MATH 342 Problem Sheet 1: Introductory

Due Monday 4th February

1. Show that 113 and 127 are prime by checking for prime factors less than $\sqrt{113}$ and $\sqrt{127}$ respectively. Find the prime decompositions of the thirteen integers between 113 and 127.

Remark: The set $G(113, 127) = \{n \in \mathbb{N} : 113 < n < 127\}$ is an example of a *prime gap*. If $p < q$ are positive primes and n is composite for all $n \in \mathbb{N}$ with $p < n < q$, then $G(p, q) = \{n \in \mathbb{N} : p < n < q\}$ is a *prime gap* of *length* $q - p$ – and contains $q - p - 1$ integers. Thus, $G(113, 127)$ is a prime gap of length 14 (and contains 13 integers).

2. Why does every non-empty prime gap have even length? Show that for any $n \in \mathbb{N}$, the numbers $n! + i$, for $2 \leq i \leq n$, are all composite. Deduce that there is a prime gap of length $\geq n$ for each $n \in \mathbb{N}$.

Remark: It is unknown if there are prime gaps of every possible even length.

3. Show that apart from 3, 5 and 7, there are no three consecutive positive odd integers which are prime.

Hint: Show that for any $p \in \mathbb{N}$, one of p, $p + 2$ and $p + 4$ must be divisible by 3.

Remark: If both p and $p + 2$ are prime then p and $p + 2$ are said to be *twin primes*. It is unknown if there are infinitely many twin primes.

4. Show that if $2^n - 1$ is prime then n must be prime.

Remark: A prime number of the form $2^n - 1$ is called a *Mersenne prime*. At the time of writing, only 47 Mersenne primes are known. It is unknown if there are infinitely many.

5. Show that $2^n - 1$ is a Mersenne prime for $n = 2$, 3, 5 and 7, but that $2^{11} - 1$ is not.

6. Show that if n is prime then $2^n - 1$ is not divisible by 7 for any $n > 3$.

Hint: Follow the example in lectures to show that $2^n - 1$ is not divisible by 3.

I will collect solutions at the lecture on Monday 4th February. Any solutions which are not handed in then, or by 5pm that day in the folder outside room 516 will not be marked.