Extended Complex Plane and Fractional-Linear Transformations

MATH206 Project D for MATH243
2009/10 academic year

Sources (the overlap of the compulsory parts from the two sources is non-empty)**:**

[P] H. A. Priestley, Introduction to Complex Analysis, Clarendon Press, Oxford, 1995.
Compulsory Sections: 6.5–6.13 (pp 84–89), 10.1–10.3 (pp 164–166), 10.9–10.13 (pp 170–172), 10.13 (p 172), 10.18 (p 176) and 10.20 (p 178).
(Copies can be requested).

[M] A. I. Markushevich, Theory of Functions of a Complex Variable, vol.I, Prentice-Hall, 1965.
Compulsory Sections: 21 (pp 79–83) and 44–47 (pp 165–177).
Optional Sections: 31–34 (pp 119-127), 43 (pp 160–165) and 48–49 (pp 178–183).
(Copies provided).

In all the problems below, the theoretical piece and tasks under the same letter are to be done by one student.

In all the problems, z and w denote respectively the source and target variables. Solutions to some exercises are not unique, but all the tasks in these exercises have more or less unique best (that is, simplest) solutions.

THEORY

A. Singularities of Holomorphic Functions (recollection: pp 84-88 and p 89 in [P], pay special attention to remark (2) on p 88).

B. The Extended Complex Plane and Stereographic Projection (Section 6.13 of [P]).

C & D & E. Möbius Transformations: Definitions, Elementary Properties, Group Properties, Invariance of the Cross Ratio (Sections 10.9 and 10.10 of [P] and Sections 44, 46 of [M]).

F. Equations of Circlines (pp 164-166 of [P]).

G. The Circle-Preserving Property of Möbius Transformations (Sections 10.11 and 10.13 of [P]; see also Sections 45, 48, 49 of [M]).

H. Mapping of a Circle onto a Circle (Section 47 of [M]). Mappings of Regions Bounded by Circles (Sections 10.14, 10.18 and 10.20 of [P]).

PROBLEMS — A

Problem 1. Classify singularities in **C** and at infinity (for the poles, determine the orders) of the function

$$\frac{z + t\pi}{\sin^2 z},$$

where $t \in \mathbb{R}$ is some fixed real number.

Problem 2. Which point on the sphere is mapped by the stereographic projection to $e^{\pi i/3}$? Find the image under the stereographic projection of the section of the sphere

$$x^2 + y^2 + (u - \frac{1}{2})^2 = \frac{1}{4}$$

by the plane $u = 3y + 1$.

Problem 3. Write equation of the circle $\gamma(-1 - i; 3)$ in the circline form. Find the image of this circle with respect to the Möbius transformation

$$w = \frac{z}{z - t - 2i},$$

where $t > 0$ is some fixed positive real number.

Problem 4. A Möbius transformation maps the points

$$\infty, 1, i$$

to the points

$$i, 0, -i,$$

respectively. Which point does it send to 1?

Problem 5. Study Example (2) from Section 10.13 of [P]. Map the disc $|z| < 1$ onto the disc $|w| < 1$ so that

$$w(\frac{1}{2}) = 0, \quad \arg w'(\frac{1}{2}) = \frac{\pi}{2}.$$

Check that the image of i lies on the unit circle.

Problem 6. Construct a Möbius transformation sending the region

$$|z| > \sqrt{2}, \quad |z - 1| > 1$$

to the interior of an angle with its vertex at the origin. Make sure that one of the sides of the angle is the positive real axis. Find its other side.

PROBLEMS — B

Problem 1. Classify singularities in $\mathbf{C}$ and at infinity (for the poles, determine the orders) of the function

$$\frac{z+it}{(tz^2+1)z},$$

where $t \in \mathbb{R}$ is some fixed real number.

Problem 2. Which point on the sphere is mapped by the stereographic projection to -4? Find the image under the stereographic projection of the section of the sphere

$$x^2 + y^2 + (u - \frac{1}{2})^2 = \frac{1}{4}$$

by the plane $x - y + u = \frac{1}{2}$.

Problem 3. Write equation of the circle $\gamma(-2+i; 1)$ in the circline form. Find the image of this circle with respect to the Möbius transformation

$$w = \frac{z+i}{z-t},$$

where $t > 0$ is some fixed positive real number.

Problem 4. Write down all Möbius transformations that map the set $\{0, 1, \infty\}$ onto itself (that is, the points $0, 1, \infty$ are mapped to the same three points but possibly in some other order, like $\infty, 0, 1$, respectively). How many such transformations exist? Prove that they form a group.

Problem 5. Study Example (2) from Section 10.13 of [P]. Map the disc $|z| < 1$ onto the disc $|w| < 1$ so that

$$w(\frac{1+i}{2}) = 0, \quad \arg w'(\frac{1+i}{2}) = \pi.$$

Check that the image of $(-i)$ lies on the unit circle.

Problem 6. Find a common pair of inverse points for the pair of circles

$$|z-1| = 1, \quad |z-2| = 3.$$

Hence find a Möbius transformation sending the region bounded by these circles onto an annulus $1 < |w| < \rho$. Determine ρ.

PROBLEMS — C

Problem 1. Classify singularities in $\mathbf{C}$ and at infinity (for the poles, determine the orders) of the function

$$\frac{(6tz-\pi)^2}{(3-4\cos^2 z)(z+1)},$$

where $t \in \mathbb{R}$ is some fixed real number.

Problem 2. Which point on the sphere is mapped by the stereographic projection to $-i$? Find the image under the stereographic projection of the section of the sphere

$$x^2+y^2+(u-\frac{1}{2})^2=\frac{1}{4}$$

by the plane $y+x+2u=0$.

Problem 3. Write equation of the circle $\gamma(i;2)$ in the circline form. Find the image of this circle with respect to the Möbius transformation

$$w=\frac{iz+1}{z+t},$$

where $t>0$ is some fixed positive real number.

Problem 4. A Möbius transformation maps the points

$$i,\infty,0$$

to the points

$$0,1,-i,$$

respectively. Which point does it send to i?

Problem 5. Study Example (2) from Section 10.13 of [P]. Map the disc $|z|<1$ onto the disc $|w|<1$ so that

$$w(-i/2)=0, \quad \arg w'(-i/2)=-\pi/2.$$

Check that the image of i lies on the unit circle.

Problem 6. Construct a Möbius transformation sending the region

$$|z|>\sqrt{3}, \quad |z+i|<2$$

to the interior of an angle with its vertex at the origin. Make sure that one of the sides of the angle is the positive real axis. Find its other side.

PROBLEMS — D

Problem 1. Classify singularities in $\mathbf{C}$ and at infinity (for the poles, determine the orders) of the function

$$\frac{z^2 - tz + 1}{(z+i)^2(z^3+1)},$$

where $t \in \mathbb{R}$ is some fixed real number.

Problem 2. Which point on the sphere is mapped by the stereographic projection to $e^{-\pi i/6}$? Find the image under the stereographic projection of the section of the sphere

$$x^2 + y^2 + (u - \frac{1}{2})^2 = \frac{1}{4}$$

by the plane $u = 2x + 1$.

Problem 3. Write equation of the circle $\gamma(-i; 1)$ in the circline form. Find the image of this circle with respect to the Möbius transformation

$$w = \frac{z+1}{z - it},$$

where $t > 0$ is some fixed positive real number.

Problem 4. Write down all Möbius transformations that map the set $\{i, 0, 1\}$ onto itself (that is, the points $i, 0, 1$ are mapped to the same three points but possibly in some other order, like $0, 1, i$, respectively). How many such transformations exist? Prove that they form a group.

Problem 5. Study Example (2) from Section 10.13 of [P]. Map the disc $|z| < 1$ onto the disc $|w| < 1$ so that

$$w(\frac{1}{2} + \frac{1}{2}i) = 0, \quad \arg w'(\frac{1}{2} + \frac{1}{2}i) = \pi.$$

Check that the image of $(-i)$ lies on the unit circle.

Problem 6. Construct a Möbius transformation sending the region

$$|z| > 1, \quad |z - \sqrt{3}i| < 1$$

to the interior of an angle with its vertex at the origin. Make sure that one of the sides of the angle is the positive real axis. Find its other side.

PROBLEMS — E

Problem 1. Classify singularities in $\mathbf{C}$ and at infinity (for the poles, determine the orders) of the function
$$\frac{e^{itz}-1}{(z+\pi)^3},$$
where $t \in \mathbb{R}$ is some fixed real number.

Problem 2. Which point on the sphere is mapped by the stereographic projection to $e^{-\pi i/4}$? Find the image under the stereographic projection of the section of the sphere
$$x^2+y^2+(u-\frac{1}{2})^2=\frac{1}{4}$$
by the plane $u = x + 3y + 1$.

Problem 3. Write equation of the circle $\gamma(1+i;\sqrt{2})$ in the circline form. Find the image of this circle with respect to the Möbius transformation
$$w=\frac{z}{z+t+i},$$
where $t > 0$ is some fixed positive real number.

Problem 4. A Möbius transformation maps the points
$$-i, 1+i, \infty$$
to the points
$$i, 0, 1,$$
respectively. Which point does it send to -1?

Problem 5. Study Example (2) from Section 10.13 of [P]. Map the disc $|z| < 1$ onto the disc $|w| < 1$ so that
$$w(\frac{1}{2}i)=0, \quad \arg w'(\frac{1}{2}i)=\pi/2.$$
Check that the image of (-1) lies on the unit circle.

Problem 6. Find a common pair of inverse points for the pair of circles
$$|z+i|=2, \quad |z+2i|=4.$$
Hence find a Möbius transformation sending the region bounded by these circles onto an annulus $1 < |w| < \rho$. Determine ρ.

PROBLEMS — F

Problem 1. Classify singularities in **C** and at infinity (for the poles, determine the orders) of the function

$$\cos\frac{t}{z},$$

where $t \in \mathbb{R}$ is some fixed real number.

Problem 2. Which point on the sphere is mapped by the stereographic projection to $1-i$? Find the image under the stereographic projection of the section of the sphere

$$x^2 + y^2 + (u - \frac{1}{2})^2 = \frac{1}{4}$$

by the plane $2u = x + 2y + 1$.

Problem 3. Write equation of the circle $\gamma(1;2)$ in the circline form. Find the image of this circle with respect to the Möbius transformation

$$w = -\frac{iz+1}{z-t},$$

where $t > 0$ is some fixed positive real number.

Problem 4. Write down all Möbius transformations that map the set $\{1, \infty, -i\}$ onto itself (that is, the points $1, \infty, -i$ are mapped to the same three points but possibly in some other order, like $\infty, -i, 1$, respectively). How many such transformations exist? Prove that they form a group.

Problem 5. Study Example (2) from Section 10.13 of [P]. Map the disc $|z| < 1$ onto the disc $|w| < 1$ so that

$$w(\frac{i}{3}) = 0, \quad \arg w'(\frac{i}{3}) = 0.$$

Check that the image of $\dfrac{\sqrt{2} + i\sqrt{2}}{2}$ lies on the unit circle.

Problem 6. Construct a Möbius transformation sending the region

$$|z| > 2, \quad |z + \sqrt{2}i| > \sqrt{2}$$

to the interior of an angle with its vertex at the origin. Make sure that one of the sides of the angle is the positive real axis. Find its other side.

PROBLEMS — G

Problem 1. Classify singularities in $\mathbf{C}$ and at infinity (for the poles, determine the orders) of the function

$$\frac{\sin(t\pi z) - 1}{z^2 - 4z - 5},$$

where $t \in \mathbb{R}$ is some fixed real number.

Problem 2. Which point on the sphere is mapped by the stereographic projection to $-2i$? Find the image under the stereographic projection of the section of the sphere

$$x^2 + y^2 + (u - \frac{1}{2})^2 = \frac{1}{4}$$

by the plane $2x + y + u = 1/2$.

Problem 3. Write equation of the circle $\gamma(i; 1)$ in the circline form. Find the image of this circle with respect to the Möbius transformation

$$w = \frac{z - i}{z + t(1 + i)},$$

where $t > 0$ is some fixed positive real number.

Problem 4. A Möbius transformation maps the points

$$0, -i, \infty$$

to the points

$$-i, 1, i$$

respectively. Which point does it send to ∞?

Problem 5. Study Example (2) from Section 10.13 of [P]. Map the disc $|z| < 1$ onto the disc $|w| < 1$ so that

$$w(\frac{1}{3}) = 0, \quad \arg w'(\frac{1}{3}) = \pi/4.$$

Check that the image of $\dfrac{3 + 4i}{5}$ lies on the unit circle.

Problem 6. Find a common pair of inverse points for the pair of circles

$$|z + 1| = 1, \quad |z - i| = 3.$$

Hence find a Möbius transformation sending the region bounded by these circles onto an annulus $1 < |w| < \rho$. Determine ρ.

PROBLEMS — H

Problem 1. Classify singularities in $\mathbf{C}$ and at infinity (for the poles, determine the orders) of the function

$$\frac{z^3 + z^2 + tz}{z^2 + z - 2},$$

where $t \in \mathbb{R}$ is some fixed real number.

Problem 2. Which point on the sphere is mapped by the stereographic projection to $2 + i$? Find the image under the stereographic projection of the section of the sphere

$$x^2 + y^2 + (u - \frac{1}{2})^2 = \frac{1}{4}$$

by the plane $x + y - u = 0$.

Problem 3. Write equation of the circle $\gamma(2; 3)$ in the circline form. Find the image of this circle with respect to the Möbius transformation

$$w = \frac{z + it}{z + 1},$$

where $t > 0$ is some fixed positive real number.

Problem 4. Write down all Möbius transformations that map the set $\{0, i, \infty\}$ onto itself (that is, the points $0, i, \infty$ are mapped to the same three points but possibly in some other order, like $\infty, 0, i$, respectively). How many such transformations exist? Prove that they form a group.

Problem 5. Study Example (2) from Section 10.13 of [P]. Map the disc $|z| < 1$ onto the disc $|w| < 1$ so that

$$w(\frac{-i}{3}) = 0, \quad \arg w'(\frac{-i}{3}) = \pi/3.$$

Check that the image of $(-i)$ lies on the unit circle.

Problem 6. Find a common pair of inverse points for the pair of circles

$$|z - i| = 1, \quad |z + i| = 4.$$

Hence find a Möbius transformation sending the region bounded by these circles onto an annulus $1 < |w| < \rho$. Determine ρ.

ORGANIZATIONAL MATTERS

The members of the group should distribute the letters A-H between themselves. The student to whom, say, the letter C has been assigned, is responsible for presenting the theoretical piece C at both presentations and also for writing up this piece. All the problems under the same letter C are to be done by this student. The same applies to A,B,...,H

PRESENTATIONS

The 1st presentation (week 6 approximately) is assumed to be done on a considerable part of the theory involved.

The final presentation should give ideas of solving all types of problems.

WRITING UP

Writing up the introduction could either be shared or done by one student. The structure of the written presentation should be as follows: the student responsible for the letter A writes up the corresponding theory (definitions and theorems) and solutions of the A-problems. The same applies to the letters B-H.

The final text should look as follows: an introduction, then the theoretical pieces (A-H), after that the solutions to the problems, first A, then B, C etc., followed by a conclusion.