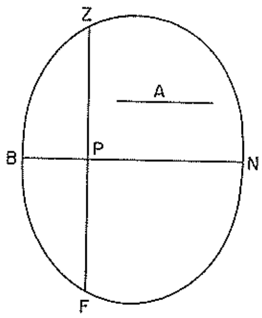
Bernoulli then indicates some other problems that can be solved by his method. They are:

(1) To find on which of the infinitely many cycloids (or circles, parabolas, etc.) passing through A with the same base AH a heavy point can fall from A to the vertical line ZB in the shortest time.

(2) To find the path of a particle moving in a medium of varying density, which curve is the same as the refraction curve studied by Huygens and himself.

(3) To find isoperimetric figures of different kinds;[10] he especially challenges his brother Johann to solve the following problem: Among all isoperimetric figures on the common base BN [Fig. 5], to find the curve BFN which—though not having itself the largest area—is such that this property belongs to another curve BZN of which the ordinate PZ is proportional to a power or a root of the segment PF or the arc BF. Johann will get 50 ducats from a gentleman known to Jakob if he solves this problem before the end of the year.

Fig. 5



21 EULER. THE CALCULUS OF VARIATIONS

After having mastered the methods that the Bernoullis had developed in the study of isoperimetric problems, Euler began to develop his own approach shortly before 1732. Where the Bernoullis had only solved specific problems, Euler began to look for a general theory. This theory, which began to take shape after 1740, appeared finally in the majestic volume entitled *Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes sive solutio problematis isoperimetrici latissimo sensu accepti* (A method for discovering curved lines having a maximum or minimum property or the solution of the isoperimetric problem taken in its widest sense; Lausanne, Geneva, 1744; *Opera omnia*, ser. I, vol. 25, 1952). The book consists of six chapters with two appendices. It does not yet present the calculus of variations in the form in which we know it—that was Lagrange's work, the importance of which Euler immediately understood when it appeared. Euler's method still has a geometric character, but Euler understood its nonessential nature: in chap. I, §32, he remarks: "It is thus possible to reduce problems of the theory of curves to problems belonging to pure analysis. And conversely, every problem of this kind proposed in pure analysis can be considered and solved as a problem of the theory of curves." Euler, however, preferred to deal with such problems in a geometric way, because by this means the method is "wonderfully

[10] Here the isoperimetric problems enter into the calculus. The theorem that of all figures of the same perimeter the circle has the largest area is ascribed to Zenodorus, who lived between 200 B.C. and A.D. 100. Some of his theorems can be found in Pappus' "Collection." See T. L. Heath, *A manual of Greek mathematics* (Clarendon Press, Oxford, 1931), 382–383.

aided and brought nearer to the intellect" (*mirifice adiuvetur atque intellectu facilis reddatur*).

Chapter I deals mainly with the type of questions that occur in the calculus of variations (the term *calculus variationum* does not appear in the book, being first employed by Euler in a paper of 1760 (1766), to indicate Lagrange's algorithm which uses δx, δy). Euler makes a difference between absolute and relative maxima and minima. In Chapter II we begin to meet the many special problems that give the book its charm. In chapter III he discusses the case in which certain other indetermined quantities occur under the integral. Chapter IV contains more special problems, chapter V discusses the relative method, and chapter VI gives more problems. The first appendix deals with elastic curves. The book abounds in examples.

The book was republished as ser. I, vol. 25 of the *Opera omnia* with a 55-page German introduction by C. Carathéodory (containing a classification of Euler's examples). There exist a partial German translation by P. Stäckel in Ostwald's *Klassiker*, No. 46 (Engelmann, Leipzig, 1894) and a complete Russian translation (Moscow and Leningrad, 1934).

We begin with a section of chapter I.

Hypothesis I. The abscissa is denoted by x, the ordinate [*applicata*] by y; further, $dy = p\,dx$, $dp = q\,dx$, $dq = r\,dx$, $dr = s\,dx$, and so on. The integral under consideration is $\int Z\,dx$, where Z must be such that $Z\,dx$ cannot be integrated; Z can be a function [*functio*] not only of x and y, but also of $p, q, r, \ldots$.

Then the principle, which Jakob Bernoulli had established, is announced in

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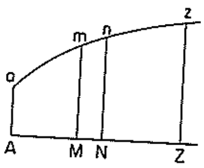
Proposition II. *Theorem.* If *amz* [Fig. 1] is a curve in which the value of the formula $\int Z\,dx$ is a maximum or a minimum, and Z is an algebraic or a determined function of $x, y, p, q, r, \ldots$, then every portion mn of this curve has the special property that, if it is referred to the abscissa MN, the value of $\int Z\,dx$ is also a maximum or minimum.

Fig. 1



The proof follows essentially the reasoning of Jakob Bernoulli (see Selection V.20(2)). One of the corollaries points out that the reasoning does not hold when in Z there appear indeterminate integrals, as $\int y\,dx$. Then follows

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Proposition III. *Theorem.* If *amz* is a curve, corresponding to the abscissa AZ, for which $\int Z\,dx$ is a maximum or a minimum, while Z contains indefinite integral expressions, then the property of a maximum or a minimum does not hold for any arbitrary part of the curve, but belongs to the whole curve corresponding to the abscissa AZ.

After the proof of this theorem, and certain corollaries, comes

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Hypothesis II. When the abscissa AZ [Fig. 2] of a curve is divided into innumerable infinitely small elements $IK, KL, LM, \ldots$, all equal to one another, and some portion AM is denoted by X, to which some variable function F

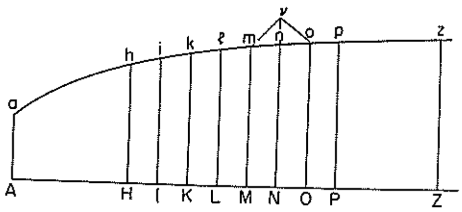


Fig. 2

corresponds, then we shall denote the values of the function F for the following points of the abscissa N, O, P, Q, and for the preceding points $L, K, I, H, \ldots$ by $F', F'', F''', \ldots$ for $N, O, P, \ldots$, and $F_{\prime}, F_{\prime\prime}, F_{\prime\prime\prime}, \ldots$ for $L, K, I, \ldots$. Thus we can indicate in an easy way, without prolix writing of differentials, the value of a subscript prime variable function at any point of the abscissa.

There follow five corollaries, which express the following identities:

$$F' = F + dF, \qquad F = F_{\prime} + dF_{\prime},$$
$$F'' = F' + dF', \qquad F_{\prime} = F_{\prime\prime} + dF_{\prime\prime},$$
$$F''' = F'' + dF'', \qquad F_{\prime\prime} = F_{\prime\prime\prime} + dF_{\prime\prime\prime},$$

and so forth, and, when the ordinates $Mm, Nn, Oo, Pp, \ldots$ are indicated by $y, y', y'', y''', \ldots$ and $Ll, Kk, Ii, \ldots$ by $y_{\prime}, y_{\prime\prime}, y_{\prime\prime\prime}, \ldots$ then, since $p = \dfrac{dy}{dx} = \dfrac{Nn - Mm}{dx}$,

$$p = \frac{y' - y}{dx}, \quad p' = \frac{y'' - y'}{dx}, \quad p'' = \frac{y''' - y''}{dx}, \quad p''' = \frac{y^{\text{iv}} - y'''}{dx},$$

$$p_{\prime} = \frac{y - y_{\prime}}{dx}, \quad p_{\prime\prime} = \frac{y_{\prime} - y_{\prime\prime}}{dx}, \quad p_{\prime\prime\prime} = \frac{y_{\prime\prime} - y_{\prime\prime\prime}}{dx}, \quad \text{etc.}$$

$$q = \frac{dp}{dx} = \frac{p' - p}{dx} = \frac{y'' - 2y' + y}{dx^2},$$

$$q' = \frac{y''' - 2y'' + y'}{dx^2}, \quad \text{etc.},$$

$$r = \frac{y''' - 3y'' + 3y' - y}{dx^3}, \quad \text{etc.}$$

Corollaries VI–VIII. If $\int Z\,dx$ is referred to the abscissa $AM = x$, then the value corresponding to the next element $MN = dx$ is $Z\,dx$. In a similar way we shall indicate the values of $\int Z\,dx$ belonging to the elements MN, MO, $OP, \ldots$ by $Z\,dx$, $Z'\,dx$, $Z''\,dx, \ldots$. Then if the expression $\int Z\,dx$ is referred to the abscissa $AM = x$, the value belonging to the abscissa AZ is

$$\int Z\,dx + Z\,dx + Z'\,dx + Z''\,dx + \cdots,$$

until we arrive at point Z.

When therefore we must find the curve for which, for the given abscissa, the value of $\int Z\,dz$ is the largest or smallest, we must obtain a maximum or minimum of this expression $\int Z\,dx + Z\,dx + Z'\,dx + Z''\,dx +$ etc.

Proposition IV. *Theorem.* When the expression $\int Z\,dx$ has a maximum or minimum for the curve *amnoz* [Fig. 2] referred to the given abscissa AZ, and we conceive another curve *amνoz* which differs from the first one only by an infinitely small amount, then the value of $\int Z\,dz$ is the same for both curves.

After demonstration and several corollaries there follows

Definition V. The *differential value* of a given expression for a maximum or minimum is the difference of the values which this expression receives on the required curve and on the curve that results from it by an infinitely small change.

One of the corollaries points out that for a maximum or minimum of $\int Z\,dx$ the differential value vanishes.

CHAPTER II

Proposition I. *Problem.* When in a curve *amz* [Fig. 2] some ordinate Nn is augmented by an infinitely small segment $n\nu$, then we must find the increase or decrease of the separate quantities determined by the curve.

To obtain the solution all quantities depending on y' are changed; the others remain fixed. For instance, $p = (y' - y)/dx$ increases by the particle $n\nu/dx$, and $p' = (y'' - y')/dx$ decreases by the particle $n\nu/dx$. Reasoning in a similar way, we find the following table of quantities that change:

Quantity:	y'	p	p'	$q_{,}$	q	q'	$r_{,,}$	$r_{,}$	r	r'
Change:	$+n\nu$	$+\frac{n\nu}{dx}$	$-\frac{n\nu}{dx}$	$+\frac{n\nu}{dx}$	$-\frac{2n\nu}{dx}$	$+\frac{n\nu}{dx}$	$+\frac{n\nu}{dx^3}$	$-\frac{3n\nu}{dx^3}$	$+\frac{3n\nu}{dx}$	$-\frac{n\nu}{dx^3}$

Among the corollaries we find one stating that from the changes in the primary quantities all the changes in the quantities that are composed of them can be found. These changes can in a sense be considered their differentials. From the ordinary differential of, say, $y'\sqrt{(1 + p^2)}$, which is $dy'\sqrt{(1 + p^2)} + y'p\,dp/\sqrt{(1 + p^2)}$, we can therefore find as the change of the function

$$+n\nu\sqrt{(1 + p^2)} + \frac{y'pn\nu}{dx\sqrt{(1 + p^2)}}.$$

Proposition II. *Problem.* When Z is a determined function of x and y alone, to find the curve az for which the value of the expression $\int Z\,dx$ is a maximum or minimum.

When $dz = M\,dx + N\,dy$, the required curve is given by $N\,dxn\nu = 0$, or $N = 0$.

Among the corollaries is the case in which Z is a function of z only, when all curves having the same axis are all solutions. When Z as a function of x and y is algebraic, the solution is algebraic. A maximum or minimum may also occur when $N = \infty$. Several examples follow; one is to find the curve for which for all curves corresponding to the same abscissa $\int (ax - yy)y\,dx = 0$ has a maximum or minimum. Answer: $ax - 3yy = 0$. Euler then discusses whether this is a maximum or a minimum, and finds the value of the integral.

Proposition III. *Problem.* When Z is a determined function of x, y, and p, so that

$$dZ = M\,dx + N\,dy + P\,dp,$$

to find among all curves corresponding to the same abscissa the curve for which $\int Z\,dx$ is a maximum or minimum.

Solution. Let amz be the required curve, and imagine the ordinate $Nn = y'$ augmented by a particle $n\nu$. Then the differential value of the expression $\int Z\,dx$, or of the equivalent expression $Z\,dx + Z'\,dx + Z''\,dx +$ etc., together with $Z_{,}\,dx + Z_{,,}\,dx + Z_{,,,}\,dx +$ etc., must be $= 0$. We obtain the differential value of the whole quantity $\int Z\,dx$, resulting from the translation of the point n to ν, when we look for the differential values of the separate terms, insofar as they have been affected by the translation, and combine them into a sum. But as a result of the translation of the point n to ν only those terms are changed that contain the quantities y', p, and p', hence only the terms $Z\,dx$ and $Z'\,dx$; since just as Z depends on x as well as on y and p, so Z' is a function of y' and p'. We must therefore differentiate those members, and substitute in their differentials for dy', dp, and dp' the above-mentioned values $+n\nu$, $+n\nu/dx$, and $-n\nu/dx$. But just as $dZ = M\,dx + N\,dy + P\,dp$, so $dZ' = M'\,dx + N'\,dy' + P'\,dp'$. The differential value of Z is therefore $P(n\nu/dx)$, that of Z' is equal to $N'\cdot n\nu - P'\cdot n\nu'/dx$, and that of $Z\,dx + Z'\,dx$, hence also of the whole expression $\int Z\,dx$, is equal to $n\nu\cdot(P + N'\,dx - P')$. But $P' - P = dP$ and for N' we may write N, so that the differential value will be $= n\nu\cdot(N\,dx - dP)$. And since we obtain the equation of the required curve by equating the differential value of the expression $\int Z\,dx$ to zero, we obtain $0 = N\,dx - dP$ or $N - dP/dx = 0$, which equation expresses the nature of the required curve. Which is what we have to find. Q.E.I.

Corollaries point out that $N - dP/dx = 0$ is always a differential equation of the second order [*gradus*], unless there is no p in P. There are therefore two constants, so that two points on the curve may be prescribed. A number of special cases are discussed. The coordinates x and y may be interchanged. Among the examples we find nos. 33, 34, 36, 38:

$Z = \sqrt{(1 + pp)}$, $y = a + nx$ (the straight line as shortest distance between two points);

$$Z = \frac{\sqrt{(1+pp)}}{\sqrt{x}}, \quad y = \int dx\sqrt{\frac{x}{a-x}}, \quad \text{the cycloid;}$$[1]

$$Z\,dy = \frac{y\,dy^3}{dx^2 + dy^2};$$[2]

$$x = \frac{a}{2}\left(\frac{3}{4p^4} + \frac{1}{pp} + 1 + \log p\right),$$

from which the curve can be constructed, using logarithms (Euler writes lp for our $\log p$).

$Z = (xx + yy)^n\sqrt{(1 + pp)}$, many cases, depending on n; for example, $n = \frac{1}{2}$ gives

$$x^2 - y^2 = 2kxy + C.$$

[1] This is the brachystochrone; see Selection V.20.

[2] This is the problem found in Newton's *Principia*, Book II, Sect. 7, Prop. 34, Scholium: to find the shape of a volume of rotation moving in a fluid with uniform velocity parallel to its axis under a pressure perpendicular to the surface and proportional to the square of the velocity in the direction of the normal to this surface. Newton without proof gave the differential equation in a geometric form; proofs were given in the *Acta Eruditorum 5* (1697), *8* (1699), and *11* (1699) by N. Fatio de Duillier, L'Hôpital, and Johann Bernoulli. The differential equation is $y\,dx\,dy^3 = a\,ds^3$; see Johann Bernoulli, *Opera omnia* (Geneva, 1744), 307–315.

Corollary III, Art. 39, finishes with the remark:

From this we obtain the following rule for the solution of problems in which the curve with a maximum or minimum of $\int Z\,dz$ is desired, where

$$dZ = M\,dx + N\,dy + P\,dp:$$

differentiate Z, place zero instead of $M\,dx$ in the differentials $M\,dx + N\,dy + P\,dp$, keep $N\,dy$ unchanged, and write $-p\,dP$ instead of $P\,dp$. Then in this way we obtain $N\,dy - y\,dP = 0$, an equation which because of $dy = p\,dx$ passes exactly into $N - dP/dx = 0$, which is the one we have already found. A method free from a geometric solution is therefore desired, from which it will be clear that in such an investigation of maxima and minima instead of $P\,dp$ we must write $-p\,dP$.[3]

Proposition IV. *Problem.* When Z is a function of x, y, p, and q, so that

$$dZ = M\,dx + N\,dy + P\,dp + Q\,dq,$$

to find among all curves corresponding to the same abscissa the curve for which $\int Z\,dz$ is a maximum or minimum.

The solution, by a reasoning along the same lines as in Prop. III, but now using

$$dy' = +n\nu, \qquad dp' = \frac{n\nu}{dx}, \qquad dq' = +\frac{n\nu}{dx^2},$$

$$dy = 0, \qquad dp = \frac{n\nu}{dx}, \qquad dq = -\frac{2n\nu}{dx^2},$$

$$dy_{,} = 0, \qquad dp_{,} = 0, \qquad dq_{,} = +\frac{n\nu}{dx^2},$$

leads to

$$n\nu\cdot dx\left(N' - \frac{P'}{dx} + \frac{P}{dx} + \frac{Q'}{dx^2} - \frac{2Q}{dx^2} + \frac{Q_{,}}{dx^2}\right) = n\nu\cdot dx\left(N' - \frac{dP}{dx} + \frac{d\,dQ_{,}}{dx^2}\right)$$
$$= n\nu\cdot dx\left(N - \frac{dP}{dx} + \frac{d\,dQ}{dx^2}\right),$$

so that the required equation is

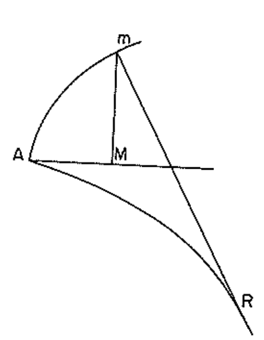
$$N - \frac{dP}{dx} + \frac{d\,dQ}{dx^2} = 0.$$

[3] This is the paragraph to which Lagrange refers; see Selection V.22, p. 407.

Again many special cases and examples are given. Example II is: To find the curve Am [Fig. 3] wich with its evolute AR and the radius of curvature mR at every point has the smallest area ARm. The answer is a cycloid. In Proposition V Euler derives for the case $dZ = M\,dx + N\,dy + P\,dp + Q\,dq + R\,dr + S\,ds + T\,dt + \cdots$ the condition

$$0 = N - \frac{dP}{dx} + \frac{d\,dQ}{dx^2} + \frac{d^3R}{dx^3} + \frac{d^4S}{dx^4} - \frac{d^5T}{dx^5} + \cdots.$$

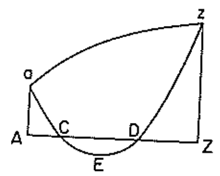
Fig. 3



In later chapters we find problems that belong to the isoperimetric type. For instance, chapter V, 41 solves the problem:

To find among all curves of the same length, connecting the points a and z [Fig. 4], the curve that encloses the largest or smallest area $aAZz$. Answer: the circle. Similarly, chapter V, 45: To find among all curves enclosing the same area $aAZz$ the curve that by rotation about the axis AZ gives the surface of smallest area. Answer: a curve of the third order, belonging to type 68 of Newton, $9b(x - c)^2 = (2b - y)^2(2y - b)$[4]

Fig. 4



The first appendix exists in an English translation by W. A. Oldfather, C. A. Ellis, and D. M. Brown, "Leonhard Euler's elastic curves," *Isis 20* (1933), 72–160; there is a German translation by H. Linsenbarth in Ostwald's *Klassiker*, No. 175 (Engelmann, Leipzig, 1910). The second appendix contains Euler's first publication of the principle of least action.

22 LAGRANGE. THE CALCULUS OF VARIATIONS

Lagrange had already studied Euler's papers when he was in his teens, and Euler's book of 1744 in particular. As a young professor at the Artillery School in Turin he began to correspond with Euler on this subject as early as 1755, when he was 21 years of age. Euler

[4] *Enumeratio linearum tertii ordinis* (1706); see Selection III.8.