MATH 342 Problem Sheet 3: Perfect and prime numbers

Due Monday 18th February

You may use modulo arithmetic wherever you like (provided it is correct, of course).

1. Show that if $n \in \mathbb{Z}$ and $k \in \mathbb{Z}_+$, then n^k is even if n is even and odd if n is odd.

2. Show that the sum of an even number of odd integers is even and the sum of an odd number of odd integers is odd.

3. Show that if $p \in \mathbb{Z}_+$ is odd and ≥ 3, and $n \in \mathbb{Z}_+$, then $(p^{n+1}-1)/(p-1)$ is even if and only if $n+1$ is even, and $(p^{n+1}-1)/(p-1) \equiv 0 \mod 4$ if and only if either $n \equiv -1 \mod 4$, or $p \equiv -1 \mod 4$ and $n \equiv 1 \mod 2$. Verify that these are correct for $(p,n) = (3,1)$, $(3,2)$, $(5,1)$ and $(5,3)$.
Hint Divide $p^{n+1}-1$ by $p-1$, and divide $p^{2k}-1$ by p^2-1 if $n+1 = 2k$ is even. It is helpful to use questions 1 and 2.

4. Show that if $N = \prod_{i=1}^{k} p_i^{n_i}$ is an odd perfect number (where the p_i are distinct odd positive primes, and $n_i \in \mathbb{Z}_+$) then it cannot be divisible by all three of 3, 5, 7. You may assume the formula which holds if N is perfect:

$$2\prod_{i=1}^{k} p_i^{n_i} = \prod_{i=1}^{k} \frac{p_i^{n_i+1}-1}{p_i-1}. \tag{1}$$

and that, if $p_1 = 3$, $p_2 = 5$ and $p_3 = 7$ then n_1 and n_3 are even, and hence ≥ 2. This follows from (1) and question 3, because the lefthand side of (1) is divisible by 2 but not by 4.
Hint: You might find it helpful to use the equation deduced from (1)

$$2\prod_{i=1}^{k}\left(1-\frac{1}{p_i}\right) = \prod_{i=1}^{k}\left(1-\frac{1}{p_i^{n_i+1}}\right) \tag{2}$$

and try to obtain a contradiction by showing the left-hand side of this equation is bound to be smaller than the right-hand side. It might also help to look first at the case when the only prime divisors of N are 3, 5 and 7.

5. Let p be prime and let $1 \leq k \leq p-1$. Show that p is coprime to $k!$. Hence or otherwise, show that p divides the binomial coefficient

$$\binom{p}{k} = \frac{p!}{k!(p-k!)}$$

Show by example that this is not always true if p is not prime.

6. Find the maximum number of zeros at the end of:

a) $376!$

b) $\binom{376}{128}$

Hint You will need to find the maximum powers of 5 and 2 dividing each of these numbers.

I will collect solutions at the lecture on Monday 18th February. Any solutions which are not handed in then, or by 5pm that day in the folder outside room 516 will not be marked.