25 October, 1675.

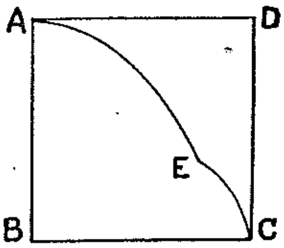
Analysis Tetragonistica Ex Centrobarycis.

[Analytical quadrature by means of centers of gravity.]

Let any curve AEC be referred to a right angle BAD; let AB $\sqcap$ DC $\sqcap a$,[8] and let the last $x \sqcap b$; also let BC $\sqcap$ AD $\sqcap y$, and the last $y \sqcap c$. Then it is plain that

$$\text{omn.}\,\overline{yx \text{ to } x} = \frac{b^2c}{2} - \text{omn.}\,\overline{\frac{x^2}{2} \text{ to } y}. \cdots\cdots (1)$$

For, the moment of the space ABCEA about AD is made up of rectangles contained by BC ($=y$) and AB ($=x$); also the moment about AD of the space ADCEA, the complement of the former is made up of the sum of the squares on DC halved $\left(=\frac{x^2}{2}\right)$; and if this moment is taken away from the whole moment of the rectangle ABCD about AD, i.e., from c into omn. x,[9] or from $\frac{b^2c}{2}$, there will remain the moment of the space ABCEA. Hence the equation that I gave is obtained; and, by rearranging it, it follows that



$$\text{omn.}\, yx \text{ to } x + \text{omn.}\,\frac{x^2}{2} \text{ to } y = \frac{b^2c}{2} \cdots\cdots (2)$$

In this way we obtain the quadrature of the two joined in one in every case; and this is the fundamental theorem in the center of gravity method.

Let the equation expressing the nature of the curve be

$$ay^2+bx^2+cxy+dx+ey+f=0, \cdots\cdots\cdots (3)$$

and suppose that $xy=z$, $\cdots$ (4), then $y=\frac{z}{x}$. $\cdots\cdots\cdots$ (5)

Substituting this value in equation (3), we have

$$\frac{az^2}{x^2}+bx^2+cz+dx+\frac{ez}{x}+f=0, \cdots\cdots\cdots (6)$$

[8] This a should be x.

[9] Here, in the Latin, "ac in omn. x" should be "a c in omn. x."

and, on removing the fractions,

$$az^2+bx^4+cx^2z+dx^3+exz+fx^2=0. \quad \ldots\ldots (7)$$

Again, let $x^2=2w$ ....(8); then, substituting this value in equation (3), we have

$$ay^2+2bw+cxy+dx+ey+f=0, \quad \ldots\ldots (9)$$

and therefore

$$x=\frac{-ay^2-2bw-ey-f}{cy+d}, \quad \ldots\ldots (10)$$

$$=\sqrt{2w}; \quad \ldots\ldots (11)$$

and, squaring each side, we have[10]

$$a^2y^2+4aby^2w+2aey^3+2afy^2+4b^2w^2+4bewy+4bfw$$
$$+e^2y^2+2fey+f^2-2c^2y^2w-4cdyw-2d^2w=0. \quad \ldots (12)$$

Now, if a curve is described according to equation (7), and also another according to equation (12), I say that the quadrature of the figure of the one will depend on the quadrature of the figure of the other, and *vice versa.*

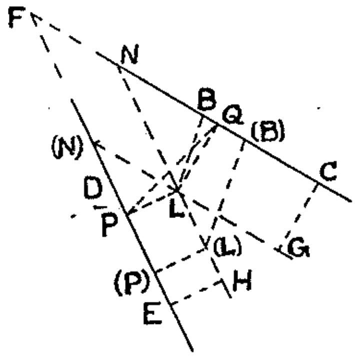
If, however, in place of equation (3), we took another of higher degree, the third say, we should again have two equations in place of (7) and (12); and continuing in this manner, there is no doubt that a certain definite progression of equations (7) and (12) would be obtained, so that without calculation it could be continued to infinity without much trouble. Moreover, from one given equation to any curve, all others can be expressed by a general form, and from these the most convenient can be selected.

If we are given the moment of any figure about any two straight lines, and also the area of the figure, then we have its center of gravity. Also, given the center of gravity of any figure (or line) and its magnitude, then we have its moment about any line whatever. So also, given the magnitude of a figure, and its moments about any two given straight lines, we have its moment about any straight line. Hence also we can get many quadratures from a few given ones. Moreover, the moment of any figure about any straight line can be expressed by a general calculation.

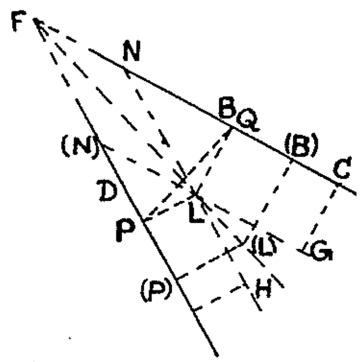
The moment divided by the magnitude gives the distance of the center of gravity from the axis of libration.

[10] In view of this accurate bit of algebra, the faulty work in subsequent manuscripts seems very unaccountable.

Suppose then that there are two straight lines in a plane, given in position, and let them either be parallel or meet, when produced, in F. Suppose that the moment about BC is found to be equal to ba^2, and the moment about DE is found to be ca^2. Call the area of the figure v; then the distance of the center of gravity from the straight line BC, namely CG, is equal to $\frac{ba^2}{v}$, and its distance from the straight line DE, namely EH, is equal to $\frac{ca^2}{v}$; therefore CG is to EH as b is to c, or they are in a given ratio.[11]



GERHARDT'S DIAGRAM.



SUGGESTED CORRECTION.

Now suppose that the straight line EH, remaining in the plane, traverses the straight line DE, always being perpendicular to it, and that the straight line CG traverses the straight line BC, always perpendicular to it, and that the end G leaves as it were its trace, the straight line G(N), and the end H the straight HN. Then, if BC and DE meet anywhere, G(N) and HN must also meet somewhere, either within or without the angle at F. Let them meet at L; then the angle HLG is equal to the angle EFC, and PLQ (supposing that PL = EH and LQ = CG) will be the supplement of the angle EFC between the two straight lines, and will thus be a given angle. If then PQ is joined, the triangle PQL is obtained, having a given vertical angle, and the ratio of the sides forming the vertex, QL : LP, also given.

When then BL is taken, or (B)(L), of any length whatever, since the angle BLP always remains the same, and in addition we have BL to LP as (B)(L) to (L)(P), therefore also BL to (B)(L) as LP to (L)(P); and this plainly happens when FL is also propor-

[11] This proves the fundamental theorem given lower down, with regard to a pair of parallel straight lines; and he now goes on to discuss the case of non-parallel straight lines.

tional to these, that is, when a straight line passes through F, L, (L),......

Hence, since we are not here given several regions, it follows that the locus is a straight line. Therefore, given the two moments of a figure about two straight lines that are not parallel,........., the area of the figure will be given, and also its center of gravity.[12]

Behold then the fundamental theorem on centers of gravity. If two moments of the same figure about two parallel straight lines are given, then the area of the figure is given, but not its center of gravity.

Since it is the aim of the center of gravity method to find dimensions from given moments, we have hence two general theorems:

If we are given two moments of the same figure about two straight lines, or axes of libration, that are parallel to one another, then its magnitude is given; also when the moments about three non-parallel straight lines are given. From this it is seen that a method for finding elliptic and hyperbolic curves from given quadratures of the circle and the hyperbola is evident.[13] But of this in a special note.

§ 5.

The next manuscript to be considered is a continuation of the preceding, and is dated the next day. Its character is of the nature of disjointed notes, set down for further consideration.

[12] The passage in Gerhardt reads:

Datis ergo duobus momentis figurae ex duabus rectis non parallelis, dabitur figurae momentis tribus axibus librationis, qui non sint omnes paralleli inter se, dabitur figurae area, et centrum gravitatis.

For this I suggest:

Datis ergo *tribus* momentis figurae ex *tribus* rectis non parallelis, *aliter* figurae momentis tribus axibus librationis, qui non *sunt* omnes paralleli inter se....

The passage would then read:

Given three moments of a figure about three straight lines that are not parallel, in other words, the moments of the figure about three axes of libration, which are not all parallel to one another, then the area of the figure will be given and also the center of gravity.

If the alternative words are *written* down, one under the other, and not too carefully, I think the suggested corrections will appear to be reasonable.

[13] Apparently, here Leibniz is referring back to the theorem at the beginning of the section.