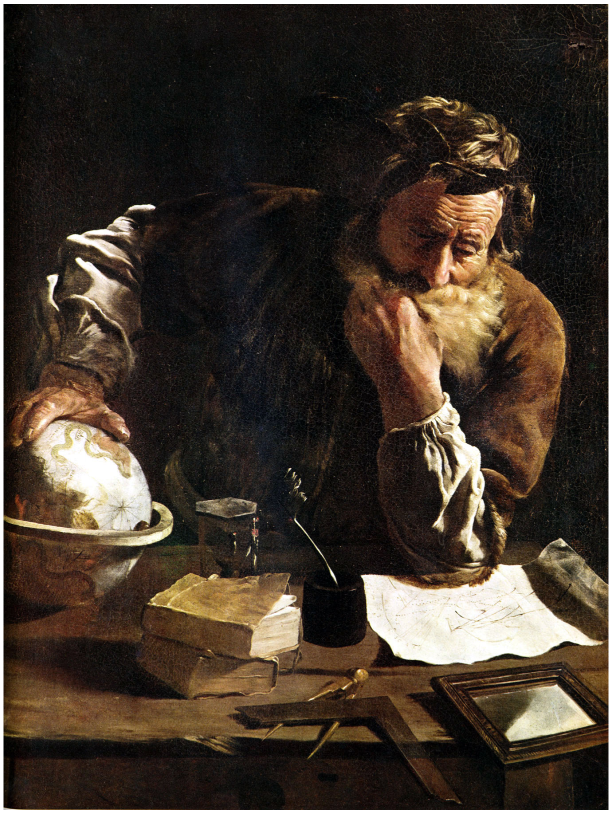
- “Calculus” is the word used to describe all mathematics involving differentiation and integration.
- “Analysis is used to describe all mathematics related to Calculus.
- Over the last two or three centuries, this has come to include the concepts of number and continuity.
- The topic this week is the early development of calculus, up to the work of Newton and Leibniz and others in the seventeenth century.
- In the analysis segment in April, we shall consider different areas of Analysis.
- Burt it would of course be possible to divide the subject area in other ways, e.g. to look at different individuals.

Kline lists four main areas which led to the introduction of calculus:

- formulae for distance, given speed or acceleration;
- tangents of curves;
- finding maxima and minima;
- finding lengths (and areas and volumes).

**The Greeks**

- Calculations areas and volumes (and lengths) was something that the Greeks developed to a fine art using the *method of exhaustion*.
- Many of these techniques are attributed to *Archimides* (287-212BC)
- Here is a depiction of Archimedes from nearly two thousand years later (16th century)



**Archimedes**

- He was born in Syracuse in Sicily, but educated in Alexandria, and spent most of his life in Syracuse.
- According to Kline, he is recognised as "the greatest mathematician in antiquity".
- His inventions are legendary:
  - the Archimedean screw ( a pump);
  - a compound pulley to launch a ship;
  - testing the debasement of a crown of gold.
- He wrote a number of texts.
- - Some of these have survived, one was only discovered in 1906 (in Alexandria);
  - some are lost;
  - some have survived only in translation;

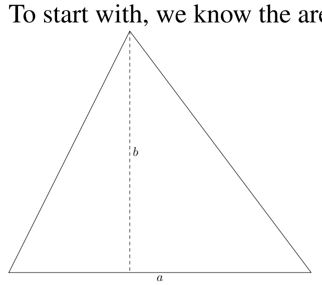
– Some of the texts, notably “On the Sphere and the Cylinder” and “On Conoids and Spheroids”, illustrate the method of exhaustion.

**An example**

- Here is an example of the method of exhaustion:
- the area of a circle is proportional to the square of its radius —
- that is, there is a constant $\pi$ such that the area of a circle of radius $r$ is $\pi r^2$, for any $r$.

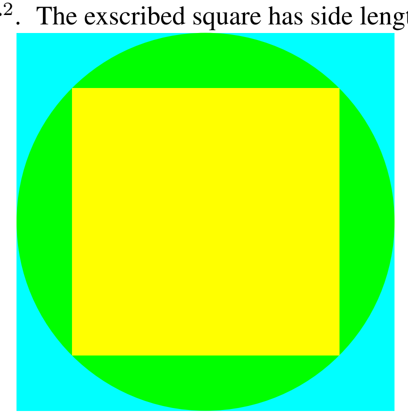
How is this done?

To start with, we know the area of any triangle.

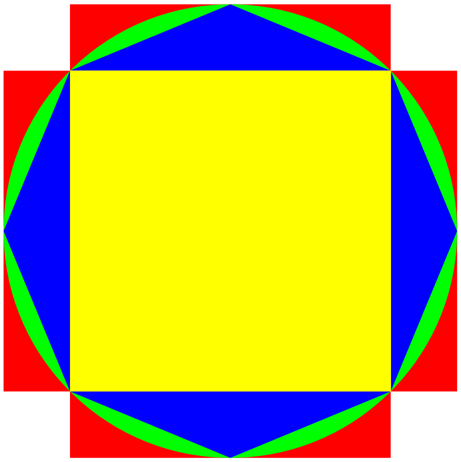


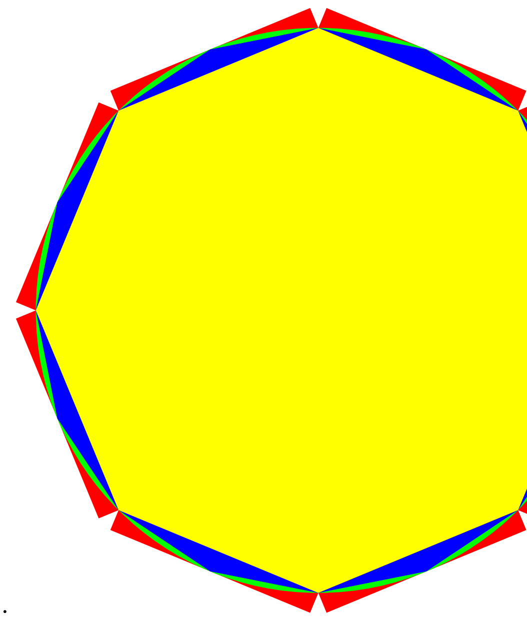

The area is $\frac{1}{2}ab$. So if we multiply all lengths by $\lambda$, the area is multiplied by $\lambda^2$.

Now take a circle of radius $r$. The inscribed square has side length $\sqrt{2}r$ and area $2r^2$. The exscribed square has side length $2r$ and area $4r^2$.

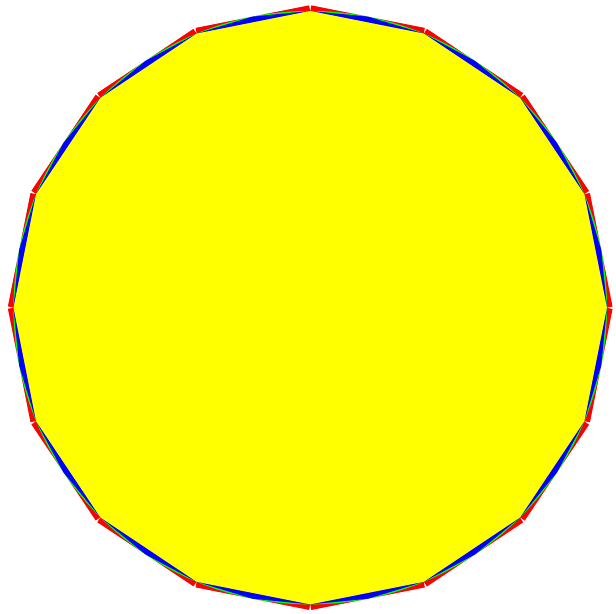


So the area of a circle of radius $r$ must be between $2r^2$ and $4r^2$ Now inscribe the regular octogon by adding an isosceles triangle on each side of the inscribed square.





What is the area of the octogon? $2\sqrt{2}r^2$. Clearly a 16-gon is better ..



and a 32-gon even better..

- Now we return to the case of the octogon.
- Let $A_1(r)$ be the area of the inscribed square and let $A_2(r)$ be the area of the inscribed octogon. (Of course $A_1(r) = 2r^2$ and $A_2(r) = 2\sqrt{2}r^2$.)
- then $A_2(r) - A_1(r)$ is the area of the four triangles that are added to the square and shown in blue.
- But the circle is completely contained in the union of the yellow square and the four rectangles which are the union of the blue and red regions.
- Each blue and red rectangle is double the area of the red triangle within it.
- So the area of the circle is between $A_2(r)$ and $A_1(r) + 2(A_2(r) - A_1(r)) = A_2(r) + (A_2(r) - A_1(r))$.
- Similarly we can inscribe a $2^n$-gon in the circle for all $n$. We inscribe a $2^{n+1}$-gon by adding an isosceles triangle to each of the sides of a $2^n$-gon.
- Let $A_{n-1}(r)$ be the area of the $2^n$-gon and $A_n(r)$ the area of the $2^n$-gon.
- $A_{n-1}(r) < A_n(r)$.
- Adding rectangles containing the isosceles triangles as we did in the case of the octogon, the area of the circle is between $A_n(r)$ and $A_n(r)+(A_n(r)-A_{n-1}(r))$.

If $A(r)$ is the area of the circle

$$A_n(r) \leq A(r) \leq A_n(r) + (A_n(r) - A_{n-1}(r)).$$

But $A_n(r) = A_n(1)r^2$ and $A_{n-1}(r) = A_{n-1}(1)r^2$. (The ratio of the areas of the $2^n$-gon inscribed in a circle of radius $r$ to that inscribed in a circle of radius 1 is $r^2$.)So

$$A_n(1)r^2 \leq A(r) \leq A_n(1)r^2 + r^2(A_n(1) - A_{n-1}(1))$$

and

$$A_n(1) \leq A(1) \leq A_n(1) + (A_n(1) - A_{n-1}(1)).$$

So

$$r^2 \frac{A_n(1)}{A_n(1) + (A_n(1) - A_{n-1}(1))} \leq \frac{A(r)}{A(1)} \leq r^2 \frac{A_n(1) + (A_n(1) - A_{n-1}(1))}{A_n(1)}$$

and

$$r^2 \frac{1}{1 + \frac{A_n(1) - A_{n-1}(1)}{A_n(1)}} \leq \frac{A(r)}{A(1)} \leq r^2 \left(1 + \frac{A_n(1) - A_{n-1}(1)}{A_n(1)}\right).$$

Since $2 \leq A_n(1) \leq 4$,

$$r^2 \frac{1}{1 + \frac{A_n(1) - A_{n-1}(1)}{2}} \leq \frac{A(r)}{A(1)} \leq r^2 \left(1 + \frac{A_n(1) - A_{n-1}(1)}{4}\right).$$

Since

$$A(1) > A_n(1) = A_1(1) + (A_2(1) - A_1(1)) + \cdots + (A_n(1) - A_{n-1}(1))$$

for any integer $k$ there can only be finitely many $n$ such that $A_n(1) - A_{n-1}(1) \geq \frac{1}{k}$. So for any integer $k$ we can find $n$ such that $A_n(1) - A_{n-1}(1) < \frac{1}{k}$ and hence, since $A_n(1) > 2$

$$r^2 \frac{1}{1 + \frac{1}{2k}} = r^2 \left(1 - \frac{1}{2k+1}\right) \leq \frac{A(r)}{A(1)} \leq r^2 \left(1 + \frac{1}{2k}\right).$$

So

$$\frac{A(r)}{A(1)} = r^2.$$

This makes use of *Archimedes' Axiom*: If

$$0 \leq x < \frac{1}{n}$$

for any positive integer $n$, then

$$x = 0.$$

**References**


- Kline, M. *Mathematical Thought from Ancient to Modern Times*, Oxford University Press, 1972.
- Netz., E., *The works of Archimedes, Translation and Commentary* Volume I: The Two Books *On the Sphere and the Cylinder*, Cambridge University Press 2004.
- http://www-history.mcs.st-andrews.ac.uk/history/