Extended Complex Plane and Fractional-Linear Transformations

MATH206 Project B and D (after MATH243)

It is very natural to extend the complex plane by adding an infinite point to it. The result is a sphere, usually called the Riemann sphere. The study of many holomorphic functions on the sphere turns out to be much more convenient than that on the original plane. That is why this point of view so popular in various branches of Mathematics and Theoretical Physics.

In the centre of the project are properties of fractional-linear (or Möbius) transformations of the Riemann sphere. These are holomorphic one-to-one invertible mappings of the sphere into itself. If we want to keep in mind the complex nature of the sphere, we can think about everything (sets, functions, etc.) on it up to such transformations only and simplify any situation by applying an appropriate transformation. For example, any circle can be replaced just by a straight line within this approach, while any triple of points can be perfectly represented by $0, 1, \infty$.

A minimal idea about the details of the project can be gained from Sections 10.9–10.12 of Priestley's textbook.

The project is for a group of (up to) 8 students. All the theory for all the members is the same; however, theory is divided into pieces, a piece for each member of the group to write it up and be responsible for at the oral presentations and during the whole group work over the project. The problems are all individual.

Contents:
Singularities of holomorphic functions.
Extended complex plane and stereographic projection.
Equations of circlines.
The group property of fractional-linear transformations.
The circle-preserving property of fractional-linear transformations.
Invariance of the cross ratio.
Mapping of a circle onto a circle.
Mappings of regions bounded by circles.

Sources (the overlap is non-empty):
H. A. Priestley, Introduction to Complex Analysis, Clarendon Press, Oxford, 1995.
(Copies of the relevant pages can be requested)
A. I. Markushevich, Theory of Functions of a Complex Variable, vol.I, Prentice-Hall, 1965.
(Copies of the pages needed will be provided)