$ being the semicircle arc. We have $|z^2 + a^2| \geq |z|^2 - |a|^2$. So

$$|f(z)| \leq \frac{1}{(R^2 - |a|^2)} \text{ for } z \in \gamma_2(R).$$

[4 marks]

So

$$|\int_{\gamma_2(R)} \frac{e^{-iz\xi}}{(z^2 + a^2)} dz| \leq \frac{\pi R}{(R^2 - |a|^2)} \to 0 \text{ as } R \to \infty.$$

[2 marks]

We have

$$(z^2 + a^2) = (z - ai)(z + ai) = 0$$

if and only if $z = \pm ai$. So the only singularity of f inside γ_R is at $-ai$. So

$$\int_{\gamma_R} f(z) dz = 2\pi i \mathrm{Res}(f(z), -ai) = 2\pi i (z - ai)^{-1} e^{-i\xi z})_{z=-ai}$$

$$= -\frac{\pi e^{-a\xi}}{a}.$$

[3 marks]
So

$$\hat{f}(\xi) = -\lim_{R\to\infty}\int_{\gamma_1(R)} f(z)dz$$

$$= -\lim_{R\to\infty}\int_{\gamma(R)} f(z)dz = \frac{\pi e^{-a\xi}}{a}.$$

[2 marks]
Now since $f(x)$ is real for real x,

$$\hat{f}(-\xi) = \int_{-\infty}^{\infty} f(x)e^{ix\xi}dx$$

$$= \overline{\int_{-\infty}^{\infty} f(x)e^{-ix\xi}dx} = \overline{\hat{f}(\xi)}.$$

So for all ξ,

$$\hat{f}(\xi) = \frac{\pi e^{-a|\xi|}}{a}$$

[2 marks]
1(ii) We have

$$g(x) = \frac{1}{3}\left(\frac{1}{x^2+1} - \frac{1}{x^2+4}\right).$$

So using (i), we have

$$\hat{f}(\xi) = \pi\left(\frac{e^{-|\xi|}}{3} - \frac{e^{-2|\xi|}}{6}\right).$$

[3 marks]
[2+2+4+2+3+2+2+3=20 marks]
Standard homework exercise - apart from (ii), which is of course a technique familiar since A-level.

2.a)

We have

$$f_1(x) = e^{-x^2} = |e^{-x^2}| \le e^{-|x|} + \chi_{(-1,1)}(x)$$

where, as usual, $\chi_{(-1,1)}$ denotes the characteristic function of $(-1,1)$. Now $\chi_{(-1,1)}$ is certainly integrable, so f_1 is integrable provided that $g(x) = e^{-|x|}$ is. This is true becaue if we take $g_n(x) = e^{-|x|}\chi_{(-n,n)}$ then $g_n(x) \le g_{n+1}(x) \to g(x)$ as $n \to \infty$. Soby the Monotone Convergence Theorem,

$$\int g = \lim_{n\to\infty} g_n = \lim_{n\to\infty} 2\int_0^n e^{-x}dx = \lim_{n\to\infty}\left[-e^{-x}\right]_0^n$$

$$= \lim_{n\to\infty} 2(1-e^{-n}) = 2 < +\infty.$$

So f_1 is integrable.
[3 marks]

We have

$$|f_2(x)| = |e^{ix}e^{-x^2}| = e^{-x^2} = f_1(x).$$

So since f_1 is integrable, so is f_2.
[2 marks]

For $x \in (0,1)$, we have $e^{-x^2} \ge e^{-1}$. So

$$\int_{-\infty}^{\infty} |f_3(x)|dx \ge \int_0^1 \frac{e^{-1}}{x}dx$$

$$= e^{-1}\lim_n \int_{1/n}^1 \frac{dx}{x} = e^{-1}\lim_{n\to\infty}[\log x]_{1/n}^1 = e^{-1}\lim_{n\to\infty}\log n = +\infty$$

using the Monotone Convergence Theorem with the sequence $x^{-1}\chi_{(1/n,1)}(x)$. So f_3 is not integrable.
[4 marks]

b)

Tonelli's Theorem. Suppose that $F : \mathbf{R}^2 \to \mathbf{C}$ is Lebesgue measurable and one of the double integrals

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |F(x,y)|dxdy < +\infty, \quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |F(x,y)|dydx < +\infty.$$

Then both of the functions

$$x \mapsto \int_{-\infty}^{\infty} F(x,y)dy, \quad y \mapsto \int_{-\infty}^{\infty} F(x,y)dx$$

are defined a.e. and integrable and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(x,y)dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(x,y)dydx.$$

[5 marks] Now we apply Tonelli's Theorem to the function

$$F(x,y) = f(x-y)g(y)e^{-inx}\chi_{(-\pi,\pi)}(x)\chi_{(-\pi,\pi)}(y)$$

where f and g are integrable on $(-\pi,\pi)$.

We have

$$\int |F(x,y)|dxdy = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |f(x-y)||g(y)|dxdy$$

$$= \int_{-\pi}^{\pi} |g(y)| \int_{-\pi}^{\pi} |f(x-y)|dxdy$$

$$= \int_{-\pi}^{\pi} |g(y)| \int_{-\pi+y}^{\pi+y} |f(t)|dt$$

(putting $t = x - y$)

$$= \int_{-\pi}^{\pi} |g(y)| \int_{-\pi}^{\pi} |f(t)|dtdy = \int_{-\pi}^{\pi} |g(y)|dy \int_{-\pi}^{\pi} |f(t)|dt < +\infty$$

(using 2π-periodicity for the first equality).

[3 marks] So we can perform the integration of $F(x,y)$ either way round and the two integrals are equal. If we do x first then again putting $t = x - y$ and using 2π-periodicity of the integrand, we have

$$\int_{-\pi}^{pi}\int_{-\pi}^{pi} F(x,y)dxdy = \int_{-\pi}^{pi}\int_{-\pi} e^{-in(y+t} f(tg(y)dtdy$$

$$= \int_{-\pi}^{pi} e^{-iny}g(y)dy \int_{-\pi}^{pi} e^{-int}f(t)dt = \hat{f}(n)\hat{g}(n)$$

as required.

[3 marks]

$3+2+4+5+3+3 = 20$ marks.

Similar to homework exercise, followed by theory from lectures. There were also homework exercises on using Tonelli's Theorem.

3. (i) To show that g and h are continuous it suffices to show the function extend continuously at 0, since the functions are defined and continuous everywhere else on $(-2\pi, 2\pi)$. For this it suffices to show that

$$\lim_{y\to 0} \frac{y}{2\sin\frac{1}{2}y}$$

and

$$\lim_{y\to 0} \frac{1}{2\sin\frac{1}{2}y} - \frac{1}{y} = \lim_{y\to 0} \frac{y - 2\sin\frac{1}{2}y}{2y\sin\frac{1}{2}y}$$

exist. The first limit is 1, a well-known limit but one can also use the Taylor series expansion of $2\sin\frac{1}{2}y$ about 0, which starts $y - y^3/24\cdots$. This can also be used for computing the second limit, because it gives

$$\lim_{y\to 0} \frac{y - 2\sin\frac{1}{2}y}{2y\sin\frac{1}{2}y} = \lim_{y\to 0} \frac{y^3/24\cdots}{y^2\cdots} = 0.$$

So the limit exists and = 0, and the function does extend continuously.
[5 marks]

(ii) We have $f(x-y) = 1+x-y$ if $0 \le x-y \le \pi$, that is, if $-\pi + x \le y \le x$, and $f(x-y) = -1$ if $-\pi < x-y < 0$, that is, if $x < y < x+\pi$.
[2 marks]

So we have

$$S_n(f)(x) = \int_{x-\pi}^{x} (1+x-y)s_n(y)dy + \int_{x}^{x+\pi} (-1)s_n(y)dy$$

$$= -\frac{1}{\pi}\int_{x-\pi}^{x} g(y)\sin((n+\tfrac{1}{2})y)dy + \left((1+x)\int_{x-\pi}^{x} - \int_{x}^{x+\pi}\right) s_n(y)dy$$

$$= -\frac{1}{\pi}\int_{x-\pi}^{x} g(y)\sin((n+\tfrac{1}{2})y)dy + \frac{1}{\pi}\left((1+x)\int_{x-\pi}^{x} - \int_{x}^{x+\pi}\right) h(y)\sin((n+\tfrac{1}{2})y)dy$$

$$+\frac{1}{\pi}\left((1+x)\int_{x-\pi}^{x} - \int_{x}^{x+\pi}\right) \frac{\sin((n+\frac{1}{2})y)}{y}dy.$$

[5 marks]

The Fourier Series Theorem says that for $0 < x < \pi$,

$$\lim_{n\to\infty} S_n(f)(x) = 1 + x.$$

[1 mark]

(iii) By our assumptions,

$$\lim_{n\to\infty} \int_{x_n-\pi}^{x_n} g(y)\sin((n+\tfrac{1}{2})y)dy = 0,$$

$$\lim_{n\to\infty} \int_{x_n-\pi}^{x_n} h(y)\sin((n+\tfrac{1}{2})y)dy = 0 = \lim_{n\to\infty} -\int_{x_n}^{x_n+\pi} h(y)\sin((n+\tfrac{1}{2})y)dy.$$

Also, $\lim_{n\to\infty} x_n = 0$. So

$$\lim_{n\to\infty} S_n(f)(x_n) = \lim_{n\to\infty} \frac{1}{\pi}\left(\int_{x_n-\pi}^{x_n} - \int_{x_n}^{x_n+\pi}\right) \frac{\sin((n+\frac{1}{2})y)}{y} dy.$$

[3 marks]

Putting $(n+\frac{1}{2})y = t$ we have $(n+\frac{1}{2})dy = dt$ and so $dy/y = dt/t$. When $y = x_n$, $t = \pi$. When $y = x_n + \pi$, $t = (n+\frac{3}{2})\pi$, and when $y = x_n - \pi$, $t = -(n-\frac{1}{2})\pi$. So

$$\lim_{n\to\infty} S_n(f)(x_n) = \lim_{n\to\infty}\left(\int_{-(n-\frac{1}{2})\pi}^{\pi} - \int_{\pi}^{(n+\frac{3}{2})\pi}\right) \frac{\sin t}{\pi t} dt$$

$$= \frac{2}{\pi}\int_0^{\pi} \frac{\sin t}{t} dt + \frac{1}{\pi}\lim_{\Delta\to+\infty}\int_{-\Delta}^{0} \frac{\sin t}{t} dt - \frac{1}{\pi}\lim_{\Delta\to+\infty}\int_0^{\Delta} \frac{\sin t}{t} dt$$

$$= \frac{2}{\pi}\int_0^{\pi} \frac{\sin t}{t} dt,$$

Where the last equality uses that $(\sin t)/t$ is an even function.

[4 marks]

$5 + 2 + 5 + 1 + 3 + 4 = 20$ marks.

Similar to homework exercise.

4.(i) If $G(z)$ is a holomorphic function of z, then for $z = x + iy$ we can write $G(z) = u(x, y) + iv(x, y)$ for real-valued functions u and v, and then the Cauchy-Riemann equations give $u_x = v_y$ and $u_y = -v_x$. Then $u_{xx} = v_{xy} = v_{yx} = -u_{yy}$ and $u_{xx} + u_{yy} = 0$. the function $(1 + z)/(1 - z)$ is holmorphic for $z \neq 1$. So we do indeed have $u_{xx} + u_{yy} = 0$ for $u(x, y) = P(r, \theta)$.
[5 marks]
(ii) For $z = re^{i\theta}$, we have

$$\frac{1 + re^{i\theta}}{1 - re^{i\theta}} = \frac{(1 + re^{i\theta})(1 - re^{-i\theta})}{(1 - re^{i\theta})(1 - re^{-i\theta})}$$

$$= \frac{1 - r^2 + 2ir\sin\theta}{|1 - re^{i\theta}|^2}$$

So taking the real part we obtain

$$P(r, \theta) = \frac{1}{2\pi} \frac{1 - r^2}{|1 - re^{i\theta}|^2}.$$

[3 marks]
Now for $0 \leq r < 1$,

$$\sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta} = \sum_{n=0}^{\infty} (re^{i\theta})^n + \sum_{n=0}^{\infty} (re^{-i\theta})^n - 1$$

$$= \frac{1}{1 - re^{i\theta}} + \frac{1}{1 - re^{-i\theta}} - 1$$

$$= \frac{1 - re^{-i\theta} + 1 - re^{i\theta} - (1 - re^{i\theta})(1 - re^{-i\theta})}{(1 - re^{i\theta})(1 - re^{-i\theta})}$$

$$\frac{= 2 - re^{-i\theta} - re^{i\theta} - 1 + re^{-i\theta} + re^{i\theta} - r^2}{(1 - re^{i\theta})(1 - re^{-i\theta})}$$

$$= \frac{1 - r^2}{|1 - re^{i\theta}|^2}.$$

[4 marks]
So

$$\int_{-\pi}^{\pi} P(r, \theta) d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta} d\theta$$

$$= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} r^{|n|} e^{in\theta} d\theta = 1$$

because for any integer $n \neq 0$,

$$\int_{-\pi}^{\pi} r^{|n|} e^{in\theta} d\theta = 0.$$

[2 marks]
(iii) The equation

$$\frac{\partial^2 P}{\partial r^2} + \frac{1}{r}\frac{\partial P}{\partial r} + \frac{1}{r^2}\frac{\partial^2 P}{\partial \theta^2} = 0$$

also holds with $P(r,\theta)$ replaced by $P(r,\theta-t)$ for any t, and then also for this multiplied by $f(t)$ and integrated from $-\pi$ to π. So the equation holds with P replaced by F.
[4 marks]
Now

$$F(r,\theta) - f(\theta) = \int_{-\pi}^{\pi} f(t)P(r,\theta-t)dt - f(\theta)\int_{-\pi}^{\pi} P(r,t)dt$$

$$= \int_{-\pi}^{\pi} f(\theta-t)P(r,t)dt - f(\theta)\int_{-\pi}^{\pi} P(r,t)dt$$

$$= \int_{-\pi}^{\pi} (f(\theta-t) - f(\theta))P(r,t)dt$$

[2 marks]
$5+3+4+2+4+2 = 20$ marks.

Theory from lectures with some calculations also carried out in a homework exercise on Laplace equation in the disc complement.

5. (i) We have

$$\widehat{g}_{a,b}(\xi) = \int_{-\infty}^{\infty} g((x-a)/b)e^{-i\xi x}dx.$$

Putting $t = (x-a)/b$ gives $x = tb + a$ and $dx = b.dt$. Since $b > 0$, the limits of integration do not change. So we have

$$\widehat{g}_{a,b}(\xi) = b\int_{-\infty}^{\infty} g(t)e^{-i(\xi b)t - ia\xi}dt = b\widehat{g}(\xi b)e^{-ia\xi}.$$

[3 marks]
(ii) We have

$$\widehat{u}_x(\xi,t) = i\xi\widehat{u}(\xi,t), \quad \widehat{u}_{xx}(\xi,t) = -\xi^2\widehat{u}(\xi,t)$$
$$\widehat{u}_t(\xi,t) = (\partial/\partial t)\widehat{u}(\xi,t).$$

So taking the transforms of (3) and (4), we obtain

$$(\partial/\partial t)\widehat{u} = (1 + i\xi - \xi^2)\widehat{u},$$

$$\widehat{u}(\xi,0) = \widehat{f}(\xi).$$

[3 marks]
The general solution to this is

$$\widehat{u}(\xi,t) = e^{t+i\xi t-\xi^2 t}\widehat{f}(\xi).$$

[1 mark]
Now we need to determine a function $g(x,t)$ such that

$$\widehat{g}(\xi,t) = e^{t+i\xi t-\xi^2 t} = e^t e^{i\xi t-(\xi\sqrt{2t})^2/2}$$

Since we are given that the Fourier transform of $e^{-x^2/2}$ is $\sqrt{2\pi}e^{-\xi^2/2}$, we have, by (i)

$$g(x,t) = e^t\frac{1}{2\sqrt{\pi t}}e^{-(x+t)^2/4t}.$$

The Fourier transform of a convolution is the product of Fourier transforms and any intgrable funciton is uniqely determined by its Fourier transform. So we have

$$u(x,t) = e^t\int_{-\infty}^{\infty} f(y)\frac{1}{2\sqrt{\pi t}}e^{-(x+t-y)^2/4t}dy.$$

[5 marks]
(iii) *Dominated Convergence Theorem.* Let $f_n : \mathbf{R} \to \mathbf{C}$ be a sequence of functions with $\lim_{n\to\infty} f_n(y) = h(y)$ for all y and $|f_n(y)| \leq g(y)$ for all y, for an integrable function g. Then h is integrable, and

$$\int h = \lim_{n\to\infty}\int f_n.$$

[4 marks]

Now apply this with

$$f_n(y) = (1/2\sqrt{\pi n}) f(y) e^{-(x+n-y)^2/4n}.$$

Since $0 < e^{-(x+n-y)^2/4n} \leq 1$ for all y and n, we have

$$|f_n(y)| \leq (1/2\sqrt{\pi n})|f(y)| \leq (1/2\sqrt{\pi})|f(y)|$$

for all integers $n \geq 1$. We can apply the Dominated Convergence Theorem because f is integrable and because

$$\lim_{n\to\infty} (1/2\sqrt{\pi n}) f(y) e^{-(x+n-y)^2/4n} = 0.$$

Of course in this case we already know that the function 0 is integrable with integral 0. Anyway we deduce that

$$0 = \int 0 = \lim_{n\to\infty} \int f_n = \lim_{n\to\infty} \int_{-\infty}^{\infty} (1/2\sqrt{\pi n}) f(y) e^{-(x+n-y)^2/4n} dy$$

$$= \lim_{n\to\infty} e^{-n} u(x, n),$$

as required.
[4 marks]
$3 + 3 + 1 + 5 + 4 + 4 = 20.$

Parts (i) and (ii) are standard homework exercises. First part of (iii) is standard theory from lectures and a handout. Second part of (iii) is unseen although there is an application of Dominated Convergence on the problem sheets.

6.

$$\widehat{f}(\xi) = \int_{-1}^{1} (1-|x|)e^{-ix\xi}dx$$

$$= \int_{0}^{1} (1-x)e^{-ix\xi}dx + \int_{-1}^{0} (1+x)e^{-ix\xi}dx$$

$$= \left[(1-x)\frac{e^{-ix\xi}}{-i\xi}\right]_0^1$$

$$+ \left[(1+x)\frac{e^{-ix\xi}}{-i\xi}\right]_{-1}^{0} - \int_0^1 \frac{e^{-ix\xi}}{i\xi}dx + \int_{-1}^{0} \frac{e^{-ix\xi}}{i\xi}dx$$

$$= \left[(1-x)\frac{e^{-ix\xi}}{-i\xi} - \frac{e^{-ix\xi}}{\xi^2}\right]_0^1 + \left[(1+x)\frac{e^{-ix\xi}}{-i\xi} + \frac{e^{-ix\xi}}{\xi^2}\right]_{-1}^{0}$$

$$= \frac{1}{i\xi} + \frac{1-e^{-i\xi}}{\xi^2} - \frac{1}{i\xi} + \frac{1-e^{i\xi}}{\xi^2}$$

$$= 2\frac{1-\cos\xi}{\xi^2}.$$

[4 marks]
We have

$$\left|2\frac{1-\cos\xi}{\xi^2}\right| \leq \frac{4}{\xi^2}.$$

We also have

$$|1-\cos\xi| = \left|\frac{\xi^2}{2} - \frac{\xi^4}{24}\cdots\right| \leq |\xi^2|$$

for $|\xi| \leq 1$. So

$$\left|2\frac{1-\cos\xi}{\xi^2}\right| \leq \frac{4}{\xi^2}\chi_{(-\infty,-1)\cup(1,\infty)} + 2\chi_{(-1,1)}$$

The righthand side is integrable. So the lefthand side is too.
[4 marks]
(ii) Since $\widehat{f}$ is integrable, and f is continuous, one of the Inverse Fourier Theorems says that

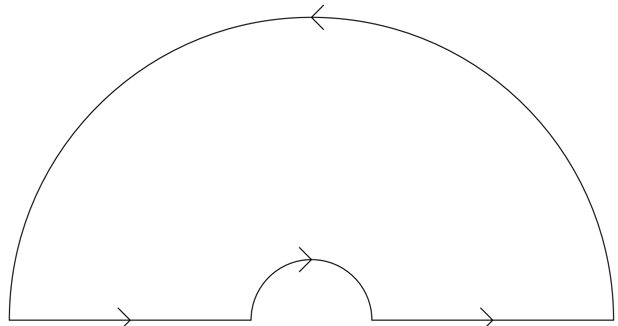
$$f(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iyt}\widehat{f}(t)dt.$$

So

$$\int_{-\infty}^{\infty} e^{iyt}\frac{1-\cos t}{t^2}dt = \begin{array}{ll} \pi(1-|y|) & \text{if } |x| \leq 1, \\ 0 & \text{if } |y| \geq 1. \end{array}$$

[2 marks]

Now let $\gamma(R,\varepsilon)$ be the contour



Then we consider

$$\int_{\gamma(R,\varepsilon)} \frac{2e^{iyz} - e^{i(yz+z)} - e^{i(yz-z)}}{z^2} dz.$$

There are no singularities inside the contour. So

$$\int_{\gamma(R,\varepsilon)} \frac{2e^{iyz} - e^{i(yz+z)} - e^{i(yz-z)}}{z^2} dz = 0.$$

[1 mark]

Let $\gamma(R)$ be the semicircle radius R and let $\gamma(\varepsilon)$ be the semicircle radius ε both oriented anitclockwise . If $y \leq -1$ and $\text{Im}(z) \geq 0$, then $\text{Re}(iyz)$, $\text{Re}(i(y-1)z)$, $\text{Re}(i(y+1)z)$ are all ≤ 0. So then

$$\left| \int_{\gamma(R)} \frac{2e^{iyz} - e^{i(yz+z)} - e^{i(yz-z)}}{z^2} dz \right| \leq \frac{4}{R^2} \text{length}(\gamma(R)) \leq \frac{4\pi}{R} \to 0 \text{ as } R \to \infty.$$

[3 marks]

On $\gamma(\varepsilon)$ we have

$$2e^{iyz} - e^{i(yz+z)} - e^{i(yz-z)} = 2 + 2iyz + 2\frac{(iyz)^2}{2} \cdots - 1 - (iyz + iz) - \frac{(iyz+iz)^2}{2} \cdots$$

$$-1 - (iyz - iz) - \frac{(iyz-iz)^2}{2} \cdots = 2yz^2 + O(z^3).$$

So on $\gamma(\varepsilon)$,

$$\left| \frac{2e^{iyz} - e^{i(yz+z)} - e^{i(yz-z)}}{z^2} \right| \leq 2|y| + O(\varepsilon).$$

So

$$\left| \int_{\gamma(\varepsilon)} \frac{2e^{iyz} - e^{i(yz+z)} - e^{i(yz-z)}}{z^2} dz \right| \leq (2|y| + O(\varepsilon)) \text{length}(\gamma(\varepsilon))$$

$$= (2|y| + O(\varepsilon))\pi\varepsilon \to 0 \text{ as } \varepsilon \to 0.$$

[4 marks]

So if $y < -1$,

$$\int_{-\infty}^{\infty} e^{iyt} \frac{1-\cos t}{t^2} dt$$

$$= \lim_{R\to+\infty,\varepsilon\to 0} \left(\int_{-R}^{-\varepsilon} + \int_{\varepsilon}^{R} \right) e^{iyt} \frac{1-\cos t}{t^2} dt$$

$$= \lim_{R\to+\infty,\varepsilon\to 0} \left(\int_{\gamma(R,\varepsilon)} - \int_{\gamma(R)} + \int_{\gamma(\varepsilon)} \right) \frac{2e^{iyz} - e^{i(yz+z)} - e^{i(yz-z)}}{2z^2} dz = 0$$

as required.

[2 marks]

$4 + 4 + 2 + 1 + 3 + 4 + 2 = 20$ marks

Similar to homework exercises on computing Fourier transforms, integrability and contour integration. An approximation to this is on the revision sheet.

7.(i) For $\mathrm{Re}(z) \geq a$,

$$\mathcal{L}(f)(z) = \int_0^\infty e^{-zx} f(x) dx.$$

[1 mark]
(ii) If $f \in L^1(0,\infty)$ then

$$|\mathcal{L}(f)(z)| = \left| \int_0^\infty e^{-zx} f(x) dx \right|$$

$$\leq \int_0^\infty |f(x) e^{-zx}| dx = \int_0^\infty |f(x)| e^{-x\mathrm{Re}(z)} dx \leq \int_0^\infty |f(x)| dx.$$

[2 marks]
(iii) We need $f(x)e^{-zx} \in L^1(0,\infty)$ (as a function of x) for all $\mathrm{Re}(z) > 0$. Now by the Cauchy Schwarz inequality,

$$\int_0^\infty |f(x)e^{-zx}| dx \leq \left(\int_0^\infty |f(x)|^2 dx \right)^{1/2} \left(\int_0^\infty e^{-2\mathrm{Re}(z)x} dx \right)^{1/2} < +\infty$$

because $f \in L^2(0,\infty)$ and $e^{-2\mathrm{Re}(z)x} \in L^1(0,\infty)$ for $\mathrm{Re}(z) > 0$, which is easily proved using Monotone Convergence:

$$\int_0^\infty e^{-2\mathrm{Re}(z)x} dx = \lim_{n\to\infty} \left[\frac{e^{-2\mathrm{Re}(z)x}}{-2\mathrm{Re}(z)} \right]_0^n$$

$$= \lim_{n\to\infty} \frac{1 - e^{-n\mathrm{Re}(z)}}{2\mathrm{Re}(z)} = \frac{1}{2\mathrm{Re}(z)}$$

[4 marks]
(iv) Plancherel's Theorem says that if $h,\ g \in L^1(-\infty,\infty) \cap L^2(-\infty,\infty)$, then $|\widehat{h}\widehat{g}| \in L^1(-\infty,\infty)$ and

$$\int_{-\infty}^\infty h(x)\overline{g(x)} dx = \frac{1}{2\pi} \int_{-\infty}^\infty \widehat{h}(\xi)\overline{\widehat{g}(\xi)} d\xi.$$

[3 marks]
Now $\mathcal{L}(f)(t+iy)$ is the Fourier transform (as a function of y) of $g(x) = f(x)e^{-tx}\chi_{(0,\infty)}(x)$. So putting $h = g$ in Plancherel's Theorem gives

$$\int_{-\infty}^\infty |g(x)|^2 dx = \int_0^\infty |f(x)|^2 e^{-2tx} dx \leq \int_0^\infty |f(x)|^2 dx,$$

while

$$\int_{-\infty}^\infty |\widehat{g}(y)|^2 = \int_{-\infty}^\infty |\mathcal{L}(f)(t+iy)|^2.$$

So this gives the result
[3 marks]

(v)

$$\mathcal{L}(\chi_{(a,b)})(z) = \int_a^b e^{-zx}dx = \left[\frac{e^{-zx}}{-z}\right] = \frac{e^{-az} - e^{-bz}}{z}.$$

[2 marks]

So $\mathcal{L}(\chi_{(0,1)})(z) = (1 - e^{-z})/z$.

[1 mark]

Now we consider $1/(z-i)$. We have

$$\lim_{z\to i} \frac{1}{|z-i|} = +\infty$$

and hence by (ii) this cannot be the Laplace transform of a function in $L^1(0,\infty)$.

[1 mark]

Also for $z = t + iy$, $0 < t \leq 2$,

$$\int_{-\infty}^{\infty} \frac{1}{|(t+iy)-i|^2}dy = \int_{-\infty}^{\infty} \frac{1}{(t^2+(y-1)^2)}dy$$

$$\geq \int_{1-t}^{1+t} \frac{1}{t^2+(y-1)^2}dy \geq \frac{2t}{4t^2} \to \infty \text{ as } t \to 0.$$

So by (iv) this cannot be the Laplace transform of a function in $L^2(0,\infty)$.

[3 marks]

$1+2+4+3+3+2+1+1+3 = 20$ marks.

Parts (i)-(iv) theory from lectures. (v) similar to homework exercises.

L^2 examples have not been set before but will be this year.

8. (i) The mean m is defined by

$$m = \int_{-\infty}^{\infty} x d\mu(x).$$

This is well defined if

$$\int_{-\infty}^{\infty} x^2 d\mu(x) < +\infty.$$

[They do not need to say this.]

The variance v is then defined by

$$v = \int_{-\infty}^{\infty} (x-m)^2 d\mu(x),$$

which is also well-defined - and finite. The Fourier transform $\widehat{\mu}$ is defined by

$$\widehat{\mu}(\xi) = \int_{-\infty}^{\infty} e^{-ix\xi} d\mu(x).$$

[4 marks]

(ii) a)

$$m = 1 \times \mu_1(\{1\}) + 0 \times \mu_1(\{0\}) + (-1) \times \mu_1(\{-1\}) = 0.$$

So

$$v = 1 \times \mu_1(\{1\}) + 0 \times \mu_1(\{0\}) + 1 \times \mu_1(\{-1\}) = \frac{1}{2}.$$

[2 marks]

b) We have

$$\int_{-\infty}^{\infty} \frac{|x|}{1+x^2} dx = \int_{-\infty}^{\infty} \frac{x^2}{1+x^2} dx = +\infty.$$

So the mean and variance are not defined in this case.

[2 marks]

(iii) We have

$$\left| \frac{\widehat{\mu}(\xi+h) - \widehat{\mu}(\xi)}{h} + \int_{-\infty}^{\infty} ixe^{-ix\xi} d\mu(x) \right| = \left| \int_{-\infty}^{\infty} \frac{e^{-ix(\xi+h} - e^{-ix\xi}}{h} + ixe^{-ix\xi} d\mu(x) \right|$$

$$\leq \int_{-\infty}^{\infty} \left| e^{-ix\xi} \frac{e^{-ih\xi} - 1}{h} + ixe^{-ix\xi} \right| d\mu(x)$$

$$= \int_{-\infty}^{\infty} \left| \frac{e^{-ih\xi} - 1}{h} + ix \right| d\mu(x)$$

[3 marks]

We have

$$(d/d\xi)\widehat{\mu}(\xi) = \int_{-\infty}^{\infty} -ixe^{-ix\xi} d\mu(x).$$

Similarly

$$(d^2/d\xi^2)\widehat{\mu}(\xi) = \int_{-\infty}^{\infty} -x^2 e^{-ix\xi} d\mu(x).$$

[2 marks]
(iv)

$$\widehat{\mu}_1(\xi) = \frac{1}{4}(e^{-ix\xi} + e^{ix\xi}) + \frac{1}{2}$$

[1 mark]

$$\widehat{g}(y) = \int_0^{\infty} e^{-x-ixy} dx + \int_{-\infty}^0 e^{x-ixy} dx = \int_0^{\infty} (e^{-x-ixy} + e^{-x+ixy}) dx$$

$$= \lim_{n\to+\infty} \left[\frac{e^{-x-ixy}}{-1-iy} + \frac{e^{-x+ixy}}{-1+iy} \right]_0^n = \frac{1}{1+iy} + \frac{1}{1-iy} = \frac{2}{1+y^2}$$

So by the Inverse Fourier Theorem

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2e^{ixy}}{1+y^2} dy = e^{-|x|}.$$

So

$$\widehat{\mu}_2(\xi) = e^{-|\xi|}$$

[5 marks]

This is not differentiable at 0, where the right and left derivatives are 1 and -1 respectively
[1 mark]
$4 + 2 + 2 + 3 + 2 + 1 + 5 + 1 = 20$ marks.

(i) and (iii) are theory from lectures. (ii) and (iv) are similar to homework exercises.